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Proceedings of the American Academy of Arts and Sciences.

Vol. 58. No. 1.— February, 1923.

VISION AND THE TECHNIQUE OF ART.

By A. Ames, Jr., C. A. Proctor and Blanche Ames.

Investigations on Light and Heat, published with aid from the
Rumford Fund.

VISION AND THE TECHNIQUE OF ART.

By A. Ames, Jr., C. A. Proctor and Blanche Ames.

Presented by Louis Bell.

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INTRODUCTORY.

There are many well known instances where very successful use has been made by artists of certain of the laws of vision as a basis of technique. The "pointillist" technique of Pissaro and Monet is probably the best example.

The artist Birge Harrison[1] has gone farthest towards recognizing the dependency of the technique of art on the laws of vision. He most forcefully and lucidly shows that a picture in its general form should be similar to our retinal impressions. Mr. Ames and his sister, Blanche Ames, who were painting together came to a similar conviction in 1912. Attempts were made to paint pictures of this nature. The difficulty of analyzing the character of images of objects upon which the eye was not focused was at once encountered. This is without doubt due to the universal and probably immemorial human practice of looking directly at, i.e., focusing upon anything we desire to judge or analyze. Mr. Ames therefore undertook to determine scientifically the characteristics of the images of those objects upon which the eye is not focused in the belief that an intellectual conception of the characteristics of such images would help in the visual recognition and analysis of them, and thus be an aid in the technique of art.

He thought that the desired information could be obtained in a few weeks — at most a few months. The scientific data upon which such information is based, however, had not been worked out. This necessitated research work upon which he has been occupied up to the present. The data collected, although representing a considerable advance, constitute hardly more than the preliminary steps towards definitely determining the characteristics of the retinal picture. Most of the data has been published in a paper by Mr. Ames and Dr. C. A. Proctor, entitled "Dioptrics of the Eye." [2]

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[1] "Landscape Painting" by Birge Harrison. Charles Scribner's Sons.

It is the purpose of the present paper to convey as clear an understanding as possible of the nature of the retinal picture and to point out conclusions to which such an understanding leads us.

PREFACE.

A consideration of the analogy between our eye and a photographic camera is helpful for a general understanding of the subject.

Broadly speaking the eye is like a camera, or more truly speaking cameras were made like the eye, the lens of the camera corresponding to the lens of the eye and the plate or film to the retina. As the character of a photograph depends upon the kind of lens and plate used, so the characteristics of our retinal picture depend first upon the nature of the lens of the eye and second upon the nature and sensitivity of the nerve structure, i.e., the retina upon which the image is formed.

Certain photographic lenses are corrected.[3] The detail in photographs taken with such lenses is clear and distinct over the entire picture and free from distortion. Other photographic lenses which are not corrected produce pictures in which the detail is indistinct and distorted in varying degrees.

The lens system of the eye is not corrected. The details in part of the image formed by it are clear, in other parts unclear and all more or less distorted.

Furthermore, the retina, instead of being equally sensitive over its entire area as is a photographic plate, varies in sensitiveness in different parts.

As a result of the effects of the lens system of the eye and the effects of the variable sensitivity of the retina the retinal picture has characteristics which make it markedly different from photographs taken with a corrected lens or even with an uncorrected lens. For though the eye has an uncorrected lens, the lack of corrections, so to speak, is markedly different from that existing in any known uncorrected photographic lens, and so produces a different effect.

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[2] Journal Opt. Soc. of Amer., Vol. V, Jan. 1921.

[3] The image of a scene formed by a simple lens (called uncorrected), such as a spectacle lens or a magnifying glass, is not an exact reproduction of the scene itself. The detail of objects at the center of the picture is slightly softened and has chromatic edges. The detail at the sides of the picture is still more softened and objects at the side are bent and distorted.

To make the detail at the center as well as at the sides perfectly sharp and to do away with distortion, so called corrected lenses were devised. They consist of a combination of two or more simple lenses of determined character and separation.

To give an understanding of the more specific characteristics of the retinal picture it is necessary to take up and describe the characteristics of the images formed of objects in different parts of the field of view, i.e., the scene at which we are looking.

We shall take up the images of objects in the various parts of the field of view in the following order: First, the images of objects at which the eye is directly looking, or, in other words, those objects which lie in the line of vision and which are in sharp focus. See A, Figure 1. This will be covered in Chapter I, Distinct Vision.

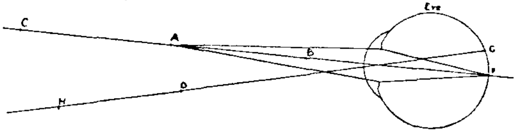

FIGURE 1. Diagram showing positions of various objects in the field of view. F B A C is the axes of vision or line of sight along which the eye is looking. A is the object on which the eye is focused. It will be imaged sharply on the fovea F. B and C are points on the line of sight inside and outside the point of focus. D is an object to one side of the line of sight. It will be imaged on the peripheral part of the retina at G.

Second, the images of objects which are on the line of vision nearer and farther away than the object in focus (see B and C, Figure 1); Chapter II, Depth of Field Axial.

Third, the images of objects which are not in the direct line of vision as D, Figure 1. These images will be described in Chapters III, IV and V.

Chapter III, Depth of Field (Lateral), will deal with the characteristic imaging of objects not in the direct line of vision and at different distances from the eye. See D and H, Figure 1.

Chapter IV, Distortion in Form, will deal with the distortion of the images of objects not in the direct line of vision.

Chapter V, Peripheral Color Sensitivity, will deal with the change in the local color of images of objects not in the direct line of vision.

In Chapter VI the effect of binocular vision will be considered in a general way.

In Chapter VII the results will be summarized and discussed.

Chapter I.

DISTINCT VISION.

Distinct vision deals with the nature of retinal images of objects at which the eye is looking directly, i.e., that are in sharp focus. As we have a very definite conception of the appearance of objects at which we look directly it may hardly seem necessary to analyze the characteristics of the images of such objects. It is believed however that such an analysis will not only be instructive but will make it easier to understand the characteristics of the images of objects upon which the eye is not focused, which will be taken up later.

Owing to the nature of the lens system of the eye the image on the retina formed by an object at which the eye is directly looking is not an exact copy of the object itself. The image is spread out and somewhat diffused. This is due primarily to three factors; chromatic aberration; spherical aberration; and irregular astigmatism. These factors and their effects will be considered in the order given.

CHROMATIC ABERRATION.

Chromatic aberration causes light of different wave length or color to come to focus at different distances behind the lens, light of shorter wave length, i.e., towards the blue end of the spectrum focusing nearer the lens. This is shown in Figure 2.

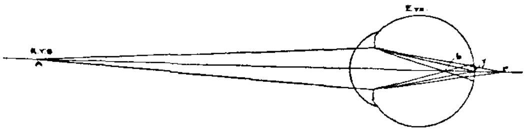

Figure 2. Diagram showing chromatic aberration of the eye.

If we have three point sources of light at A, one red, one yellow, and one blue, the image of the blue source will be formed at (b) the image of the yellow source at (y), while the red image will be at (r). Figure 3 shows the shape of image bundles as formed in Mr. Ames' eye by point sources of red, yellow and blue light. These of course are much enlarged; the lens system of the eye is to the left.

It will be seen that the blue bundle lies to the left or nearer the lens than the yellow bundle and that the yellow bundle lies to the left of the red.

The eye normally focuses for yellow light, i.e., so that the small cross section of the yellow bundle falls on the retina. If the eye looks at red, yellow, and blue point sources at the same time the retina would be in the position relative to the three bundles as shown in Figure 3. It will be noted that the smallest cross section of the blue bundle lies in front of the retina, that of the red behind.

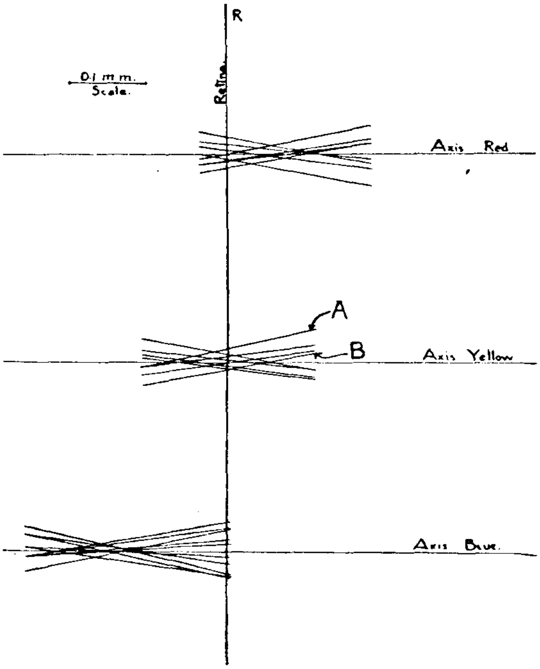

FIGURE 3. Image bundles formed by light from red, yellow, and blue point sources. Their displacement right and left shows the Chromatic Aberration of the eye. The different distances from the retina at which the individual rays, in each bundle, cut the axis shows the Spherical Aberration.

It will also be noted that where the red and blue bundles cut the retina they are much larger in diameter than the yellow bundle. This is also shown in Figure 4 which is a photograph of the magnified image of a point source of white light taken with a lens which has approximately the same chromatic aberration as the eye. A white light source is composed of light of all wave lengths. The camera was focused to get as sharp an image as possible of the yellow light and then the three pictures were taken, the top one through a red screen, the middle one through a yellow,[4] the lower one through a blue. The top photograph therefore shows the image formed by the red rays in the white light source, the middle one that formed by the yellow rays, and the bottom one that formed by the blue rays.

Colored images of the same relative difference in size are apparent to the eye looking at a point source through monochromatic screens or filters if the eye is kept focused for yellow. Without any screens or filters the eye receives an image of the nature obtained by combining the three images in Figure 4. It would have a bright center tending towards yellow in color, surrounded by larger and larger rings of shorter and longer wave lengths, the blue rings extending out farther than the red.

A comparison of the image formed by the eye and that formed by a lens corrected for chromatic aberration is of interest. Such a lens is designed so that light of all wave lengths focuses at the same distance from the lens. This is shown in Figure 5 where it will be seen that the narrowest part of the bundles for light of different wave length, instead of being focused at different distances from the lens, all focus at the same distance. Figure 6 is a photograph of a point source of white light taken with a corrected lens. The photographs were made in the same way as those in Figure 4. It will be noticed that where the images taken through the red, yellow and blue screens with the uncorrected lens are all of different size similar images formed by a corrected lens are substantially all the same size. That is, this correction causes the images of an object in focus to be much sharper or clearer than those formed in the eye.

We have been speaking so far only of point source objects. If the object is either a line or an edge, for instance a dark edge of a window against a light sky, the diffusion circles we have been describing take the form of diffusion edges. The distribution of color in these diffusion edges follows the same laws as govern that in the diffusion circles. This is shown clearly in Figure 7, which is a photograph taken through a lens having approximately the same chromatic aberration as the eye,

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[4] As the filters actually used in taking this and following pictures were Eastman Kodak Co. films Red No. 25, Green No. 58 and Blue No. 48, it would be more exact to use the word "green" instead of "yellow." The difference in wave length between the green and yellow however is such that there would be no appreciable differences in the appearance of the photographs whether a green or yellow filter was used. The term yellow will therefore be used for sake of simplicity and clearness.

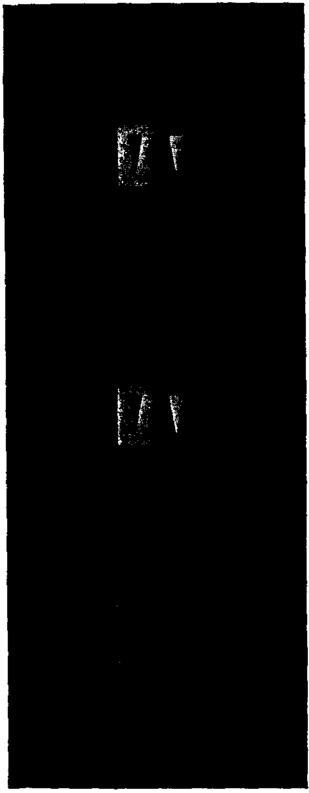

Fig. 7.

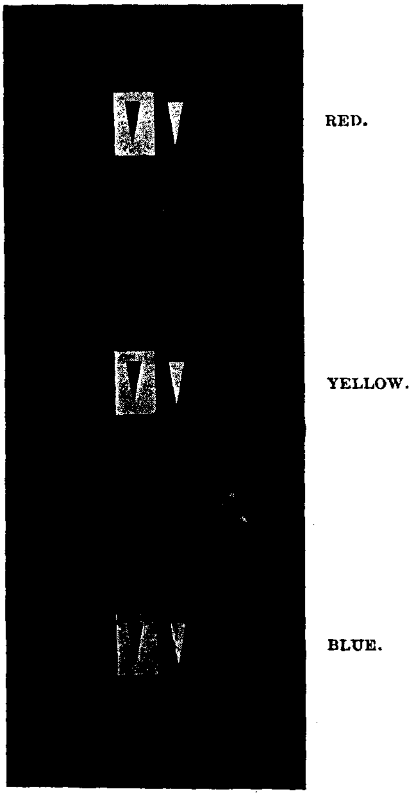

Fig. 8.

FIGURE 7. Three color photograph of black and white wedges taken with lens having the same chromatic aberration as the eye. The base of the wedges which were about six feet distant measured ¼ inch.

FIGURE 8. Same as Figure 7 taken with a lens corrected for chromatic aberration.

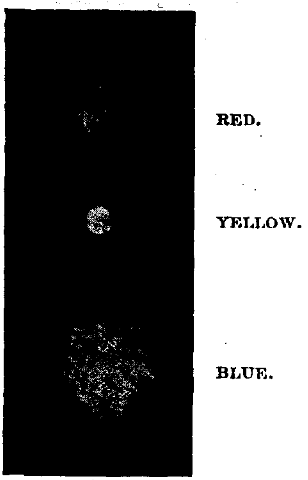
RED.
YELLOW.
BLUE.

Fig. 4.

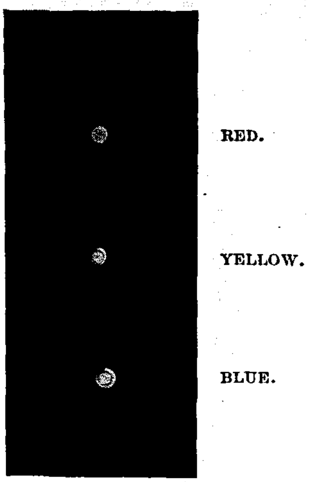
RED.
YELLOW.
BLUE.

Fig. 6.

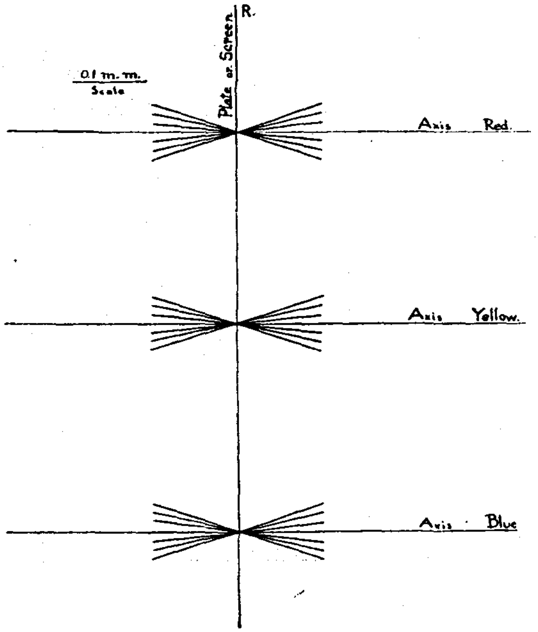
R.
Plate or Screen.
0.1 m.m.
Scale
Axis Red.
Axis Yellow.
Axis Blue

Fig. 5.

Figure 4. Three color photograph of a white light point source taken with lens having same chromatic aberration as the eye. Magnification 110 diameters.

Figure 5. Image bundles which would be formed by light from red, yellow and blue point sources passing through a lens perfectly corrected for chromatic and spherical aberration.

Figure 6. Three color photograph of a white light point source with a lens corrected for chromatic aberration. Magnification 110 diameters.

of a white wedge on a black background and a black wedge on a white background. The base of the wedges was one-fourth inch and their distance six feet from the camera.

The top picture was taken through a red, the middle through a yellow, and the bottom one through a blue filter. The camera was focused to give a sharp image with the yellow filter. The top picture shows the kind of image that is formed on the retina by the red light, the middle one the kind of image that is formed by the yellow light and the bottom one the kind of image that is formed by the blue light. If you imagine these superimposed, which is what takes place on the retina, the combined picture will have a blue diffused edge extending over the black and a less wide red edge. The color of the white near the edges will be slightly yellowish due to the subtraction of the blue and red.

With a lens free from chromatic aberration no such effect is produced. This is shown by Figure 8, which is a photograph of the same objects taken at the same distance and in the same way, with a corrected lens. The images of all the wedges in this case are sharp and clear and of the same size. They will all exactly superimpose and no chromatic edges will be formed.

Under ordinary circumstances unless the attention is especially called to them these chromatic rings and edges formed in the eye are not seen. This is due, it is believed, partly to the fact that the red rings overlie the green, which being complementary colors form white light, and to the fact that the blue is so spread out that it is relatively weak. However, if one looks carefully for these rings or edges they can be seen around an arc light at night, in the blue haze or halo on the dark background.

The red and blue chromatic circles or edges can be seen separately by looking at a dark object, such as a window sash against a bright sky at a distance of three to six feet and shutting off the light from half the pupil by passing a card or piece of paper close to the eye, the edge of the paper being kept parallel to the window sash. One side of the frame will have a red orange edge, the other a bluish edge. If the card is brought in from the other direction the color of the edges will reverse. Without the card the colors overlie each other and become much less visible for the reason given above. However, once the phenomenon has been noticed a soft floating purplish edge becomes apparent even without the card.

A very striking example of chromatic aberration, and one which gives a very good idea of its magnitude, is apparent when one looks at the purple railroad switch lamps used, for example, by the Boston and Albany Railroad. At close range these lamps appear purple, but as one moves away the light appears to have a red center surrounded by a blue disk or halo; the farther off one goes the larger the blue halo appears.

SPHERICAL ABERRATION.

Another factor that has a bearing on the nature of the retinal image of an object on which the eye is focused is the spherical aberration mentioned above. In a lens free from spherical aberration the rays that come through different zones of the lens, for example those that come through the lens near its center and near its edge, all focus down to a point. This is shown in Figure 5 where rays near the outside of the cone and those near its center all go through one point at the apex of the cone. In a lens which has spherical aberration this is not true. The rays from different zones of the lens do not pass through the same point. This is shown in Figure 3, where it will be seen that ray A, for instance, crosses the axis far to the left of ray B. Due to this fact the image of a point source formed in the eye will be larger and with softer edges than that formed by a corrected lens.

IRREGULAR ASTIGMATISM.

There is still another factor that has a bearing on the nature of retinal images of an object on which the eye is focused. That is irregular astigmatism. This term covers all irregularities in the shape and distribution of light in the image due to such things as opaque substances or irregularities of densities in the lens system. A most marked example of this factor is the star shape appearance, known to everyone, of a small source of light. If the eye were not subject to irregular astigmatism of some sort the image of a small source, though it might be affected by chromatic and spherical aberration, would be circular. Such retinal images, however, are always star shaped.

SUMMARY.

The three aforementioned factors, chromatic aberration, spherical aberration, and irregular astigmatism, cause the retinal image of an object upon which the eye is focused to have a characteristic appearance both as to the amount of detail which is visible and in the appearance of all edges.

From the point of view of the technique of art the question arises is it necessary that these characteristics be reproduced in the depicting of an object upon which the eye is focused?

The care used by the better artists to paint broadly, i.e., not to get, even in the most detailed parts of their pictures, any more detail than is apparent to them at the distance at which they stand from their model or scene and their quite common practice of softening and treating their edges, is evidence that to be technically satisfactory from an artistic point of view the detail of an object on which the eye is focused should be depicted with the same characteristics with which it is imaged upon the retina. This evidence is further supported by the fact, as will be shown later, that the characteristics of the images of objects upon which the eye is not focused, must be reproduced to bring out the appearance of depth and to make the pictures pleasing.

CHAPTER II.

DEPTH OF FIELD (AXIAL).

In this chapter we will deal with the nature of images of objects that are in the line of vision nearer and farther away from the observer than the object upon which the eye is focused.

The images of such objects will have markedly different characteristics from those of objects in focus.

These characteristic differences are due primarily to two factors. First, Depth of Field of the lens system; second, Chromatic Aberration.

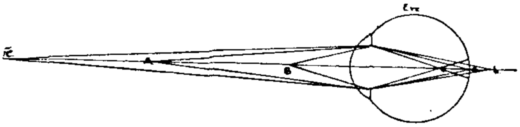

FIGURE 9. Diagram showing diffusion of images of point source objects not in focus.

DEPTH OF FIELD.

For sake of brevity and simplicity a full explanation of Depth of Field will not be gone into. It is evident however that an object which lies nearer or farther from the observer than the object upon which he is focused will be imaged not on the retina but behind it or in front of it. This is shown in Figure 9.

The image of point source object A upon which the eye is focused is imaged on the retina at (a), that of B behind the retina at (b), that of C in front of the retina at (c). Where the image bundle which forms the image (b) cuts the retina it is a cone of considerable size. The object B will appear as a diffusion circle. The image bundle which focuses down to the image C in front of the retina is spread out again and also appears as a diffusion circle. If instead of a point source the objects at B and C are objects with edges the edges will have the well known appearance that is seen on an object that is out of focus, i.e., a pencil held near the eye while the eye is focused on a distant object. The nearer the objects B and C approach A the smaller will be the size of these diffusion circles and the more similar all their images become. The magnitude of this effect depends upon the focal length of the lens system and the size of the aperture. In the eye this means, broadly speaking, the length of the eye and the size of the pupil. It is very marked where either the objects in or out of focus are close to the eye but decreases as they are moved away and becomes imperceptible when they all are at a distance of thirty feet or more.

CHROMATIC ABERRATION.

As was described in Chapter I the effect of chromatic aberration is to cause light of different wave length or color to focus at different distances from the lens. See Figure 2.

If we move the blue point source at A, Figure 2, towards the eye, the eye being kept focused on A, it will cause the image of the blue source to move back towards the retina. A position B, Figure 10, will be found where a blue light will focus sharply on the retina while the eye is still focused on the point at A. If we move the red light away from the eye a similar position R, Figure 10, will be found where the red light will also be in focus.

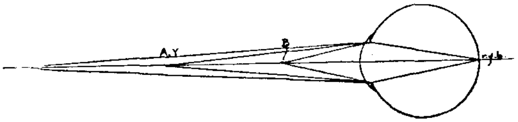

FIGURE 10. Diagram showing how images from sources of different colors situated at different distances from the eye can all be in focus at the same time.

That is all three lights will be in focus at the same time although they are at very different distances from the eye. For example if the eye is focused on a yellow light at a distance of six feet it will see

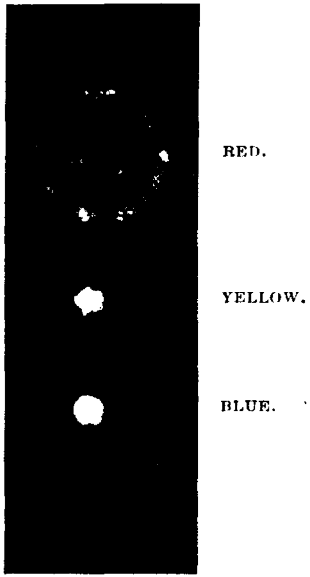
RED.
YELLOW.
BLUE.

Fig. 11.

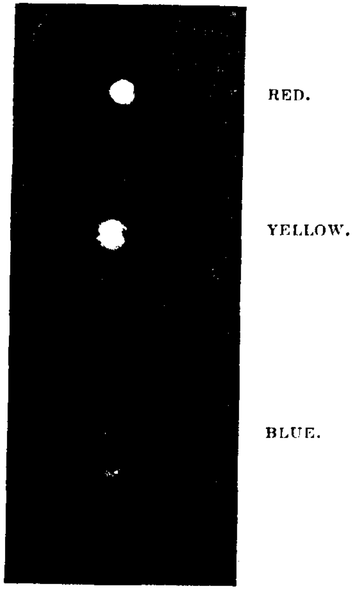
RED.
YELLOW.
BLUE.

Fig. 12.

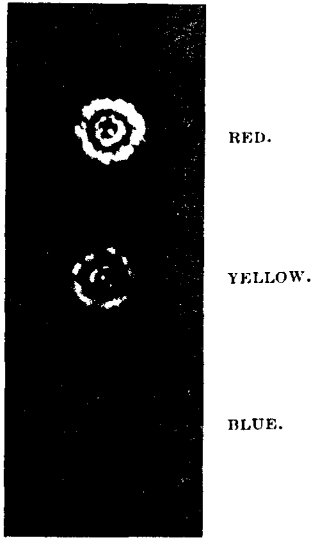
RED.
YELLOW.
BLUE.

Fig. 13.

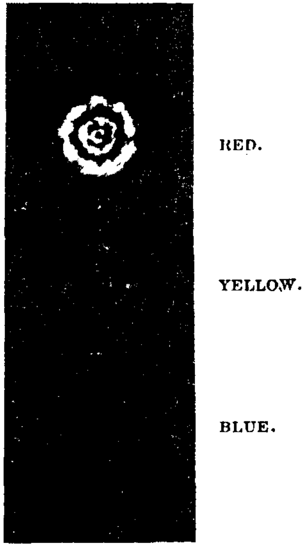
RED.
YELLOW.
BLUE.

Fig. 14.

FIGURE 11. Three color photograph of a white light point source taken with a lens having the same chromatic aberration as the eye. The lens was focused on an object six feet away. The light source was three feet away. Magnification 110 diameters.

FIGURE 12. Same as Figure 11 except that the light source was eighteen feet away.

FIGURE 13. Same as Figure 11 taken with a lens corrected for chromatic aberration.

FIGURE 14. Same as Figure 12 taken with a lens corrected for chromatic aberration.

sharply at the same time a blue light about three feet away and a red light at about twenty feet away.

We described in Chapter I the characteristic chromatic diffusion circles and edges of an object in focus. We saw that a white light point source as at A, Figure 10, would form an image with a yellowish center surrounded by red and blue diffusion circles. A white light point source at B, Figure 10, however, will form quite a different image. The blue in the white light being in focus will, as we have just shown, form a sharp image. The yellow will be out of focus and form a yellow diffusion circle around the blue and the red will be still more out of focus and form a larger red diffusion circle which will extend outside that of the yellow.

Figure 11, is a photograph of a point source of white light taken with a lens which has approximately the same chromatic aberration as the eye. The lens was focused to give a sharp image of the yellow rays in a white light source about six feet away. The pictures are of a white light point source about three feet away. As in Figure 4 the top image was taken through a red, the middle one through a yellow, and the bottom one through a blue filter. The combined image which is the appearance the white light point source would have to the eye is markedly different from that shown in Figure 4.

A white light point source at R, Figure 10, will form a still different kind of image. The red in the white light being in focus will form a sharp image. The yellow will be out of focus and form a yellow diffusion circle around the red and the blue being still more out of focus will form a larger blue diffusion circle.

Figure 12 is a photograph similar to those described in Figure 4 and Figure 11, but with the objects twenty feet away. The images formed are very different from those shown in Figure 4 and 11.

Images of white light point sources situated at other distances along the axis will vary from those shown above, their characteristics depending on the position of the fixation point and their distance from it.

A lens corrected for chromatic aberration forms very different images of similar white light point sources. Figures 13 and 14 are photographs taken with a corrected lens. The photographs were made in the same way as those shown in Figures 11 and 12 of white light point sources in similar positions. As the point in the focus of a corrected lens is in focus for all colors so a point out of focus is out of focus for all colors and to the same extent. The diffusion circles for red, yellow and blue light in Figures 13 and 14 are therefore all about the same size. The combined image although enlarged and fuzzy will appear white, i.e., without colored edges, no matter in what direction or distance the white light point source is from the point in focus.

As described in Chapter I if instead of a point source edges are used, as a dark object against a light background, the above described diffusions make themselves evident in the form of chromatic edges.

Figure 15 shows a photograph taken with a lens, which has approximately the same chromatic aberration as the eye, in the same way and of the same objects as described in Figure 7. The lens was focused as described for Figure 11 for yellow at six feet, the objects being three feet away. The combined images are marked by the orange red edges extending over the dark and the blueness of the white along its edges due to the subtraction of the red and yellow light which has diffused over the black.

Figure 16 is a similar photograph, the focus of the lens remaining the same, the objects being placed at a distance of about twenty feet. In this case the combined images are characterized by the green and blue extending over the dark and the redness of the white along its edges.

It is regretted extremely that these pictures cannot be reproduced in color as they not only show these characteristics much more clearly, but are very beautiful.

Figure 17 is similar to Figure 15 taken with a corrected lens. As would be expected from what has been said of Figures 13 and 14 the images taken through the different filters are all diffused to the same extent. As a result the composite picture shows diffused but no colored edges.

Once one's attention has been called to them, these characteristic chromatic edges in the retinal images are very easily seen. The aberration of the red rays over the dark, characteristic of images of objects which lie inside the focus, was first noticed in the warm color of black specks on a windowpane viewed from a few feet when the eye is focused on the distant sky. All of the characteristic colored edges can be easily seen with the following arrangement. Against the white wall of a room or a piece of cardboard or sheeting put up at the distance of about twenty feet a black object, preferably one that comes down to fine black points, as iron grill work, black wedges of paper will do. At six feet distance put up any small object to focus on in line with the distant objects. At three feet distance in the same line put up another black object preferably like the first. If the eye is kept focused on the object at six feet the dark edges of the near object appear to have a reddish orange tinge next to which the light appears colder or bluer. The edges of the object at twenty feet, in fact the whole surface if the

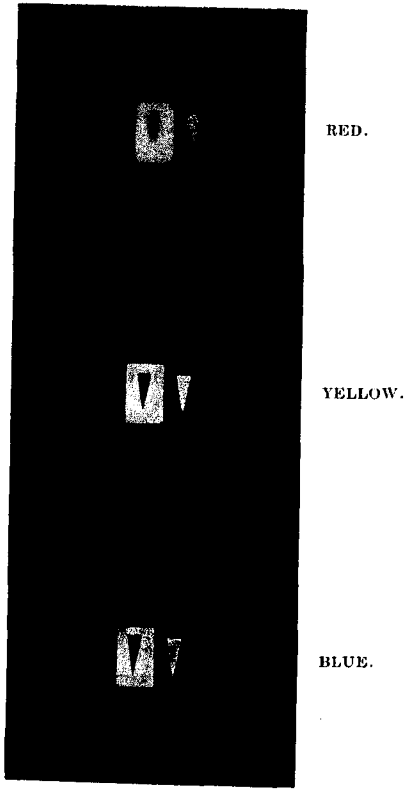

Fig. 15.

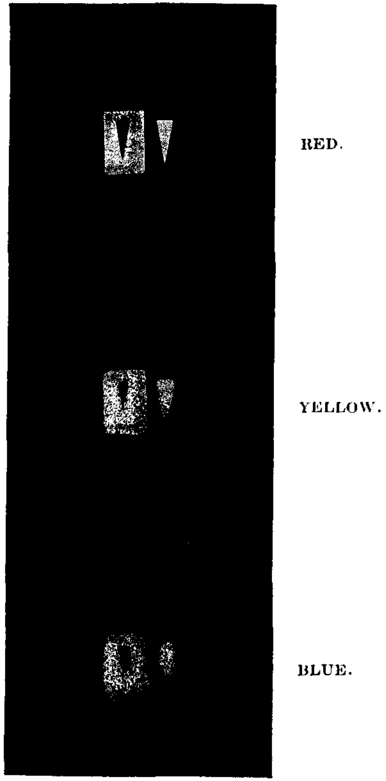

Fig. 16.

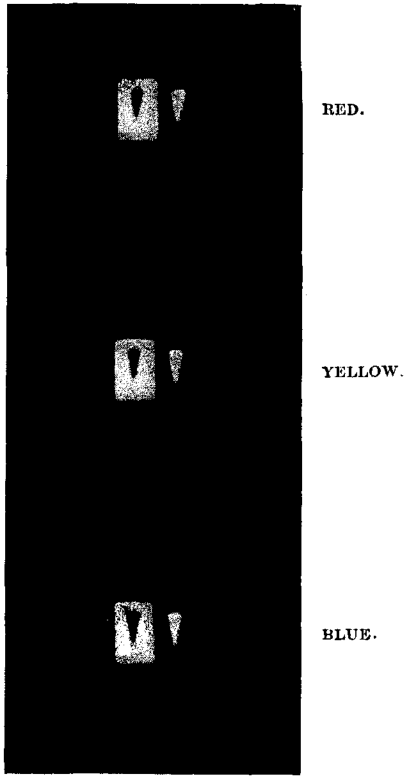

Fig. 17.

Figure 15. Three color photograph of black and white wedges taken with lens having same chromatic aberration as the eye. The lens was focused on an object six feet away, the wedges were three feet away. Width of base of wedges $\frac{1}{8}$ inch.

Figure 16. Same as Figure 15 except that wedges were eighteen feet away. Width of base of wedges $\frac{3}{4}$ inch.

Figure 17. Same as Figure 15 taken with a lens corrected for chromatic aberration.

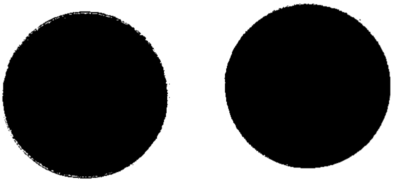

Fig. 17a — No. 1.

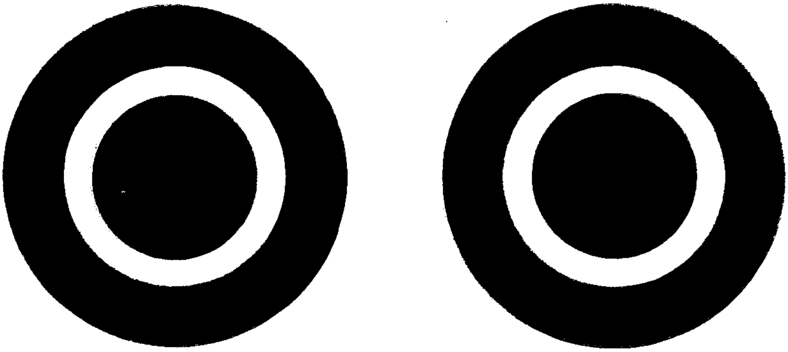

Fig. 17a — No. 2.

dark objects are narrow enough, appear very bluish while the lights next to the dark objects appear pale orange. Of course if the focus of the eye is not kept fixed on the central object these appearances will not be visible for they only exist on objects that are not in focus. It may take some practice to keep from changing the focus as the natural tendency is to focus upon the object which one is trying to analyze.

The colored edges on both far and near objects are best seen when the objects are at the relative distances above described. Red edges on the dark become more evident with a more distant fixation and blue on the dark with a nearer one.

Where both the fixation and the nearer object are twenty feet or more away both objects are so nearly in the same focus that the difference in colored edges is hard to distinguish.

These facts lead naturally to the assumption that with a given fixation the characteristic colored edges of objects nearer and farther than the fixation object inform us of their relative distance, that is, if an object has red edges we judge it to be nearer to us than the fixation object while if it has blue objects we judge it to be farther away. If this is so objects depicted in a picture with red edges should appear nearer than those with blue edges. This is exactly what was found to be the case as is shown by Figure 17a. It will be seen that the circles with the reddish edges appear to be on a nearer plane than those with the blue edges.[5]

SUMMARY.

The effect of depth of focus which produces the fuzzy edges of objects nearer or farther than the object plane has been long recognized in photography.

This is shown in the photographs in Figures 18 and 19. That in Figure 18 was taken with a small aperture, F 16, which causes objects at all distances to be imaged sharply, while that in Figure 19 was taken with a large aperture, F 4.5, which causes objects nearer and farther than the focus point to the imaged with a greater softening of edge. The greater effect of depth in the photograph in Figure 19 as compared with that in Figure 18 is very evident.

Gleichen [6] goes into the subject at length and points out that a natural effect of depth can only be produced by a lens which has the same depth of field as the eye.

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[5] The illusion is more apparent if one eye is closed. This prevents the functioning of our binocular depth perception by which we tend to recognize the true distance of both figures.

[6] "Die Grundgesetze der naturgetreuen photographischen Abbildung." Halle 1910. "Uber Helligkeit und Tiefe inbesondere bei der naturgetreuen Photographischen Abbildung." Zeit, fur Wiss. Phot. Vol. 9, 1911, p. 241.

He does not, however, deal with the effects of chromatic aberration. The depth effect due to it is believed to be much more marked than that due to simple depth of field.

In simple depth of field the diffusion of edge due to an object being out of focus gives no indication as to whether the object is nearer or farther than the focus point. Chromatic aberration on the other hand causes objects beyond the focus point to be imaged in a characteristically different way from those inside. This in turn gives a real basis for monocular depth perception apart from the relative sizes of objects or disappearing or other perspective. As shown by Figure 17a this effect of depth can be obtained in a picture if the objects are depicted with their characteristic chromatic edges. A marked effect of depth has thus been obtained in pictures painted by Blanche Ames.

Many paintings by great masters have been looked at to find whether this characteristic edging has been made use of. The treating of edges as stated before is quite common, and in certain paintings, by Millet for instance, warm or reddish edges are found around near objects while objects in the distance are colder and bluer. It is not felt, however, that the evidence is sufficient to conclude that it has been used consistently.

CHAPTER III.

DEPTH OF FIELD (LATERAL).

In this chapter we will deal with the characteristics of images of objects which are situated not on the line of vision. The characteristic form of such images is due primarily to an aberration called oblique astigmatism. (This should not be confused with corneal astigmatism.)

Oblique astigmatism causes the rays of light, from a point source not on the axis, to focus into a ray bundle of complex form. As explained above the ray bundle from a point source on the axis focuses in the form of a cone to a point where the rays cross to spread out into another cone. The ray bundle from a point source not on the axis forms a more complicated figure. In its pure form in a simple uncorrected lens it focuses first to a line, see aa, Figure 22, which lies in a position tangential to a circle about the axis of the lens. It then crosses and narrows in its long dimension and lengthens in its short

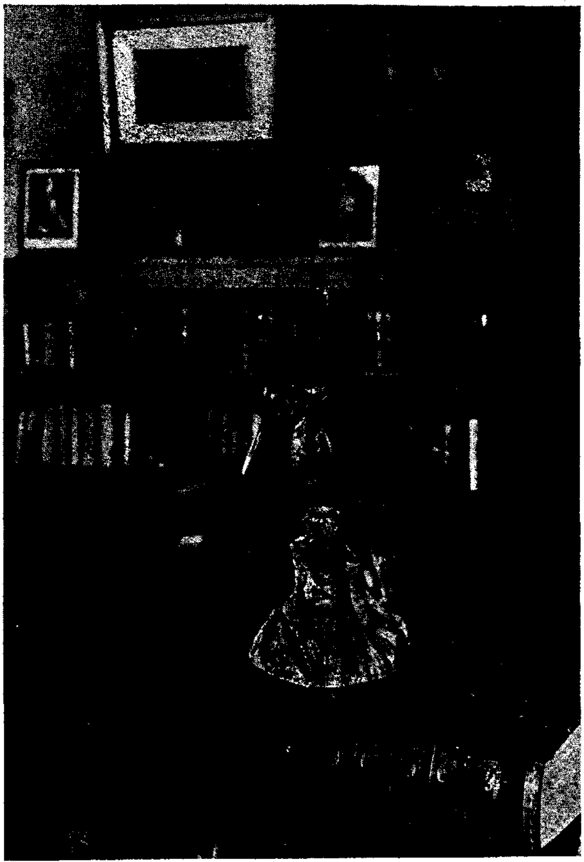

FIGURE 18. Photograph taken with a small aperture, i.e. F 16.

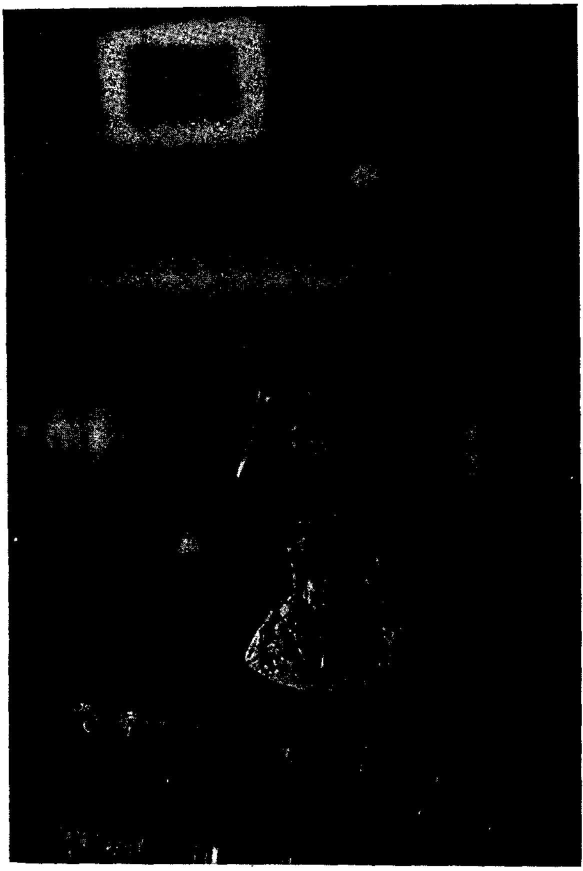

FIGURE 19. Same view as in Figure 18 taken with a large aperture, i.e. F 4.5.

one until it becomes a line again, see bb, Figure 22, which is perpendicular to the first line.

The image of every point source not on the axis has this peculiar form. The farther the source from the axis the greater the separation between the two parts of the image which have the form of lines. If a sensitive plate or ground glass screen is placed behind the lens the form of the image that is apparent depends upon the position of the screen. If it is placed far back, i.e., to the left of bb, Figure 22, the image will be in the form of a radial oval, i.e., radial to a circle about the axis of vision, in the horizontal plane this would be horizontal; if at bb, in the form of a radial line; if half way between aa and bb, the form of a circle; if at aa, the form of a tangential line, i.e., tangential a circle about to the axis of vision, in the horizontal plan this would be vertical; if still nearer the lens, in the form of a tangential oval. If the screen is held stationary relative to the lens, a similar change in form of image can be noted by moving the point source from a distance to a position near the lens, always keeping it at the same angular obliquity.

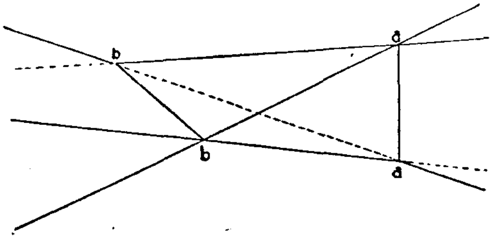

FIGURE 22. Diagram showing characteristic shape of the image bundle of a point source not on the axis.

If, instead of a point, a line source is used, a similar imaging takes place. Every point on the line source is stretched tangentially or radially, depending upon the position of the line source. It can be seen that if the stretching of the various parts of the line source is along the length of the line source itself, the image of the line will appear perfectly sharp and slightly elongated. That is, if a line source is tangential to the axis, its image will be sharp when the source is so positioned that the part of the image that forms a tangential line, i.e., aa, Figure 22, falls on the screen. If it is in a position radial to the axis, its image will be sharp when that part of its image that forms a radial line falls on the screen.

In all simple lenses this characteristic image formation is more or less confused by coma, a one-sided blur, and by chromatic aberrations. The magnitude of these aberrations in the eye has not yet been measured.

This may all be made a little clearer by a brief description of how Mr. Ames measured the oblique astigmatism in his own eye. A point in space is fixated with one eye, i.e., by focusing on a point of light, the line of vision and the focus of the eye is not allowed to vary. At one side of the line of vision (at an angle of obliquity of five degrees), three narrow tangential lines (in this case vertical) of yellow light were moved back and forth. It was found that when these lines were at a certain distance they could be distinguished as separate. At any other distance they could not be distinguished. The distance at which they could be distinguished was such as to cause that part of the image designated as aa, Figure 22, to fall on the retina which made the tangential lines appear most sharp. This point was found to be nearer than the fixation point. Similar points were found at varying degrees of obliquity from the axis where the three narrow lines appeared most sharp. In this way a surface in space was determined where yellow tangential lines were most visible. The shape of this surface which is called the primary astigmatic object field for yellow is shown in plan on Figure 23 by the solid lines extending from the point marked "fixation point" back towards the eye.

In the same way a surface in space was determined where yellow radial lines were most visible. This is called the secondary astigmatic object field for yellow and is shown in plan by the dotted lines extended in shape of a ram's horns from the point marked "fixation point" outward. Figure 23.

Corresponding fields for red, blue and green monochromatic light were found and are shown in Figure 23.

If this fixation point is changed although these primary and secondary astigmatic object fields keep their general shape they shift forward and backward.

Heinrich[7] made a similar experiment with a black thread. He moved a black vertical thread (as he worked in a horizontal plane this would be a tangential line) which was placed to one side of the line of vision, back and forward until it appeared most like the thread on which the eye was fixiated. He found a field similar in shape and position to that which the writer found for the primary astigmatic field for yellow.

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[7] British Journal of Psych., Vol. 3, p. 66.

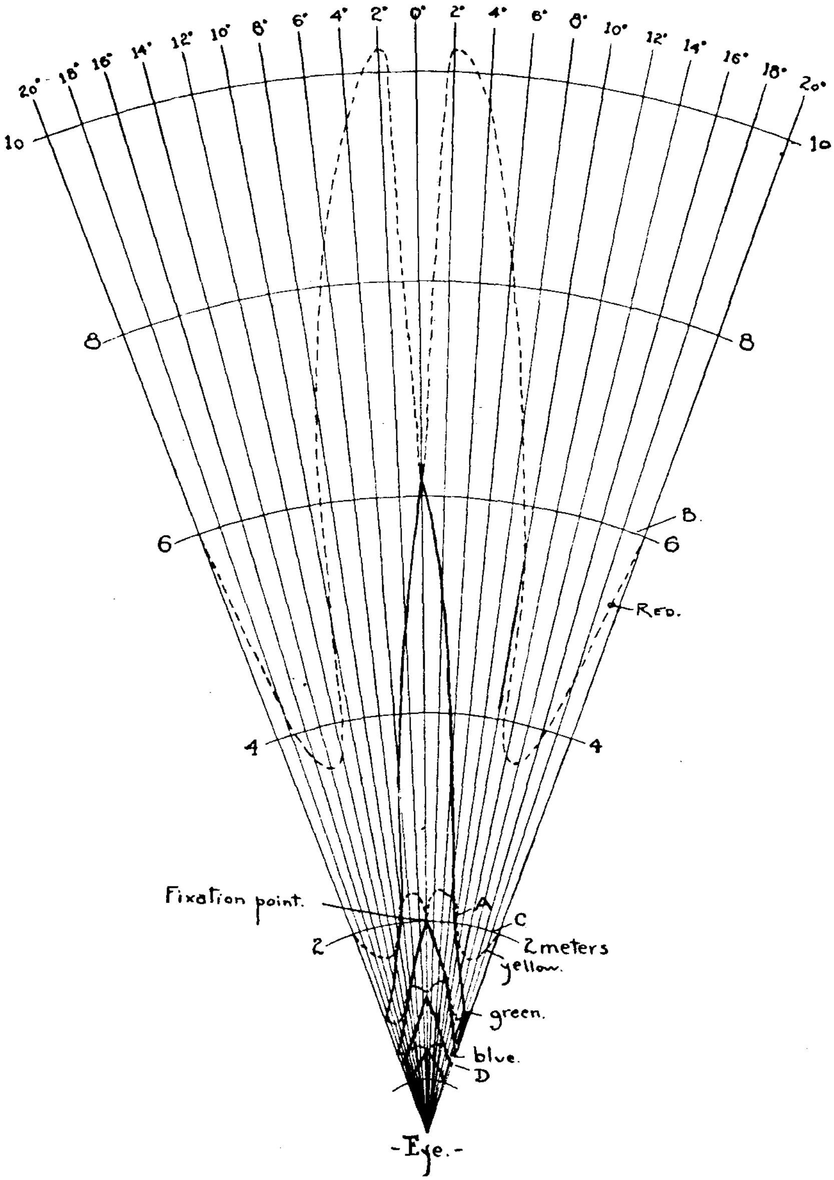

20°
18°
16°
14°
12°
10°
8°
6°
4°
2°
0°
2°
4°
6°
8°
10°
12°
14°
16°
18°
20°
10
10
8
8
6
6
B.
Red.
4
4
Fixation point.
A
C
2
2 meters
yellow.
green.
blue.
D
-Eye.-

Figure 23. Position of primary and secondary astigmatic object fields for different colors.

This means that with any given fixation, at a given angle of obliquity, there are certain positions in space where tangential black and white lines appear most sharp and other positions where radial black and white lines appear most sharp. It also means that with the same fixation there are still other positions where tangential lines of a particular color will appear most sharp and still other positions where radial lines of the same color appear most sharp. The positions in space therefore where tangential and radial colored lines appear most sharp depend upon their color.

The different position of the primary and secondary astigmatic fields for different colors has a further effect of causing the images of black and white objects to have characteristic chromatic edges due to their position in space relative to the fixation point. This can be best shown by the following photographs taken with a lens which has approximately the same oblique astigmatism and chromatic aberration as the eye.

Figure 24 (b) is of a white light point source situated at C, Figure 23, i.e., in the secondary field for yellow at an angular obliquity of about eighteen or twenty degrees. As the lens is focused at the distance marked "fixation point" this is in the plane of the focus. The top picture shows the image formed by the red rays in the white light source; the middle that formed by the yellow; the bottom one that formed by the blue. As the point source is in the secondary field for yellow the yellow light is stretched in a radial direction. Being near the primary field for red the red is beginning to be stretched in a tangential direction. And being beyond the secondary field for blue the blue is stretched in a radial direction forming a diffused radial oval.

Figure 24 (a) is a photograph of a white light point source situated at B; Figure 23, i.e., in the secondary astigmatic field of red light. The red light is therefore stretched in a radial direction. The point source being beyond the secondary fields for both yellow and blue light they are imaged as diffused radial ovals, the blue more diffused than the yellow.

Figure 24 (c) is of a white light point source situated at D, Figure 23, i.e., near the primary field of red and yellow and the secondary of blue, consequently we see the red and yellow stretched in a tangential direction, the blue in a radial one.

The images of white light point sources in similar positions formed by a lens corrected for oblique astigmatism and chromatic aberration are quite different. Figure 24 (e) is such a photograph of a white light point source at C, Figure 23, i.e., in the plane of the focus. The lens

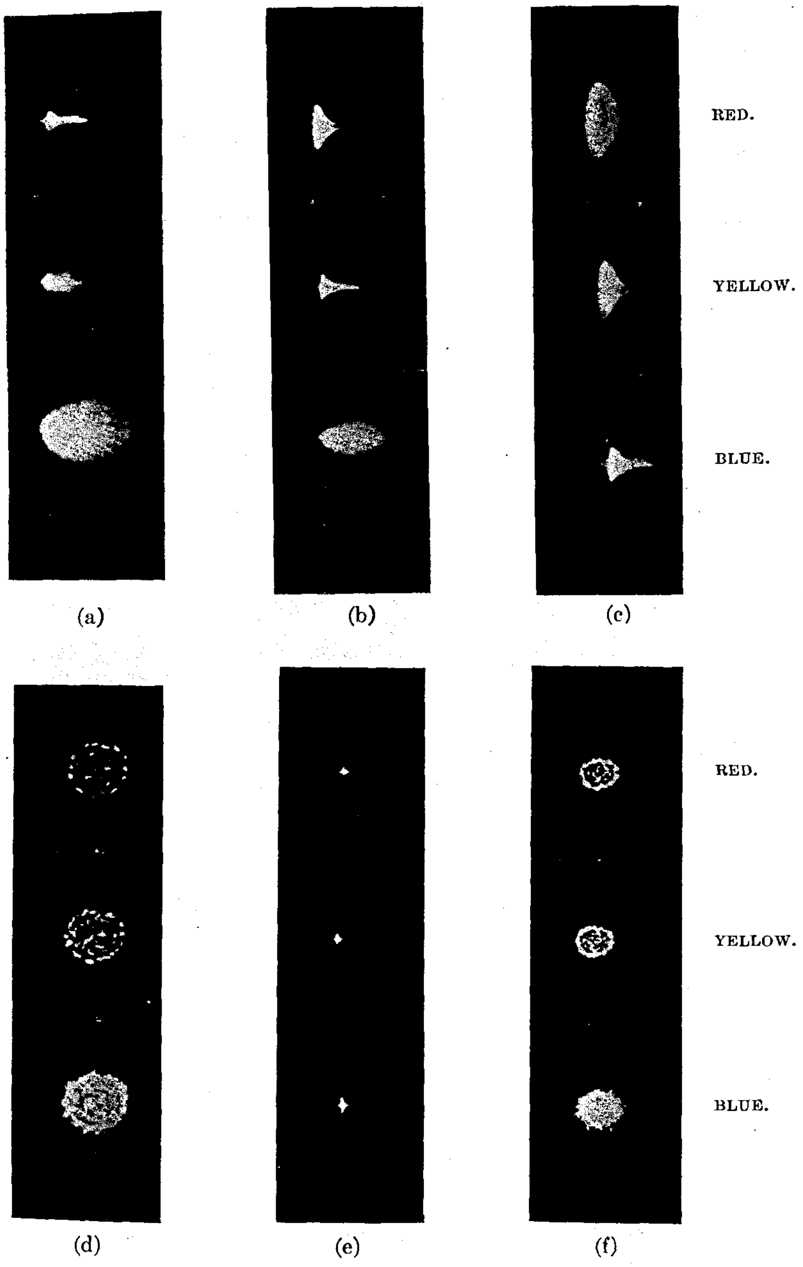

RED.
YELLOW.
BLUE.
(a)
(b)
(c)
RED.
YELLOW.
BLUE.
(d)
(e)
(f)

FIGURE 24. Three color photographs showing characteristic image forms of white light point sources at different distances. (a) (b) and (c) were taken with a lens having approximately the same chromatic aberration and astigmatism as the eye. (d) (e) and (f) were taken with the sources at the same distances with a corrected lens. Magnification 28 diameters.

being corrected for oblique astigmatism means that the image lines aa and bb, Figure 22, are brought together to a point on the image plane. There is therefore none of the characteristic stretching in tangential and radial directions which was so evident in the oblique images formed by the other lens. The lens also being corrected for chromatic aberration there is no substantial difference in the form of the images for the different colors. They are all imaged as small spots of light of about the same size. The marked difference in imaging from that which occurs in the eye as shown in Figure 24 (b) should be noted.

Figure 24 (d) and (f) are photographs made with a corrected lens of white light point sources at B and D, Figure 23. Due to the corrections the images do not show any stretching in a radial or tangential direction and they are all the same shape and size for different colors. The combined images will therefore appear simply as diffused white spots. The marked difference between this imaging and that shown in Figure 24 (a) and (c) should be noted.

It should also be noted here, that while in the imaging of point sources by the lens having substantially the same aberrations as the eye the images have characteristic forms and colored fringes due to their distances from the lens, in the imaging by the corrected lens, although there is a diffusion which may indicate that the point source is not in the object plane, there is nothing to indicate which side of the object plane it is or how far it is from it.

In order to show the characteristic chromatic edges produced in the images of black and white objects, colored photographs of a black cross on a white background and of a white cross on a black background were taken with a lens having approximately the same chromatic aberration and oblique astigmatism as the eye, and also with a corrected lens. They are shown in Figures 25, 26, 27, 28, and 29. The photographs taken with the corrected lens will be considered first as they show the exact shape of the black and white crosses.

Figure 28 was taken with a corrected lens of the crosses placed at C, Figure 23. The blue, yellow and red images are seen to be all of the same size and shape, the combined image will therefore be white and black with no chromatic edges.

Figure 25 was taken with a lens having approximately the same chromatic aberration and astigmatism as the eye, of the crosses at C as in Figure 28. Being beyond both the primary and secondary astigmatic fields for blue the blue is generally diffused. Being in the secondary field for yellow the yellow horizontal or radial lines, both black and white, are relatively sharp, the vertical or tangential ones, diffused. Being nearer the primary than the secondary field for red, the vertical lines are sharper than the horizontal ones. The chromatic edges that exist from the combined figures can be visualized by the amount the different colored edges extend in the different directions. It is regretted again that these figures cannot be reproduced in color which shows the effects much more clearly. The marked difference in characteristic edges in this figure compared with Figure 28 is however plainly evident.

Figure 29 was taken with the corrected lens of the crosses placed at B, Figure 23. A photograph of the crosses placed at D is so similar that it is not shown.

Figures 26 and 27 were taken with a lens having approximately the same chromatic and astigmatic aberrations as the eye of the crosses placed at B and D, Figure 23. The characteristic diffusion and accentuation of vertical and horizontal lines for the different colors is evident and a little study will show that it takes place in conformity with the position of the primary and secondary object astigmatic fields for the different colors.

The marked difference in characteristic imaging between these figures and Figure 29 should be noted.

In all these photographs the objects were at an angular obliquity of between eighteen and twenty degrees. If the angular obliquity had been less the characteristic imaging would be different due to the difference in the relative positions of the primary and secondary astigmatic object fields for the different colors.

SUMMARY.

The foregoing demonstrates that the retinal image of an object in space has characteristics both as to shape and colored edges due to the object's particular position in space relative to the observer and his fixation point. In other words with a given fixation the image of a particular object in the field of view has characteristics which are peculiar to the image of an object at its particular angular obliquity and distance.

That these characteristics are of sufficient magnitude to be recognized is evidenced by the fact that those doing research in this line have been able to discover and measure them. The accentuation of radial and tangential lines is observable in landscape views. This accentuation can be observed in Figure 30. Hold up the page so that you can either look by its edge at some distant object or at the small

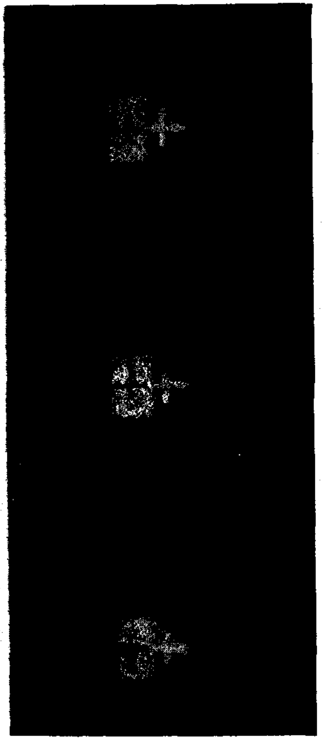


Fig. 25.

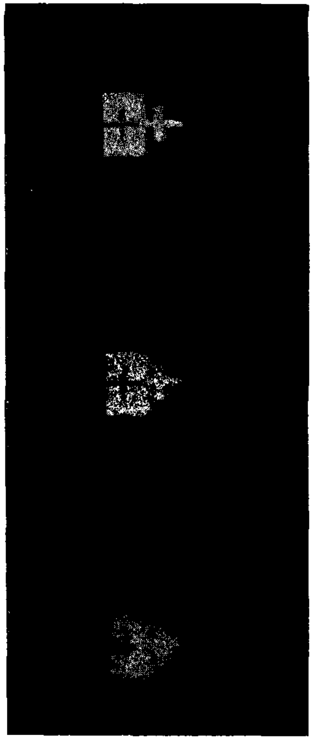


Fig. 26.

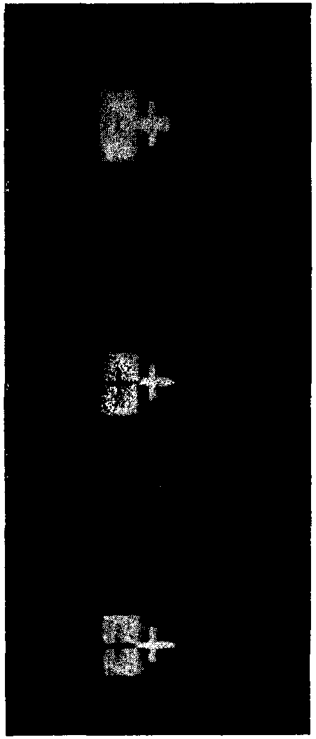


Fig. 27.

FIGURE 25. Three color photographs of black and white crosses taken with a lens having approximately the same chromatic aberration and astigmatism as the eye. The camera was focused at point marked "Fixation Point" Figure 23, the crosses were at C. Width of arms of crosses $\frac{1}{12}$ inch.

FIGURE 26. Same as Figure 25 except that crosses were positioned at B, Figure 23. Width of arms of crosses $\frac{1}{4}$ inch.

FIGURE 27. Same as Figure 25 except that crosses were positioned at D, Figure 23. Width of arms of crosses $\frac{1}{24}$ inch.

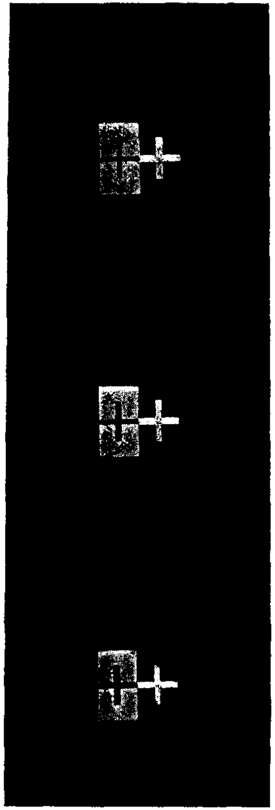
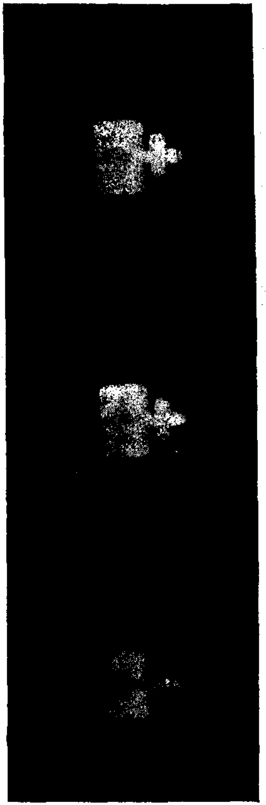


Fig. 28.



RED.
YELLOW.
BLUE.

Fig. 29.

Figure 28. Same as Figure 25 taken with a corrected lens.
Figure 29. Same as Figure 26 taken with a corrected lens

cross near the edge. If a distant object is focused the vertical line in the large cross is more evident than the horizontal line. If the little cross on the edge of the page is fixated the opposite is the case. Care must be taken not to let the accommodation or line of vision vary.

These facts lead naturally to the assumption that with a given fixation the characteristic imaging of oblique objects in space informs us of their distance; that is if an object has sharp tangential edges we judge it to be nearer than an object with soft tangential edges and if it has sharp radial edges we judge it to be farther away than an object with soft radial edges. If this is so, objects depicted in a picture with sharp tangential edges should appear nearer than those depicted with soft ones and objects depicted with sharp radial edges should appear farther away than objects depicted with soft ones. This is exactly what was found to be the case as is shown by Figures 30a and 30b. Carefully observing 30a it will be seen that the circles with the sharp edges, whether inside or outside appear nearer than the circles with the soft edges. In Figure 30b it will be seen that the central portions of figure I which are sharp edged appear to be more distant than the circumference where the edges are soft, while in figure II where the radial edges near the center are soft and those near the outside sharp the center seems to be on the same plane as or nearer than the circumference.[8]

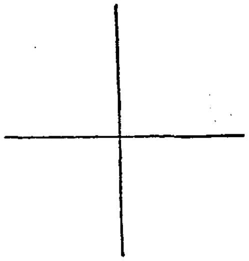


FIGURE 30. Figure for observing change of appearance of radial and tangential lines with variation of focus.

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[8] These illusions are more apparent if one eye is closed for the reasons given in the footnote page 15.

The judgment of two hundred and two students at Dartmouth College were obtained on the effect of these illusions. On the figures in 30a 77.3% judged that the circles with the sharp edges appeared nearer than those with the soft ones. 19.8% made the opposite judgment and 2.9% got no effect.

On the figures in 30b, I and II, 82.2% judged that the central portions of Fig. I appeared more concave than that in Fig. II. 17.8% made the opposite judgment.

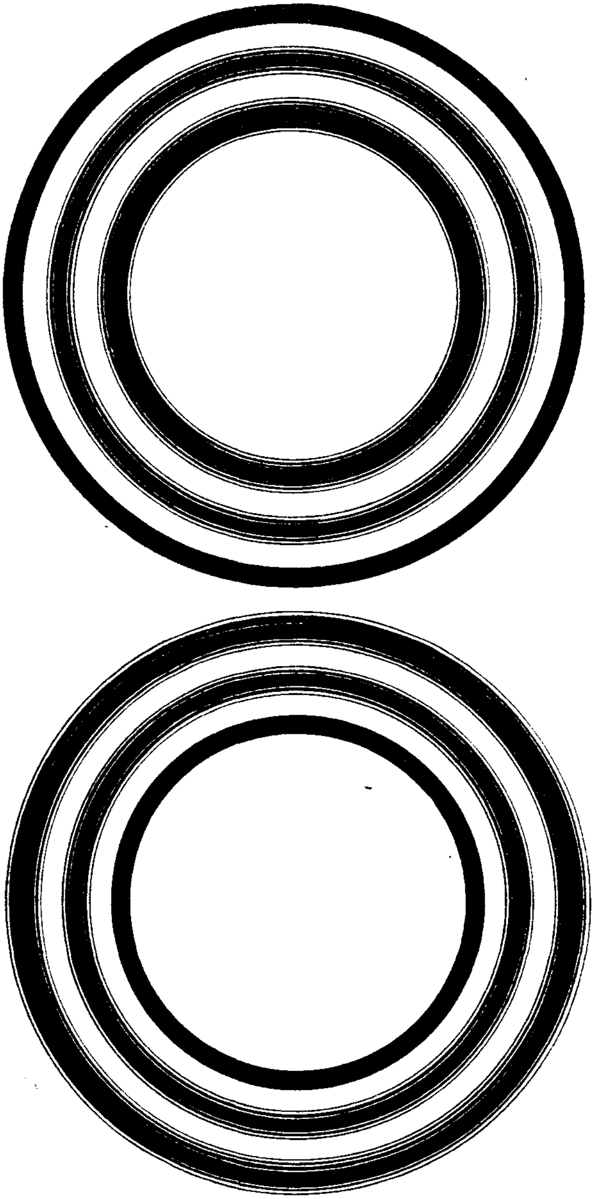

FIGURE 30a. Figures showing effect of depth produced by sharp and soft tangential edges.

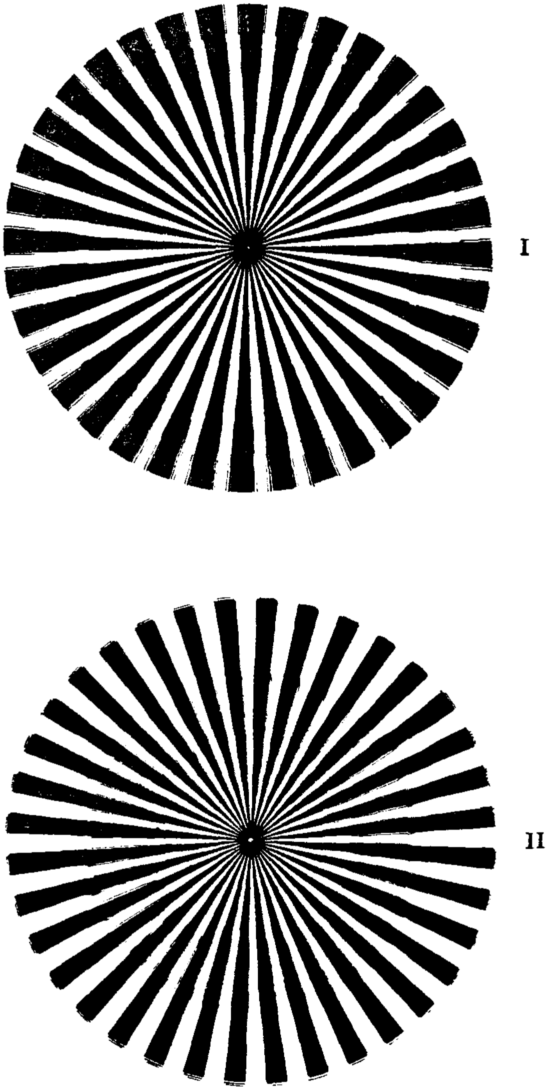

FIGURE 30b. Figures showing effect of depth produced by sharp and soft radial edges.

In paintings made by Mrs. Oakes Ames in which objects on the sides of the pictures were depicted with these characteristics a marked sense of depth is given by the objects taking their proper relative distances. The accentuating of tangential and radial lines in their proper planes is found in many paintings, especially those of Turner in whose work it is apparent in the accentuation of tangential lines inside the focus and of radial lines on and behind the object plane of the scene he is painting. The accentuating of tangential lines in the foreground is found in many works of art and might be called almost a trick of composition to produce an effect of depth.

Although a large part of the above described effects are due to color and therefore cannot be reproduced in black and white, the photographs in Figures 31 a and b taken with a lens having approximately the same oblique aberrations as the eye, and 32 a and b taken with a corrected lens, are shown to give an idea of the general differences obtained. The photographs in Figure 31 a and b are believed to give a greater effect of depth and to be generally more pleasing than those in Figure 32 a and b.[9]

The accentuation of the tangential lines and softening of the radial lines in the foreground are very apparent in the candle stick in Figure 31a. The accentuation of the radial lines and the softening of the tangential lines in the background and plane of the focus are apparent in the books in Figure 31a and in the objects on the right and left of Figure 31b.

The difference in the size of the books in Figures 31a and 32a is due to the distortion in Figure 31a. This distortion is approximately the same as exists in the eye. This difference in size may help to cause Figure 31a to give a greater sense of depth than Figure 32a.

CHAPTER IV.

DISTORTION.

There is still another factor that affects the nature of the images of an object situated on one side of our line of vision. That is distortion. It can be defined as that characteristic of a lens which causes variation in distance between points in the image field which in the object field are equidistant. The eye is subject to so-called barrel distortion,

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[9] It is regretted that the general loss of detail due to reproduction masks much of the effect which is apparent in the original photographs.

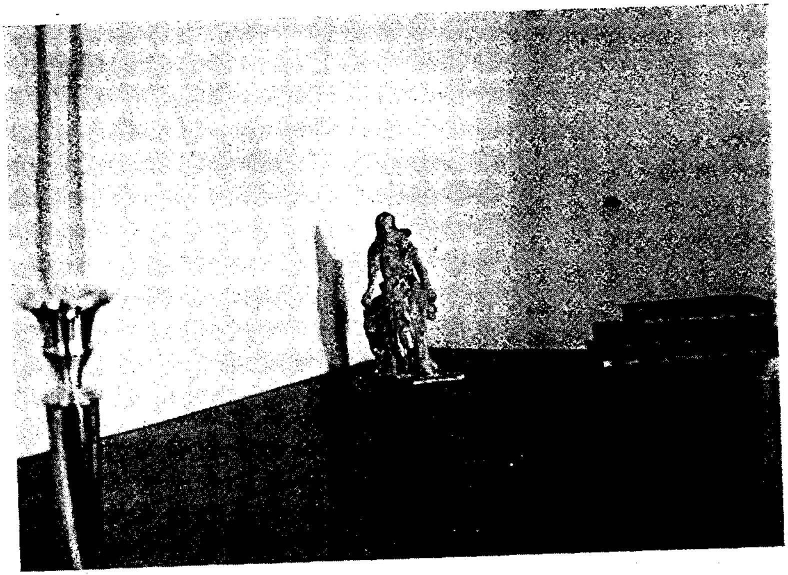

Fig. 31a.

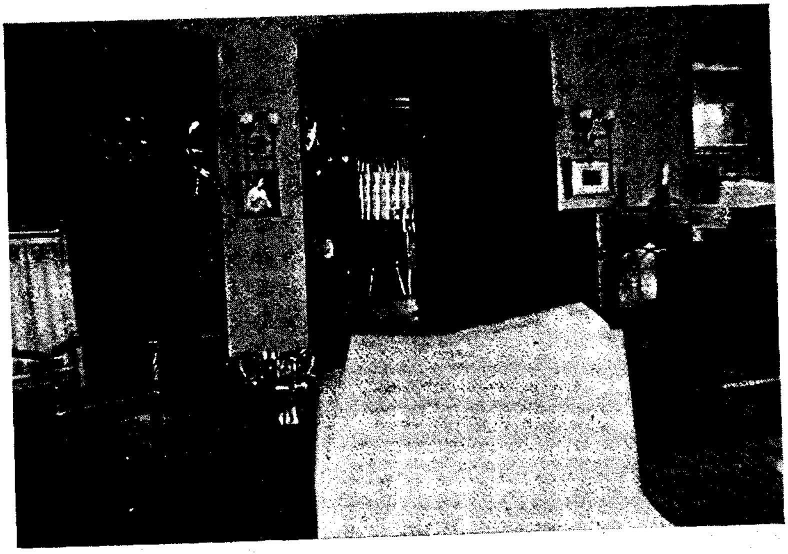

Fig. 31b.

FIGURE 31a and b. Views taken with lens having approximately the same aberrations and distortions as the eye.

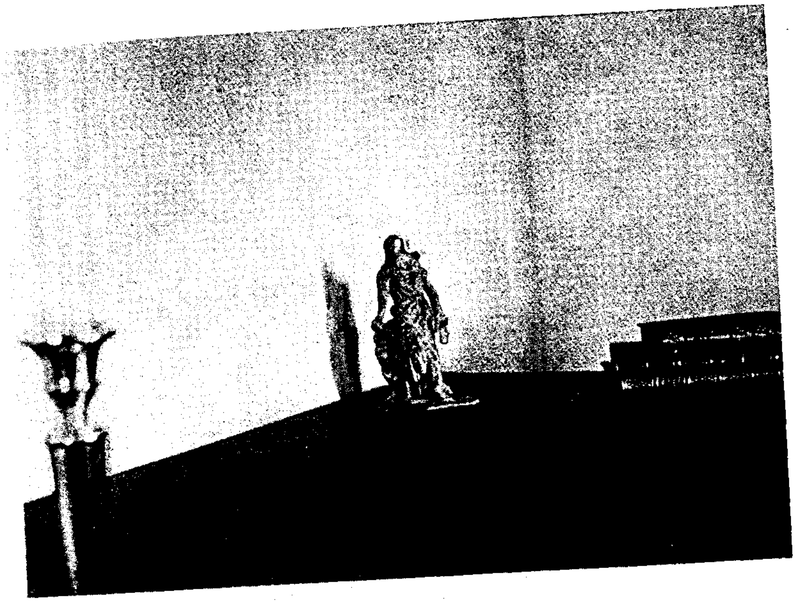

Fig. 32a.

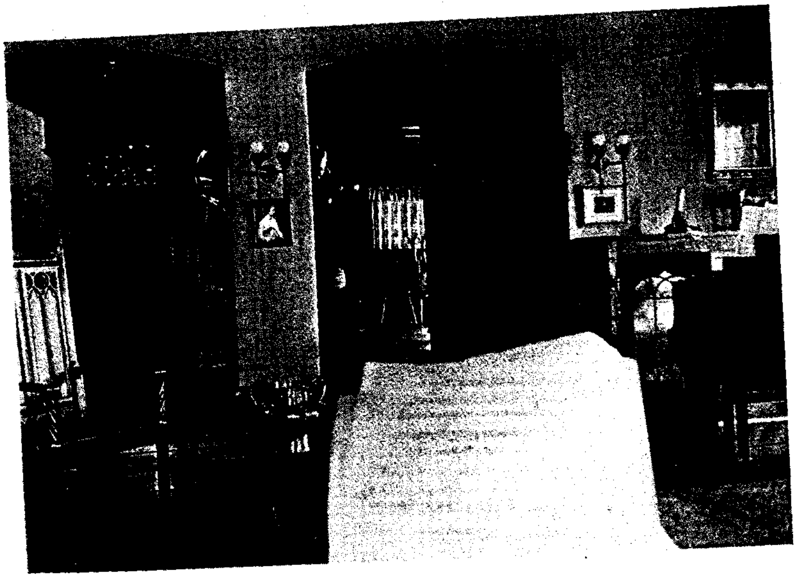

Fig. 32b.

Figure 32a and b. Same views as in Figure 31 taken with a corrected lens.

which means that a series of points subtending equal angles in space are not imaged at equal distances on the retina. The images of equidistant points at a distance from the axis are closer together than those near the axis, the farther from the axis the closer they are together.

Figure 33 shows the approximate distortion which exists in the eye. If the eye looks at the center of a rectilinear grid, similar to the one in the figure, held at such a distance that when looking at its center the corner will subtend an angle of thirty-two degrees with the visual axis, the picture that is formed on the retina will not be a reproduction of the rectilinear grid but will take the form of the barrel shaped grid shown in Figure 33. By comparing the two grids it will be seen that points that are equidistant on the rectilinear, as shown by the intersection of the cross lines, are not equidistant in the barrel shaped grid. They are practically the same distance apart near the axis but the farther away from the axis you go the smaller these distances become.

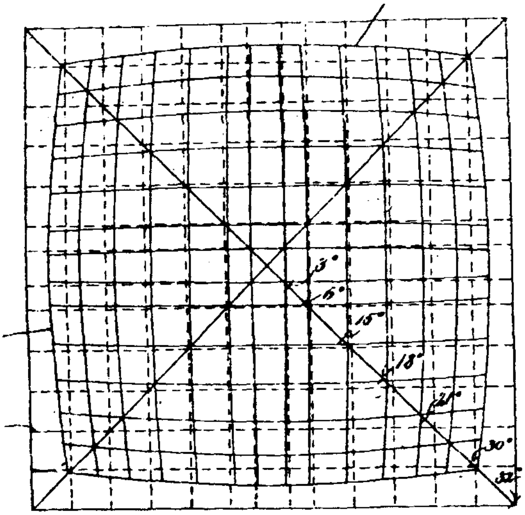

FIGURE 33. Curves showing the "barrel" distortion of a rectilinear grid that takes place in the eye.

The effect of this is twofold. It causes straight lines that do not pass through the axis of vision, to be bowed outward in their central portions. It also causes objects away from the axis to be imaged in smaller relative size than those near the axis.

This effect applies to all objects depending in amount upon their angular obliquity regardless of their distance away.

This distortion is very easily seen by looking at the middle of any rectilinearly bounded space such as the side of a room and, without allowing the axis of vision to change, noting the curvature of the boundary lines.

It is in fact the most easily noticed of the characteristics of our retinal picture.

SUMMARY.

The effect of this distortion on the nature of the images of objects off the axis is very marked. It causes straight lines in a scene to become curved and take very different positions relative to each other from those laid down by the laws of disappearing perspective. At the same time it causes objects to one side of the line of vision to become relatively smaller.

In corrected photographic lenses this distortion is corrected so that with such a lens a photograph of the rectilinear grid shown in Figure 33 would also be rectilinear with all the squares the same shape and size. Figure 34, which is a photograph taken with a lens having the same distortion as the eye, shows the characteristic barrel distortion while Figure 35 shows the same scene taken with a corrected lens. The barrel distortion is shown in Figure 34 in the curvature of the lines in the tiling, of the edge of the tank and of the balcony. This distortion produces a more natural effect. This is most noticeable by comparing the ceilings in the two photographs. That in Figure 34 seems to arch over properly, while that in Figure 35 flares upward. It is also evident in the way the lower left hand corner of the tank is rendered. By comparison the corner in Figure 35 seems to fall away and gives the effect of the water not being level.

It is of the greatest interest and significance, that of all the characteristics of the retinal picture distortion has been most commonly used by the great painters. That the "rectilinear" effect is not satisfactory has long been recognized. W. R. Ware in his book [10] on perspective in a chapter entitled "Cylindrical, Curvilinear or Panoramic Perspective" points out that the laws of ordinary disappearing perspective must be departed from in order that the depicting of certain features on the sides of the field of view shall appear satisfactory. He shows this is especially necessary in the case of certain architectural compositions,

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[10] Modern Perspective, W. R. Ware, University Press, Cambridge, Mass.

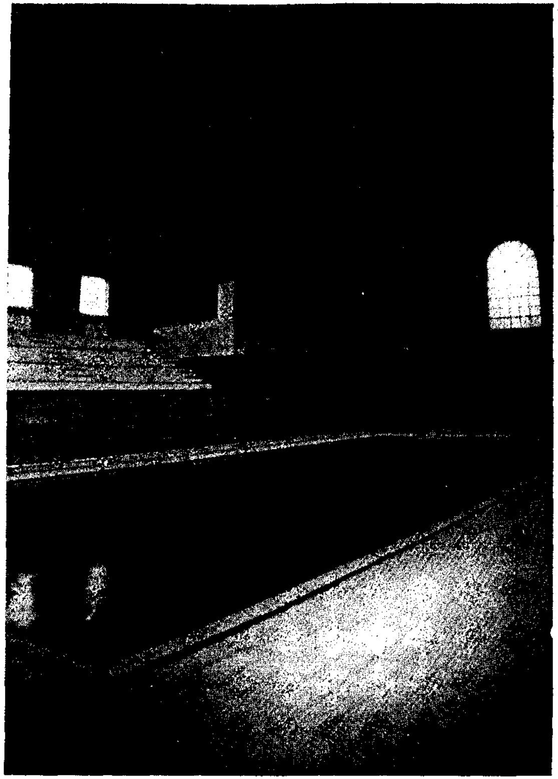

Fig 34.

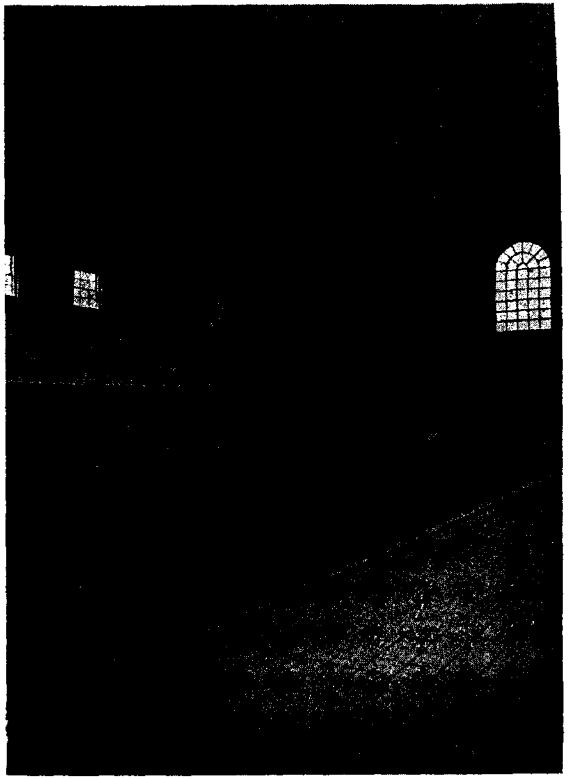

Fig. 35.

Figure 34. Architectural interior taken with lens having approximately same distortion as the eye.
Figure 35. Same view as Figure 34 taken with a corrected lens.

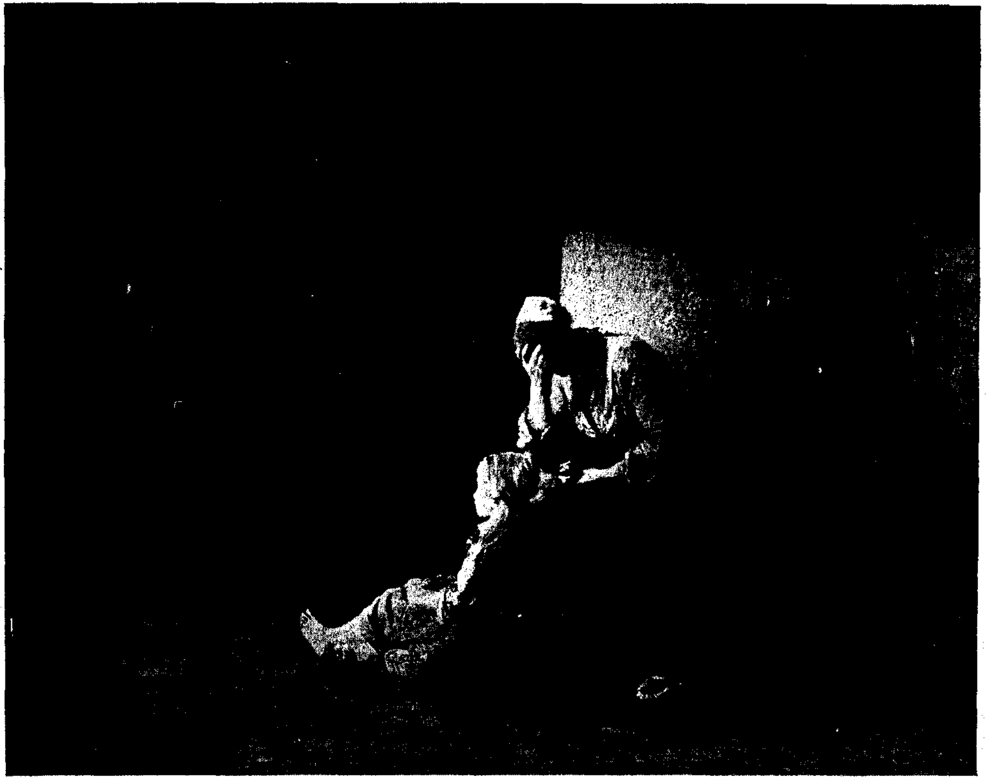

FIGURE 36. Copy of painting by Israels showing "barrel" distortion.

and shows further many instances where this has been done by great artists. He shows further that satisfactory results can be gotten by introducing what practically amounts to a "barrel distortion." It is not evident, however, that he recognizes the fundamental principles on which curved perspective is based.

The writers have made what can hardly be called more than a casual search and has found distortion in the following works:

Leonardo da Vinci	"Last Supper"
Puvis de Chevanne	"St. Genevieve"
Rembrandt	Numerous instances
Israels	"The Day before Parting"[11] and "The Convalescent" and other pictures
Millet	"Cliffs of Gouchy"[11]
Turner	Numerous instances
Whistler	Etching Venice Scene, "The Palaces" First Venice Set 1880
De Hoogh	"Interior of a Dutch House"[11]
Van Vleet	"Church Interior"[11]
Inness	"The Greenwood"

The lack of its consistent use in most cases causes one to believe that those who used it did so intuitively to make the picture "look right." Israels, however, used it so consistently that he probably was conscious of the law and the same is probably true of Rembrandt. The only example of it that has been found in the works of living artists is in Sir William Orpen's painting "The Peace Conference." As only a photographic copy of this picture has been seen there is a possibility that the distortion was in the photograph. Figure 36 is a reproduction of Israels's "The Day before Parting." The distortion is most evident in the tiling in the floor though it can be seen in various other parts of the picture.

One would think that the representation of straight architectural features by a curved line coming next to the straight edge of the frame of the pictures as it often does would be very noticeable. It is not however. Not that it is not perfectly evident when one's attention is called to it, but it does not attract one's attention. Its evident effect in many cases is to prevent certain parts of the picture from looking as if they were falling out.

In using distortion the detail, edges, etc., of the distorted features should be depicted approximately in the way they would be imaged on the retina. This was shown by the obtrusive unnatural appearance of a painting which Blanche Ames and Mr. Ames made in about 1912 in which they put in the approximate distortion which exists in the eye but painted all the detail as it appeared while looking directly at it. In later pictures painted by Blanche Ames in which the detail of the distorted features approximated in its characteristics the way it is imaged upon the retina the distortion ceases to be noticeable and gives a pleasing and natural effect.

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[11] These pictures are at the Boston Art Museum.

Another interesting effect due to distortion results from the fact that objects away from the optical axis are imaged in smaller relative size than those near the axis. This effect is very evident in Figures 31(a) and 32(a). In Figure 31(a) taken with the eye lens the statuette is much larger relative to the books than it is in Figure 32(a), although the statuette in both pictures is the same size.

With a larger angular field this relative difference in size is still greater. In the eye where the field is between four and five times that in the figures the effect is very marked. This is probably the reason why a distant mountain appears so much larger to us when we look at it than it does in a photograph taken with a corrected lens.

CHAPTER V.

SENSITIVITY OF THE RETINA.

As stated in the introduction, the character of the picture we get on the retina is determined not only by the kind of image that is formed by the lens system of the eye but also by the nature of the sensitive surface upon which the image falls, that is by the sensitivity of the retina.

To gain a thorough knowledge of our retinal picture it is necessary therefore to know the sensitivity of the retina in its various parts both to light and to color.

Unfortunately relatively very little is known as to the sensitivity of the retina as a whole. Considerable is known about the sensitivity of the fovea, i.e., that part of the retina which is on the axis of vision, but very little definite knowledge exists as to the sensitivity of the peripheral parts.

A great deal of work has been done on the limits of the color fields, i.e., as to the limit of obliquity at which different colors are visible.

The validity of that work, however, is put in question for reasons given in the article on "Dioptrics of the Eye" [12] by Dr. Proctor and Mr. Ames.

As far as is known no quantitative measurements, with one exception which will be considered later, have been made of the color sensitivity of different parts of the retina. The late Dr. J. W. Baird and Mr. Ames undertook to make such measurements at Clark University in 1913, but found that the aberrations of the lens system of the eye would first have to be determined. This led to the work described in "Dioptrics of the Eye." Mr. Ames hopes to carry out these quantitative measurements later.

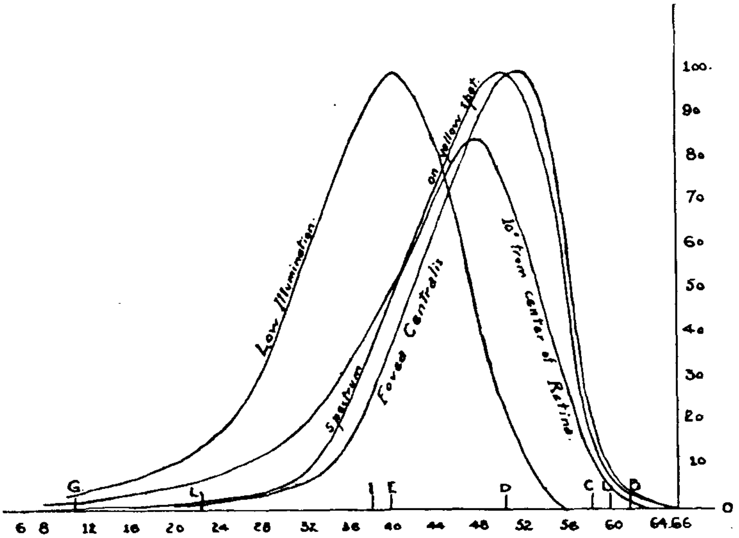

FIGURE 37. Curves showing sensitivity of the retina in various parts to light of different color.

Dr. Baird and Mr. Ames did find that blue appeared much more saturated on the periphery than it did on the fovea. This is in conformity with measurements made by Abney.[13] He measured the sensitivity of the retina to lights of different wave lengths both at the fovea and at an angle of ten degrees. His results are shown in Figure 37. It will be seen that blue appears brighter at ten degrees than at the fovea. The amount brighter that blue light of different wave length appears is shown in Figure 38.

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[12] Loco Set.
[13] "Researches in Color Vision," p. 94, Longmans Green & Co., 1913.

This greater peripheral intensity of blue is probably primarily due not so much to the difference in sensitivity of the retina in the two regions as to the absorption of blue light at the fovea by the yellow spot. The yellow spot lies over the fovea, covering an angular area of about six degrees horizontally and four vertically. Its effect is to absorb light of short wave lengths, i.e., blue.

The effect on our retinal picture of this difference in sensitivity is to cause those parts which are outside the yellow spot to appear more blue. This effect was very evident to Dr. Proctor and Mr. Ames while measuring the astigmatic fields for blue light, the blue light appearing many times brighter when it was a few degrees off the axis than when looked at directly. The sensation is also commonly experienced in the falling off of the apparent blueness of something one sees out of the side of the eye when one turns to observe it directly.

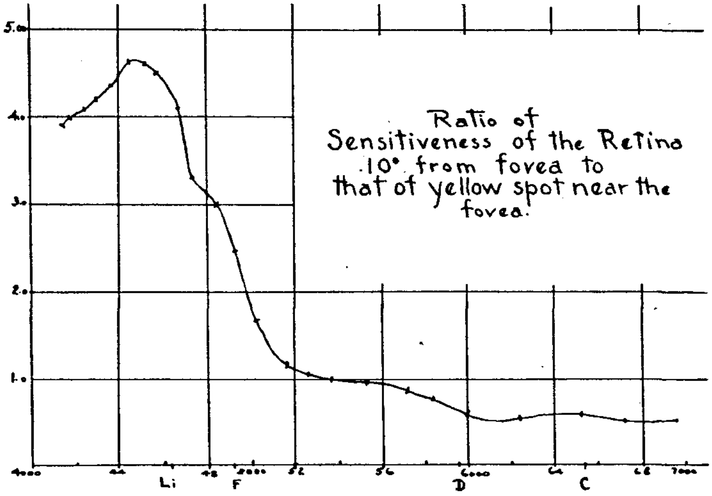

Figure 38. Curve showing ratio of sensitivity of the retina 10° from the fovea to that at the fovea.

To approximate this effect photographically a yellow spot of approximately the proper absorption and of about six degrees in angular size was put in the middle of the blue focal plane filter which with a red and green filter was used to take three color photographs. The exposure for blue was then made sufficiently longer so that the color rendering would be normal in the center of the picture while on the outer parts the blue would be stronger in approximate accordance with the results found by Abney. A lens having the approximate aberrations and distortion of the eye was used.

The results were very interesting. The effect was two-fold; first: to render all the colors outside of the "yellow spot" bluer; second: to make more apparent the aberrations in the images formed by oblique rays and thus cause a greater softening of the outer parts of the picture. In the photographs in which the color rendering all over the picture was the same as we get in our yellow spot the aberrations of light from the blue end of the spectrum, although they existed, were so low in intensity as to produce no effect. When, however, the intensity of the blue light was made greater these aberrations became apparent.

These effects are primarily chromatic and show very poorly in black and white reproductions. The difference in the apparent aberration however is evident in the photographs shown in Figures 39a, b, and c. These were taken with a lens having the approximate aberrations of the eye. In Figure 39 (a) no filter used. In Figure 39 (b) a filter having the approximate absorption of our yellow spot covered the entire picture. In Figure 39 (c), which represents the conditions we get in our eye, a similar filter covered an angular area corresponding to that of the yellow spot.

It will be seen that the whole of the picture in Figure 39 (a) is much softer than that in 39 (b). This is due to the greater amount of the aberrated blue light which struck the plate in Figure 39 (a) and which is absorbed by the filter used in taking 39 (b). In Figure 39 (c) the center portions are sharp due to the local action of the yellow filter while the outer part is soft due to its absence. It is very evident that Figure 39 (c) which approximates the conditions we get on our retina is much more pleasing [14] than Figure 39 (a) or 39 (b). Figure 39 (d) is a photograph of the same scene taken with a corrected lens. This seems to be pretty conclusive evidence that the purpose of the yellow spot is to counteract the strong chromatic aberration in the eye by reducing the brightness of the light from the blue end of the spectrum so that it ceases to be apparent.

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[14] It is regretted that a very marked effect that is apparent in the original photograph is lost in the reproductions.

SUMMARY.

The above facts show the marked effect that variations in sensitivity of the retina have on the nature of our retinal picture.

The slightly brighter warmer centers in some of Corot's pictures suggest the effect produced by the yellow spot. But besides his work the only evidence that has been found that the above described effects have been made use of by artists is in their very common practice of rendering shadows in out-of-door scenes much bluer than they appear when one looks directly at them. As far as is known this has not been limited to the outer parts of their pictures. The blue appearance of shadows which are imaged on the side of the retina are, however, very easily seen, and as this effect holds true over the greater part of the field of vision it was probably found that pictures look better with blue shadows all over them than without any blue shadows at all.

As has been stated our knowledge of the sensitivity of the retina is very limited. We already know, however, that our capacity to distinguish detail away from the center of focus is largely due to the structure of the retina. It is probable that a further knowledge would give suggestions as to the laws which control the difference of local values of which we are conscious on the different parts of the retina.

There are also of course other effects such as contrast, simultaneous and successive, and after images which must have a marked influence on our retinal picture. Their use in pictures raises the thought of the possibility of suggesting eye motion.

CHAPTER VI.

BINOCULAR VISION.

The fundamental idea in undertaking the research work which is the basis of this article was that pictorial art should be similar to our mental visual images, and, since our mental visual images are probably similar to our retinal pictures, valuable suggestions could be obtained from a knowledge of the characteristics of our retinal picture. Our mental visual impression, however, is not derived from a single retinal picture but from two, as we normally look with two eyes.

The whole subject of binocular vision is too long and complicated to be considered here. It was believed, however, that some of the characteristics of binocular vision under particular conditions could be reproduced in a single picture such as a photograph or painting.

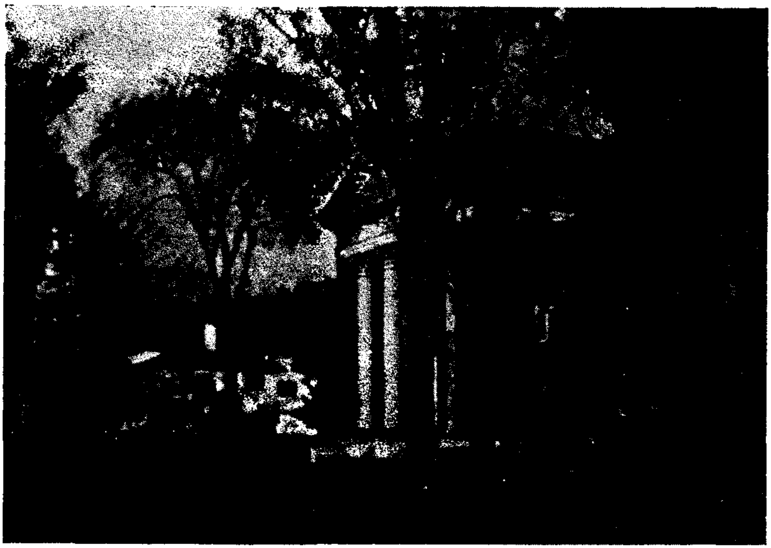

FIGURE 39a. View taken with a lens having the same chromatic aberration as the eye.

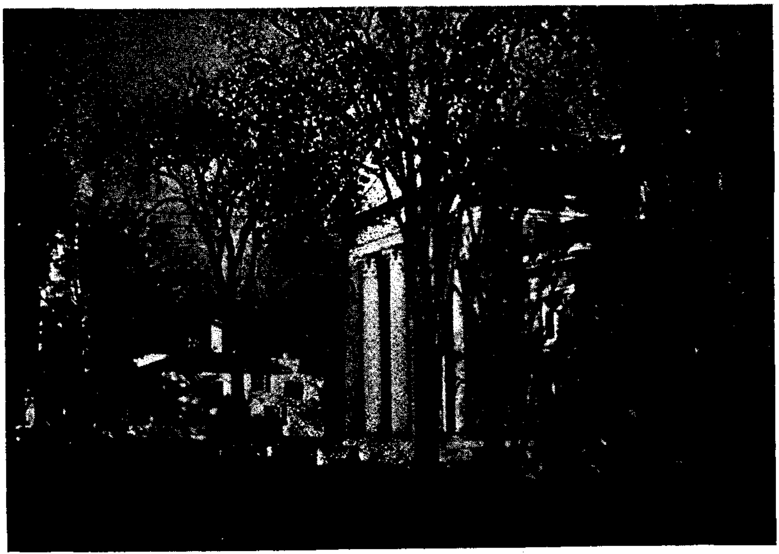

FIGURE 39b. Same view as in 39a taken with lens having the same chromatic aberration as the eye and a focal plane filter having the approximate absorption of the "yellow spot" covering the whole picture.

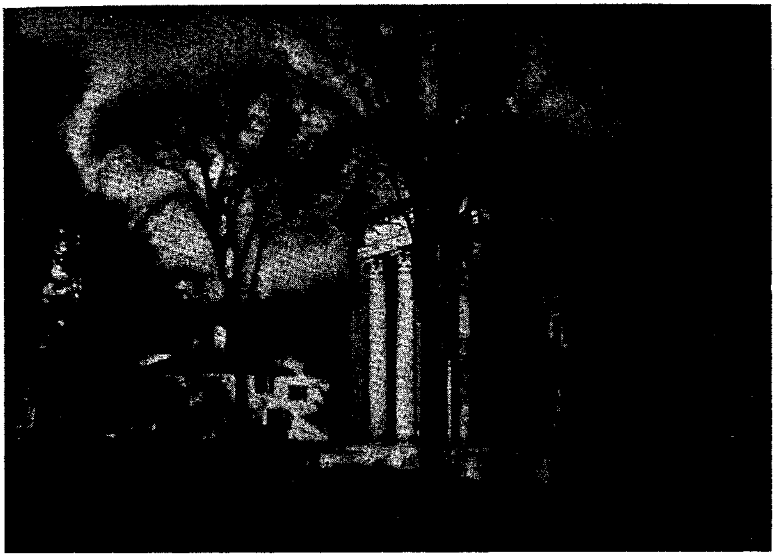

Figure 39c. Same as 39b. The focal plane filter covering the area covered by yellow spot thus approximating the conditions we get in one eye.

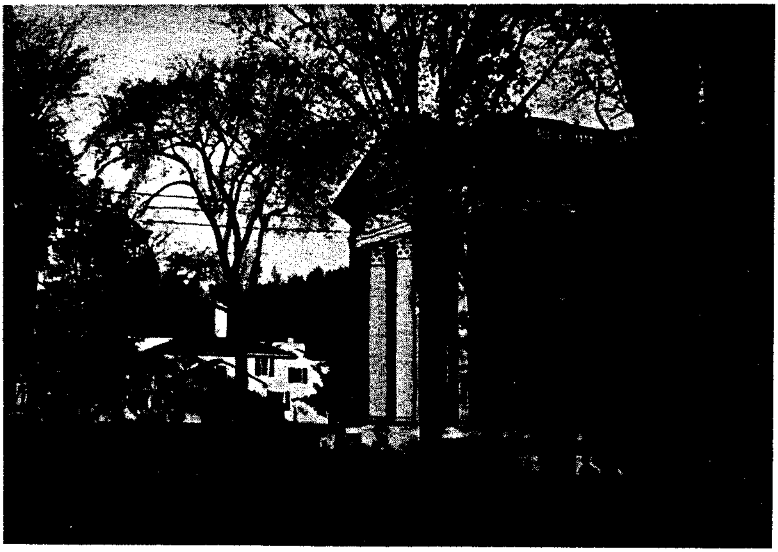

Figure 39d. Same view taken with a corrected lens and no filter.

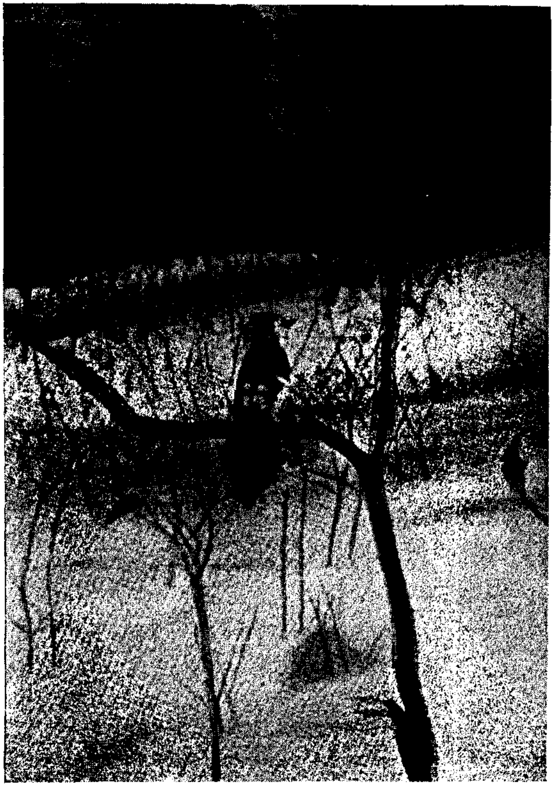

Fig. 41.

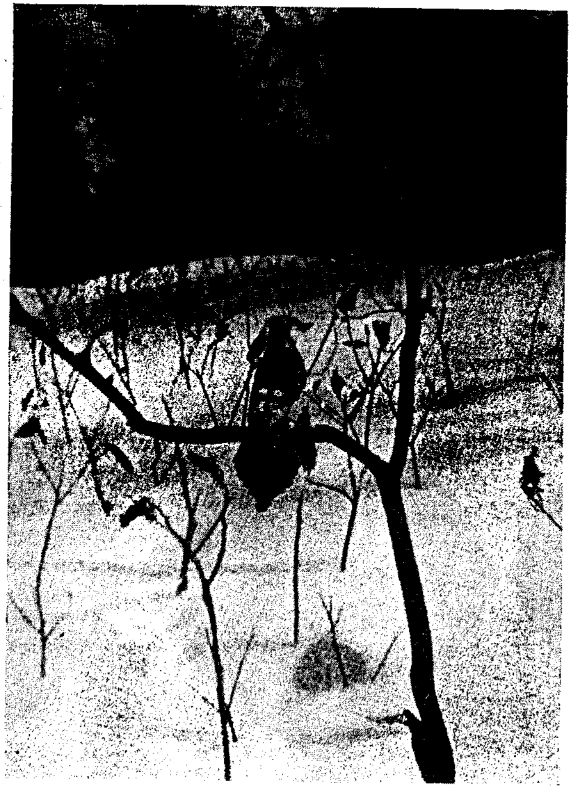

Fig. 42.

FIGURE 41. "Binocular" photograph.
FIGURE 42. Same view as in Figure 41 "monocular" photograph.

The conditions chosen were where the background behind the object at the point of convergence was of an indeterminate nature such as a mass of branches or foliage. The absence of any marked contours under such conditions would not call for the suppression of parts of either retinal image.

Leaving out the effect of ocular movement and the fusion of doubled images the brain under such conditions may be considered as receiving two superimposed pictures of the object field as seen from each eye.

To reproduce this effect a camera was devised which, by means of a reflector and half silvered prism, produced superimposed pictures of the landscape as viewed from two points of view,— the distance between which was the same as that between the eyes. The detail in these pictures superimposed where the axis of the two systems crossed, as the two monocular images do at the point of convergence. The details in all other parts of the pictures were more or less doubled due to the parallax of the two systems.

Figure 41 shows such a "binocular" photograph. Figure 42 shows an ordinary photograph of the same scene.

The following characteristics will be noted in the "binocular" picture:

First, there is a "broadening" of everything in a horizontal direction. At the convergence point this is due to seeing more of the sides of an object. At other points in the scene it is due to the doubling in a horizontal direction resulting from the parallax. This effect of the "broadening" of a scene when viewed binocularly can be noticed by anyone by first observing the scene with one eye and then with two.

Second, there is an increase in contrast values between the lights and darks in the objects at the convergence point relative to that in other parts of the picture. This is due to the fact that at the convergence point the darks and lights superimpose and so reinforce each other while in all other parts of the picture they tend not to superimpose and so counteract each other. Probably some such effect as this exists in our binocular impression.

Third, there is a doubling up of the images of objects not at the convergence point, the extent of the separation of the doubled images depending upon their distance from that point. The seeing of objects not at the convergence point in doubled images is supposed to be one of the factors that gives us our idea of relief, the extent of the doubling suggesting the distance of the object from the convergence point. The impression we receive on our mind from these doubled images is different from that shown in Figure 41 due to the modifying effects of the antagonism of the visual fields which suppresses one set of images, and to other physiological factors. Our impression, however, is probably more like the effects shown in Figure 41 than like a monocular impression, as is shown by the greater effect of depth, less flat appearance of Figure 41 as compared with Figure 42.

SUMMARY.

It is well known among artists that a different effect is produced from painting with one eye than with two and that to get satisfactory results two eyes must be used. There are unquestionably certain effects in the binocular impression that can be reproduced in a single photograph and still further effects that can be reproduced in a painting, where factors such as antagonism of the visual fields can be dealt with. There are other effects due to ocular movements which cannot be reproduced in a single picture but which may be possible of reproduction in motion pictures.

Chapter VII.

GENERAL SUMMARY AND DISCUSSION OF RESULTS.

From the foregoing description of the characteristics of retinal images of objects in various parts of the visual field it is possible to determine fairly definitely the nature of the retinal picture as a whole. It can be described, in general terms, as being a picture in which objects at the center of interest, or focus point, are depicted in considerable detail, but not with microscopic detail.

Objects in the field of view, nearer or farther from the observer than the center of interest, are depicted with less detail and with chromatic edges the color of which depends upon the position of the objects relative to the center of interest.

Objects lying to one side of the line of vision are also less clearly depicted, the lack of clearness increasing with the angle of obliquity, the accentuation of detail and edges in such objects depending upon their position relative to the center of interest. Speaking generally this accentuation is in a tangential direction if the objects are situated nearer to the observer than the center of interest and in a radial direction if they are on the plane with or behind it. Such oblique objects also have characteristic chromatic edges depending upon their position relative to the center of interest.

All oblique objects are distorted and changed in shape varying in amount with their obliquity. This distortion is shown in the bowing out in their central portions of straight lines which do not pass through the center of interest and a reduction in size of oblique objects.

And finally the color of the picture in its outer parts is bluer than at its center.

In our ordinary habit of vision, when looking at a scene, we focus on some particular part or object in it due to its special interest or beauty to us; we hold that focus for a moment or two and then look at another center of interest or another or look away entirely. With each fixation of the eye a retinal picture of the kind just described is formed. We therefore receive on our retinas a series of such pictures.

MENTAL VISUAL IMAGES.

The question arises: What is the nature of the mental visual images which we have of actuality? Without doubt, our brain receives a series of impressions similar in character to our retinal pictures. But how are those impressions registered in our consciousness and memory. There are two general possibilities. One that our visual memory consists of a series of pictures similar in nature to our retinal pictures. The other that, by some mental process, these serial impressions are combined and form a memory impression similar to actuality as we know it exists intellectually, i.e., with the detail all over the picture sharp and clear and with no colored edges or distortion.

Although there is no known psychological work on the analysis of mental visual images to substantiate the conclusion, what evidence there is indicates that our mental visual images consist of a series of images similar to our retinal pictures. This was the opinion of the well known psychologist, Dr. J. W. Baird, who mentioned as a reason for such belief the relatively definite character of the center and nebulous character of the outer parts of our mental visual images both when we are awake and when we are dreaming. Such a view is further substantiated by the fact that the indefiniteness in those parts of a scene that are not at the focus point and other characteristics of our retinal picture give a sense of depth and relief. A sacrifice of these characteristics would mean a sacrifice of effect of depth in our mental visual images which would seem most improbable.

Furthermore the existence of a mental visual image of a scene similar in detail to the scene itself could only be based on a visual knowledge of all the detail in the scene. This could only be acquired by passing the fovea or clear seeing part of the eye over every part of the scene which of course, is never done in the ordinary habits of vision. It also assumes some process of mental synthesis of particular parts of a series of impressions; of the existence of such a process we have no evidence.

It is believed that it can be concluded that our mental visual images of actuality consist of a series of images similar in general character to the picture we receive on our retina, or more accurately a series of combined pictures such as we receive on our two retinas.

METHODS OF DEPICTING NATURE.

In the arts, painting, drawing, sculpture, photography, our purpose is to depict nature. There are in general two ways in which this can be done.

First, a reproduction of the actuality can be attempted. By this is meant as close a reproduction as possible of all the objects in the scene in every measurement and detail. In sculpture the well known wax figures do this most successfully, though much work in marble and clay does so very closely. In the pictorial arts it has been most closely approximated by photographs taken with a corrected lens. Many paintings in which all objects have been depicted in full detail, as they appear on the fovea of the eye when directly observed, also very closely approximate actuality. By this is meant that the objects are so placed in the picture and the details of the objects are so depicted that, if the picture is viewed from the proper distance, all the objects will lie in the same angular direction as they do in the scene itself; while the depicting of each object is such as to produce to the eye looking directly at it the same appearance that the object itself would produce were the eye looking directly at it.

In a photograph this is accomplished by using a corrected lens; that is, one which has been so designed and constructed that every object in the field is imaged with as great detail as possible and the images of objects in the image plane have the same relative lateral positions to each other as the objects themselves. The same result is accomplished in a painting or drawing in which the artist depicts every part of the scene as it appears to him while looking directly at it. Every object will then be represented in full detail whether near, far, or on one side of the field of view, and the depicted objects will lie in the same relative lateral positions to each other as the objects themselves.

While such representations of nature are a reproduction of actuality in the accuracy with which the detail and relative lateral positions of objects are depicted, they do not reproduce the positioning of objects situated at different distances. This is impossible because a picture is, of necessity, on a flat surface where everything depicted must be the same distance from the observer, whereas nature exists in tri-dimensional space.[15] Due to this fact, the picture as a whole can never give the same effect to one looking at it as one gets from looking at nature. For in looking at one part of a natural scene, objects at all other distances take on characteristic appearances due to their different distances. In a picture where they are all reduced to one plane this is not true.

The second way in which the depicting of nature can be attempted is, instead of trying to reproduce actuality itself, to attempt to reproduce the impression that nature makes on the human consciousness, i.e., to reproduce mental visual images. The general characteristics of our mental visual images are, as it is believed it has been shown, similar to those of the retinal pictures.

Such depicting of nature can be approximated photographically by means of a lens which produces the same characteristic imaging as the lens system of the eye, and a plate whose sensitivity over its various parts is similar to that of the retina. It can be approximated in paintings and drawings if the artist keeps focused on whatever he picks out as the center of interest and depicts everything else as it appears to him while keeping his eye focused on that point. Artists who have very clear and lasting mental visual images closely approximate it by copying those images directly without regard to actuality.

As the retinal picture lies on a surface and as the canvas or paper on which it is depicted is also a plane surface, there does not seem to be the same fundamental limitations in reproducing it that exists in attempting to reproduce tri-dimensional actuality. In reproducing the subjective binocular impression in a single picture there do exist, however, the limitations which arise from the impossibility of reproducing those parts of the impression which we get from eye movement and motor impulses.

For all artistic purposes, it is believed that the attempt should be to reproduce not the actuality but the impression which it makes on us. Three general facts may be given in support of this.

First, the use by so many of the great painters of characteristics of the retinal picture which is the strongest evidence of the artistic value of pictures of this type. Second, photographs taken with a lens which approximately reproduces the characteristics of the retinal picture are more pleasing than those taken with a corrected lens. Third, photographs taken with corrected lenses are the most perfect reproduction of actuality which we have, much more accurate than can probably ever be accomplished with brush or pencil, yet this type of picture is admittedly a complete failure from an artistic point of view, indeed its failure seems to be due to the fact that it does reproduce actuality so accurately.

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[15] Stereoscopic photographs do reproduce the effect of the third dimension. They are not however satisfactory from an artistic point of view, and as we are dealing only with single pictures will not be considered here.

The following three argumentative reasons are also given:

First, the general accepted belief that artistic expressions are subjective. The purpose of the great artist is to make others see nature as he sees it. He could convey no more by reproducing actuality to those who look at his picture than they would get by looking at the scene itself. He has to put into his picture nature's impression on himself, the beauty and the truth he sees. Second, the subtle variations and differences which cause him to see the scene beautifully are alterations in his mental visual images due to personal psychological factors. The depicting of such subtle differences could be much more easily accomplished in a picture which in its general type was similar to his subjective impression than one which was not. Third, the purpose of art is to awaken subjective associative processes in those who look at it. This is especially evident in portraiture. The natural way to cause us to recall our mental visual images or start a train of them in motion is to present to us a picture similar to them.

When we look at a picture of this type we recognize that it is an attempt to reproduce not actuality but our impression of actuality. Where we see the objects farther away than the object in focus depicted without much detail we do not think that they in fact did not have detail in them or perhaps that an intervening mist existed which obscured the detail but we know that such objects were farther away than the object in focus. And similarly with the distortion of line on the side of the field of view, we do not think that a building, for instance, so depicted is in fact curved. We recognize that such is the character of our subjective impression of a building on one side of our field of view. That is we pass our fovea or the part of our eye that gives us distinct vision over the picture and recognize its various parts as being similar to our mental visual image, just as we can direct our attention to various parts of such an image.

There is possibly a third way to attempt to depict nature, which may be considered a modification of the first method above described.

In such a picture, all objects not at the center of interest would be painted so that, when the center of interest in the picture was looked at, the picture as a whole would make an impression similar to that produced by the scene itself. Such pictures as photographs taken with corrected lenses attempt to do this but, as has been stated, fail in that they do not give a proper suggestion of depth.

In this third type this deficiency might be met by accentuating the characteristic imaging of all objects not at the focus point to suggest their position in tri-dimensional space. For instance, near dark objects, in the line of focus, would be shown with red edges, distant ones with blue; and, on the sides of the picture, the radial accentuation of distant objects and the tangential accentuation of near objects would be shown.

In making these peripheral accentuations it would have to be borne in mind that they would be modified by the oblique aberrations of the eye as they would lie on the peripheral part of the picture and be imaged on the periphery of the retina. Allowances would therefore have to be made. As the picture plane lies near the secondary astigmatic field the radial accentuations would have to be relatively slight and tangential accentuations relatively marked. No distortion would be put in such a picture as the eye itself would introduce it, if the picture was viewed from the proper distance.

Granting that the proper accentuation of the radial and tangential and chromatic edges could be made, which would be very difficult, it is questioned whether such a picture would be satisfactory.

The proper impression could only be produced when the center of interest of the picture was looked at. If any other part of the picture was looked at that part would not only appear like nothing ever seen before but the rest of the picture would then cease to produce the proper impression.

THE RETINAL PICTURES AND ARTISTIC PHOTOGRAPHY.

As has been stated cameras in their inception were copied after the eye. It can be argued that the value of photographs for pictorial purposes rests on two quite different bases. One that it lies in reproducing in black and white, or in photographs in color, the same general picture we get on our retina. The other that it lies in reproducing in light and shade detail and color the effects that exist in nature.

On the first basis it can be said that the results, when looked at, are satisfactory because they appear to us similar to the impression we have received. On the second basis it can be said they are satisfactory because in looking at them our eye is affected in the same way as it is affected by nature.

In spite of the inherent impossibility already pointed out that the effect of objects in tridimensional space cannot be reproduced on a flat surface, it seems to have been the second basis that was followed in the evolution of the art of photography. This led to the development of corrected lenses which would image everything in the field of view in full detail with no distortion.

After such lenses had been perfected it was found that the photographs taken with them, although having unquestioned value for scientific and other purposes, were not satisfactory from an artistic point of view.

Then followed the use of so called "soft focus" lenses, and various manipulations in printing and enlarging to get away from the hard full detail effect produced by corrected lenses. Those desiring artistic effects bought up old lenses such as were used in making daguerreotypes and had lenses designed in which various aberrations were left uncorrected. As a result there has of late years been a most marked advance in the artistic side of photography.

The method of development has however been one of "cut and try" and as far as is known, with the exception of the work done by Gleichen mentioned above, no fundamental laws have been followed. Innumerable methods and lenses have been tried and only those producing pleasing effects have survived.

It is believed that the advance in artistic effect has been due to the fact that the results obtained were more similar to the subjective impression and that future developments in photography on the artistic side will come from approximating as closely as possible the retinal pictures and mental visual images.

THE RETINAL PICTURES AS THE BASIS OF THE TECHNIQUE OF ART.

In the chapters describing the characteristics of retinal images of objects situated in different parts of the field of view examples were given of paintings by artists in which these characteristics appear. The lists do not mention a far greater number of works that are in general suggestive of the retinal picture without showing the special characteristics noted. Of these Corot is the best example. Whistler, Manchini and Abbot Thayer and many others are also examples.

This similarity to the retinal picture is shown in a tendency to accentuate the center of interest and lose detail elsewhere. This is especially evident in black and white work and in etchings. An explanation of this may be that it is easier to accomplish in black and white work. For in that part of the picture where it is desired to lose detail, detail is simply left out, the white paper suggesting blankness. Where color is used this cannot be done. Some color must be put on the canvas and then the difficulty arises of getting on the right colors in the right way.

It is believed that anyone going through a gallery, with the points of views here set forth in mind, will be impressed by the fact that the works of many of the best men show a suppression of detail in those parts of the pictures which are not the center of interest. And there will not be the slightest question that, in most cases, the entire picture is not painted as it would appear if every part of the scene were looked at directly.

It is also very interesting to note that it is almost a general rule that the early work of most of the great masters was, so to speak, tight and hard, photographic. There is a very good example of this in an early picture by Corot in the Boston Art Museum. Later their style or technique changes. Their work is done more broadly especially the outer parts and their pictures get a center of focus and, to use a stock term, compose.

It cannot be questioned that the change is a departure from a photographic reproduction of the scene. But the question may still remain: what are the laws that govern this change in style or technique? The commonly accepted belief is that, if there are any laws, they are purely aesthetic or psychical, and that the artist puts in and leaves out and changes solely according to the dictates of his personal taste. In view of what has been shown, especially the use by so many great painters of distortion, see Chapter IV, it is believed that the improvement in technique of these artists was due to a development of their vision. Consciously or unconsciously they approximated the scene as it would have appeared to them had they kept their focus upon the center of interest.

The difficulty which arises in this method of painting is to know how to reproduce the impression that one gets from an object at which one is not looking. That the capacity to recognize and analyze such impressions can be developed is shown by fact that some of the characteristics of such impressions have been represented by numerous artists. That this is very difficult is shown by the fact with all the genius we must attribute to the great artists, there is, it is believed, no artist who has intuitively recognized and depicted all the characteristics above mentioned; and in spite of the fact that distortion, the most apparent of these characteristics, has in the past been used by many great artists it has been recognized and used as far as the writers know by but one artist living today, i.e., Sir William Orpen.

In fact so little do we know as to what we see that the ordinary person, including art students and many painters, do not know that we see clearly only those objects upon which we focus. They believe that the whole of the field of view is clear because when they are interested in the question of the clarity of any part of it they look directly at it and are unconscious of the eye movement.

Granting that it is desirable for the painter to be able to recognize and depict the character of his retinal impression it is believed that there is no question but that he can be greatly helped by an intellectual knowledge of the characteristics of his retinal picture. The knowledge that he never sees the whole of the scene with equal clearness will, after he has tried a few times, awaken his consciousness to the fact that he sees objects away from the center of focus less clearly. The knowledge that the characteristic edges and shapes of those objects not at the center of interest can be seen only if the eye is kept focused on the center of interest will enable him to see these characteristics. The knowledge of distortion and the perception following therefrom will cause him to become conscious of its existence. Similarly the knowledge of the characteristic chromatic edges of objects nearer and farther than the focus point, of the accentuation of radial and tangential lines and of the greater brightness of blue at a slight obliquity will lead to his conscious perception of these phenomena. Apart from the matter of developing his perceptions, the knowledge of the characteristic imaging of objects in various parts of the field of view will enable him to produce an effect of depth without the use of, or to supplement, the means now used. It will also enable him to produce a natural center of interest.

The objection has been raised that such an intellectual knowledge would be harmful to an artist by destroying the "innocence" of his eye. That is in his knowledge of and expectation of seeing these things, he would see them where they do not exist, or would over-accentuate them. If the eye were "innocent" in the first place such an objection might have weight. But it is not. Take for instance the matter of the detail which we see in objects. We know that all objects both those in focus and those nearer and farther are sharp and clear, so we think we see them sharp and clear while in fact we see the objects out of focus with fuzzy edges.

And likewise in the matter of distortion. We know that the side of a building is straight. That knowledge destroys the innocence of our vision and makes us think we see it straight out the side of our eye when in fact we see it curved. "Innocence" is already gone. To restore truth to the eye, we have to learn that under certain conditions we see chromatic edges on objects which are in fact sharp and curved lines in place of straight ones. The same line of argument applies to the other characteristics of the retinal pictures.

The development of art seems to have been a struggle to put down what we see and not what we know or think we see. The Egyptians for instance in drawing an eye as seen from in front on a face in profile were putting down not what they saw but what they knew. So they put in the same picture what they saw from two different points of view. We are confident that today we paint things as we see them. But on consideration are we not making just as bad a mistake when we paint an object in the foreground and one in the distance with equal clarity, or when we paint straight lines on the side of the picture straight. It is just as impossible for us to see both near and distant objects equally sharp at the same time or to see straight lines on the side of our field of view as straight, as it was for the Egyptian to see the eye from the front view in the face from the side view.

The conclusion must be that the "innocence" of vision can be developed by intellectual suggestions. Such suggestions which help the artist to know what he sees will also bring within the grasp of his intellectual consciousness, so that he can definitely perceive it, what formerly he could only feel intuitively and will leave his intuition free to reach out for the subtler expression of truth and beauty.

In the ultimate analysis what the great artist expresses through his work is a matter of his own mental and spiritual visual images of beauty and truth. Their character depends upon his personality and psychology. Of necessity these factors will modify the characteristics of his retinal pictures, to what extent it is impossible of course to say. It is believed however that the technical structure of his work is based upon his retinal impressions. The similarity of his retinal image to those of the rest of mankind will insure a universality of understanding and appreciation.

It should be clearly understood that it is not suggested that the methods of depicting the various parts of a scene which have been described will in themselves make a great work of art. They are simply matters of technique, the grammar so to speak of the expression. The quality that makes a great work of art is what the artist expresses through his work. Masterpieces expressing great truth and beauties have been done with entirely different bases of technique, the work of the "Primitives" as compared with that of the Barbizon School, for instance. Genius will express itself through any technique, but certain techniques will give greater possibilities for depth and subtlety of expression than others just as a modern piano, or violin or full orchestra gives the musical composer more freedom than the shepherd's pipes.

It is believed that everything that has been said applies to scuplture as well as to painting, taking into consideration of course that sculpture is tridimensional.

Its usefulness in architectural drawings seems to be pretty conclusively proved by Figures 34 and 35. As to its application to architecture itself, the writers do not feel qualified to speak. The belief that it is applicable can only be based on the assumption that in architecture we desire a subjective impression of a preconceived rectilinear arrangement. It would seem that under certain circumstances this might be desirable. The possibility immediately comes to one's mind that there may be a connection between the curves in Greek and Gothic architecture and the characteristics of our retinal image.

Before closing it might be interesting to suggest a line of thought to which the determination of the more specific characteristics of our retinal pictures might lead. In the first place the characteristics of our retinal pictures are greatly affected by external physical conditions. Take for instance retinal pictures of a scene in the day time and at night. Due to the enlarged pupil with which the night scene would be viewed and the greater sensitivity of the retina to blue light, as shown by the left hand curve in Figure 37, the characteristics of our retinal picture of the night scene will be very different from those of the day scene. It is believed that there is no doubt that pictures depicting these characteristics would most strongly suggest the external physical conditions which give rise to them. Further, the eye is a most delicately sensitive organ and is without doubt, affected by our bodily and likewise by our grosser emotional states. It is conceivable that the characteristics of our retinal pictures are also affected, and that specific bodily and emotional states are accompanied by specific changes in the characteristics of our retinal pictures. This in turn opens up the possibility that if the characteristics produced by a particular bodily or emotional state were depicted in a picture, the picture would suggest that particular state to those looking at it.

CONCLUSIONS.

1. Every object in space is imaged on the retina with characteristics of form, color, accentuation of line and chromatic edges, due to its particular position relative to the focus point.
2. These characteristics, of which the observer may or may not be conscious, suggest to him the position of the object in space relative to his fixation point.
3. The reproduction in a picture of these characteristics of images of objects causes the depicted objects to appear in the same relative positions that they occupied in space.
4. A pictorial representation of nature to be technically satisfactory from an artistic point of view should be similar to our subjective impression. It should not attempt to reproduce actuality.
5. Our subjective impressions are, in their general character, similar to the pictures we receive on our retinas while holding one center of focus.
6. A pictorial representation of nature to be technically satisfactory from an artistic point of view should be similar in its general characteristics to the pictures we receive on our retinas while holding one center of focus.
7. An intellectual knowledge of the characteristics of the retinal image of objects not at the center of focus helps one to become visually conscious of such characteristics.

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THE BOUNDARY PROBLEMS AND DEVELOPMENTS ASSOCIATED WITH A SYSTEM OF ORDINARY LINEAR DIFFERENTIAL EQUATIONS OF THE FIRST ORDER.

BY GEORGE D. BIRKHOFF AND RUDOLPH E. LANGER.

THE BOUNDARY PROBLEMS AND DEVELOPMENTS ASSOCIATED WITH A SYSTEM OF ORDINARY LINEAR DIFFERENTIAL EQUATIONS OF THE FIRST ORDER

By George D. Birkhoff and Rudolph E. Langer.[1]

Introduction.

It is the purpose of this paper to develop in outline the theory of a system of n ordinary linear differential equations of the first order containing a parameter and subject to certain boundary conditions. Toward this end the notation of matrices is used. For the convenience of the reader the paper opens with a brief review of the fundamentals of matrix algebra and the integration and differentiation of matrices. This is followed by an expository discussion of the homogeneous and non-homogeneous differential matrix equations of the first order. The major portion of the treatment is devoted, however, to the homogeneous differential vector equation with a complex parameter in its coefficient, and to the system composed of such an equation and suitable boundary conditions. The solutions of the equation for large values of the parameter are discussed and the formal development of a vector of arbitrary functions into a series of solutions of the system is obtained. The paper closes with the proof of the convergence of this development under appropriate conditions, which, in the ordinary notation, establishes the possibility of simultaneously expanding n arbitrary functions in terms of the characteristic solutions of a properly restricted differential system of the type

$$y_i'(x) = \sum_{k=1}^{n} \{a_{ik}(x)\lambda + b_{ik}(x)\} y_k(x),$$

$$\sum_{k=1}^{n} \{\alpha_{ik} y_k(a) + \beta_{ik} y_k(b)\} = 0, \quad i = 1, 2, \ldots n.^{2}$$

When reduced to a single equation of the nth order this includes as a special case the expansions obtained by Birkhoff in 1908.

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[1] Much of the material preceding the proof of convergence is due to Birkhoff, having been developed by him in lectures at Harvard University in the fall of 1920. The reorganization of this material into its present form, the treatment of the irregular case, and the proof of convergence are due to Langer.

[2] For other developments in this field and more complete references see the following papers in the Transactions of the American Mathematical Society: Birkhoff, *On the Asymptotic Character of the Solutions of Certain Linear*

SECTION I.

Definitions.[3]

An array of elements of the form

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \ldots . a_{1n} \\ a_{21} & a_{22} & \ldots\ldots . a_{2n} \\ a_{31} & \multicolumn{2}{c}{\ldots\ldots\ldots\ldots . a_{3n}} \\ \multicolumn{3}{c}{\ldots\ldots\ldots\ldots\ldots\ldots} \\ \multicolumn{3}{c}{\ldots\ldots\ldots\ldots\ldots\ldots} \\ a_{n1} & a_{n2} \ldots & \ldots\ldots . a_{nn} \end{pmatrix}$$

in which the number of rows equals the number of columns is called a square matrix, and is denoted either by (a_{ij}) or by A. Two such matrices are said to be equal when, and only when, every element of the one is equal to the correspondingly situated element of the other.

The sum of two matrices, (a_{ij}) and (b_{ij}), having the same number of rows and columns, is by definition the matrix $(a_{ij} + b_{ij})$, from which it follows that matrix addition is both commutative and associative, i.e., $A + B = B + A$, and $A + (B + C) = (A + B) + C$.

The product of two n rowed matrices A and B is defined by the identical equation

$$(a_{ij})\,(b_{ij}) \equiv \left(\sum_{k=1}^{n} a_{ik} b_{kj}\right),$$

on the basis of which it is easily verified that matrix multiplication is both associative and distributive, i.e.

$$A(BC) = (AB)C,$$
$$A(B + C) = AB + AC.$$

Differential Equations Containing a Parameter, and *Boundary Value and Expansion Problems of Ordinary Linear Differential Equations*, vol. 9 (1908), p. 219 and p. 373.

Wilder, *Expansion Problems of Ordinary Linear Differential Equations with Auxiliary Conditions at More than Two Points*, vol. 18 (1917), p. 415.

Hopkins, *Some Convergent Developments Associated with Irregular Boundary Conditions*, vol. 20 (1919), p. 245.

Hurwitz, *An Expansion Theorem for a System of Linear Differential Equations of the First Order* (about to appear in vol. 22 (1921)).

Langer, *Developments Associated with a Boundary Problem not Linear in the Parameter* (about to appear in vol. 22 (1921)).

[3] For a more ample discussion of the theory of matrices see Bôcher, M., *Introduction to Higher Algebra*. New York; The Macmillan Co., 1907.

That it is in general not commutative is a consequence of the fact that $\sum_{k=1}^{n} a_{ik}b_{kj}$ is in general not equal to $\sum_{k=1}^{n} b_{ik}a_{kj}$. The rearrangement of the factors in a matrix product is, therefore, as a rule, not permissible.

The multiplication of a scalar into a matrix has simply the effect of multiplying each element of the matrix by the scalar. Thus if k is a scalar, then $kA = Ak = (ka_{ij})$. Conversely, any factor common to all the elements of a matrix can be factored from the matrix.

Two special matrices must be mentioned, namely $O = (0)$, the *zero matrix*, and $I = (\delta_{ij})$, the *unit matrix*, where $\delta_{ij} = 0$ when $i \neq j$, $\delta_{ii} = 1$. These matrices satisfy respectively the relations

$$AO = OA = O$$

and
$$AI = IA = A.$$

The determinant formed from the elements of a matrix without changing the order of the array is called the *determinant of the matrix.* The alternative notations $| a_{ij} |$ or $| A |$ for the determinant of the matrix (a_{ij}) will be used.

Given a matrix A it follows that if $| A | \neq 0$ then there exists a unique solution in the x's for each of the linear systems

$$\sum_{k=1}^{n} a_{ik}\, x_{kj_0} = \delta_{ij_0}, \qquad i = 1,\ 2,\ \ldots n,$$

where j_0 ranges from 1 to n. This means that there exists a unique set of n^2 quantities x_{ij} such that

$$\sum_{k=1}^{n} a_{ik}\, x_{kj} = \delta_{ij}, \quad i, j, = 1, 2, \ldots \mathrm{n},$$

i.e. there exists a unique matrix (x_{ij}) such that

$$A \cdot (x_{ij}) = I.$$

This matrix (x_{ij}) is denoted by the symbol A^{-1} and is called the *inverse of* A. From its derivation it is seen to satisfy the relation $AA^{-1} = I$.

Either of the relations, $AX = O$ or $XA = O$, leads, on the assumption that $| A | \neq 0$, to the conclusion $X = O$, as is evident from the theory of the systems of linear equations to which the matrix equations are equivalent. It follows from the relations $AA^{-1} = I$ and $IA = AI$, however, that $AA^{-1}A = IA = AI$, i.e., $A(A^{-1}A - I) = O$. Hence if A is any matrix for which $|A| \neq 0$, it is seen that

$$A^{-1}A - I = O, \text{ i.e., } A^{-1}A = I = AA^{-1}.$$

In accordance with the following definitions, namely

$$\int_{\alpha}^{\beta} A(x)\,dx = \left(\int_{\alpha}^{\beta} a_{ij}(x)\,dx\right)$$

and

$$\frac{dA(x)}{dx} = \left(\frac{da_{ij}(x)}{dx}\right),$$

a matrix is seen to be integrable or differentiable if and only if this is true of each of its elements. It is also clear that if C is a matrix of constants, then $\frac{dC}{dx} = O$, while for any product

$$\frac{d}{dx}AB = A\frac{dB}{dx} + \frac{dA}{dx}B.$$

SECTION II.

The equation $Y'(x) = A(x)\,Y(x)$.[4]

Consider any matrix of functions $Y(x)$ which satisfies (i.e. is a solution of) the equation

$$Y' = AY. \tag{1}$$

In accordance with the rules for determinants we have

$$\frac{d\,|Y|}{dx} = \begin{vmatrix} y'_{11} & \cdots & y'_{1n} \\ y_{21} & \cdots & y_{2n} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ y_{n1} & \cdots & y_{nn} \end{vmatrix} + \begin{vmatrix} y_{11} & \cdots & y_{1n} \\ y'_{21} & \cdots & y'_{2n} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ y_{n1} & \cdots & y_{nn} \end{vmatrix} + \cdots + \begin{vmatrix} y_{11} & \cdots & y_{1n} \\ y_{21} & \cdots & y_{2n} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ y'_{n1} & \cdots & y'_{nn} \end{vmatrix},$$

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[4] For the general theory of matrix differential equations see Schlesinger, L., *Vorlesungen über lineare Differentialgleichungen.* Leipzig; B. G. Teubner, 1908.

whence, substituting from the system

$$y'_{ij} = \sum_{k=1}^{n} a_{ik}\, y_{kj}, \quad i, j, = 1, 2, \ldots n$$

(which is equivalent to equation (1)) we obtain

$$\frac{d\,|\,Y\,|}{dx} = \begin{vmatrix} \sum\limits_k a_{1k}\, y_{k1} \ldots \sum\limits_k a_{1k}\, y_{kn} \\ y_{21} \ldots\ldots\ldots\ldots y_{2n} \\ \ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots \\ y_{n1} \ldots\ldots\ldots\ldots y_{nn} \end{vmatrix} + \begin{vmatrix} y_{11} \ldots\ldots\ldots\ldots y_{1n} \\ \sum\limits_k a_{2k}\, y_{k1} \ldots \sum\limits_k a_{2k}\, y_{kn} \\ \ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots \\ y_{n1} \ldots\ldots\ldots\ldots y_{nn} \end{vmatrix}$$

$$+ \ldots + \begin{vmatrix} y_{11} \ldots\ldots\ldots\ldots y_{1n} \\ y_{21} \ldots\ldots\ldots\ldots y_{2n} \\ \ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots \\ \sum\limits_k a_{nk}\, y_{k1} \ldots \sum\limits_k a_{nk}\, y_{kn} \end{vmatrix},$$

i.e. $\dfrac{d}{dx}\,|\,Y\,| = a_{11}\,|\,Y\,| + a_{22}\,|\,Y\,| + \ldots + a_{nn}\,|\,Y\,|.$

Let us suppose, now (i), that the elements of $A(x)$, $Y(x)$, and $Y'(x)$ are each continuous in an interval $a \leqq x \leqq b$, and (ii), that at some point x of this interval $|\,Y\,| \neq 0$. Then throughout a neighborhood of the point in question

$$\frac{d\,|\,Y\,|}{|\,Y\,|} = \sum_{k=1}^{n} a_{kk}\, dx,$$

or, integrating,

$$|\,Y\,| = ce^{\int \Sigma\, a_{kk}\, dx}.$$

Inasmuch as the right-hand side of this equation cannot vanish if $c \neq 0$, it is seen that the hypothesis that $|\,Y\,| \neq 0$ for some x leads to the conclusion that $|\,Y\,|$ differs from zero for all x. Thus we infer the

THEOREM: If $Y(x)$ is a matrix of functions which satisfies equation (1), while $|\,Y(x_0)\,| = 0$, $a \leqq x_0 \leqq b$, then $|\,Y(x)\,| \equiv 0$, $a \leqq x \leqq b$.

A matrix of functions $Y(x)$ of this type which satisfies equation (1) and whose determinant $|Y(x)| \neq 0$, is called a *matrix solution* of equation (1). We assume at least one such solution to exist.

THEOREM: If $Y(x)$ is any (matrix) solution of equation (1), and C is any matrix of constants (for which $|C| \neq 0$), then $\bar{Y}(x)$, defined by $\bar{Y}(x) = Y(x)C$, is also a (matrix) solution of equation (1).

Proof: We have
$$\frac{d\bar{Y}}{dx} = \frac{d}{dx} YC$$
$$= \frac{dY}{dx} C + Y\frac{dC}{dx}$$
$$= \frac{dY}{dx} C.$$

But by hypothesis
$$\frac{dY}{dx} = AY.$$

Hence
$$\frac{d\bar{Y}}{dx} = AYC,$$

i.e.
$$\frac{d\bar{Y}}{dx} = A\bar{Y}.$$

Moreover, since the product of two matrices is derived in the same manner as that of two determinants, it follows that $|\bar{Y}| = |Y||C|$. Hence $|\bar{Y}| \neq 0$, if $|C| \neq 0$ and $|Y| \neq 0$. Q.E.D.

THEOREM: Given any matrix of constants Y_0 for which $|Y_0| \neq 0$, then there exists a matrix solution of equation (1) which for any preassigned x, say $x = x_0$, reduces to the matrix Y_0.

Proof: It has already been shown that when $Y(x)$ is a matrix solution of equation (1), then $Y(x)C$ is also such a solution, where C may be any matrix of constants whose determinant is not zero. Then in particular C may be chosen as the matrix $Y^{-1}(x_0)\, Y_0$, whereupon it follows that $Y(x)\, Y^{-1}(x_0)\, Y_0$ is also a matrix solution of equation (1). This solution, however, obviously reduces to the matrix Y_0 when $x = x_0$. Q. E. D.

THEOREM: If $Y(x)$ is a matrix solution of equation (1) then $Y(x)C$ is the most general solution of equation (1).

Proof: Let $\overline{Y}(x)$ be any solution of equation (1) whatever. Then the relation $\overline{Y}(x) = Y(x)\,\Phi\,(x)$ defines a matrix $\Phi(x)$, and substituting from this expression in equation (1) we have

$$\frac{d}{dx}Y\Phi = AY\Phi,$$

i.e.
$$Y\frac{d\Phi}{dx} + \frac{dY}{dx}\Phi = AY\Phi.$$

But by hypothesis
$$\frac{dY}{dx} = AY.$$

Hence
$$Y\frac{d\Phi}{dx} + AY\Phi = AY\Phi,$$

i.e.
$$Y\frac{d\Phi}{dx} = 0.$$

Inasmuch as $|\,Y(x)\,| \neq 0$, it follows that $\frac{d\Phi}{dx} = 0$, and $\Phi(x) \equiv C$.

Q. E. D.

A mere interchange of the rôles played by the rows and the columns of the matrices involved transforms the discussion carried out thus far for equation (1) to the corresponding discussion for the equation $Y' = YA$. The facts in the two cases may, therefore, be established by precisely the same methods.

The pair of related equations

(1) $$Y' = AY,$$

(2) $$Z' = -ZA,$$

are said to be *adjoint*, either being the adjoint of the other.

THEOREM: If $Y(x)$ is any matrix solution of equation (1), and $Z(x)$ is any matrix solution of equation (2), then

$$Z(x)\;Y(x) \equiv C.$$

Proof: From the relation

$$\frac{d}{dx}ZY = \frac{dZ}{dx}Y + Z\frac{dY}{dx}$$

it follows, upon substituting from equations (1) and (2), that

$$\frac{d}{dx} ZY = (-ZA)\, Y + Z(AY).$$

Hence $\quad \frac{d}{dx} ZY = 0$, and $Z(x)\, Y(x) \equiv C$. Q. E. D.

Converse Theorem: If $Y(x)$ is any matrix solution of equation (1), and the matrix $Z(x)$ is defined by the relation $Z(x)\, Y(x) \equiv C$, where $| C | \neq 0$, then $Z(x)$ is a matrix solution of equation (2).

Proof: We have $\quad \frac{d}{dx} ZY = 0,$

i.e.
$$\frac{dZ}{dx} Y + Z \frac{dY}{dx} = 0.$$

In virtue of equation (1), therefore,

$$\frac{dZ}{dx} Y + ZAY = 0,$$

and it follows, since $| Y | \neq 0$, that

$$\frac{dZ}{dx} = -ZA.$$

Moreover, $| Z | = |Y^{-1}| \cdot | C | \neq 0.$ Q. E. D.

Any pair of solutions $Y(x)$ and $Z(x)$ of equations (1) and (2) respectively which satisfy the relation $Z(x)\, Y(x) \equiv I$, are said to be *associated solutions*. Thus if $Y(x)$ is any matrix solution of equation (1) the associated matrix solution of equation (2) is $Z(x) = Y^{-1}(x)$.

A differential equation is said to be *self-adjoint* when and only when it is identical with its adjoint after interchange of rows and columns. A necessary and sufficient condition that the equation (1) be self-adjoint is readily seen to be that $a_{ij} = -a_{ji}$ for all i and j.

SECTION III.

The equation $Y'(x) = A(x)Y(x) + B(x)$.

THE EXISTENCE THEOREM: Given the equation

$$Y' = AY + B, \tag{3}$$

where $A(x)$ and $B(x)$ are matrices of continuous functions, $a \leqq x \leqq b$, then there exists a unique matrix $Y(x)$ whose elements are functions which are continuous together with their first derivatives, $a \leqq x \leqq b$, and which satisfies equation (3) as well as the condition $Y(x_0) = Y_0$, the matrix Y_0 being any prescribed matrix of constants.

Proof: By means of the relation

$$Y_m(x) = Y_0 + \int_{x_0}^{x} \{A(t)\, Y_{m-1}(t) + B(t)\} dt$$

it is possible to define the following infinite sequence of matrices $Y_1(x)$, $Y_2(x)$, ... $Y_m(x)$, ... which satisfy the relations

$$\begin{array}{llll} Y_1'(x) & = A(x)\, Y_0 + B(x), & Y_1(x_0) & = Y_0, \\ Y_2'(x) & = A(x)\, Y_1(x) + B(x), & Y_2(x_0) & = Y_0, \\ \multicolumn{2}{l}{\dots\dots\dots\dots\dots\dots} & \multicolumn{2}{l}{\dots\dots\dots} \\ Y_m'(x) & = A(x)\, Y_{m-1}(x) + B(x), & Y_m(x_0) & = Y_0. \\ \multicolumn{2}{l}{\dots\dots\dots\dots\dots\dots} & \multicolumn{2}{l}{\dots\dots\dots} \end{array}$$

Then setting $\quad Y_m(x) - Y_{m-1}(x) = U_m(x)$

in the identical equation

$$Y_m(x) \equiv Y_0 + (Y_1(x) - Y_0) + (Y_2(x) - Y_1(x)) + \dots + (Y_m(x) - Y_{m-1}(x))$$

we have

$$Y_m(x) \equiv Y_0 + U_1(x) + U_2(x) + \dots + U_m(x).$$

Moreover, $\quad U_m(x_0) = 0,$

while

$$\begin{aligned} U_m'(x) &= Y_m'(x) - Y_{m-1}'(x) \\ &= A(x)\, \{Y_{m-1}(x) - Y_{m-2}(x)\} \\ &= A(x)\, U_{m-1}(x)\, . \end{aligned}$$

Denoting by the symbol $\ll$ the fact that each element of the matrix of constants on the right represents a value as large as the largest numerical maximum attained by any element of the matrix on the left, we have, since $A(x)$ and $U_1(x)$ are continuous in the closed interval $a \leqq x \leqq b$,

$$U_1(x) \ll (k),$$
$$A\ (x) \ll (\alpha)\ .$$

From this it follows that

$$U_2'(x) = A(x)\, U_1\ (x) = \left(\sum_{l=1}^{n} a_{il}\, u_{lj}^{(1)}\right) \ll (n\alpha k)$$

and integrating

$$U_2(x) \ll (kn\alpha \mid x - x_0 \mid).$$

Likewise $\quad U_3' = AU_2 = \left(\sum_{l=1}^{n} a_{il}\, u_{lj}^{(2)}\right) \ll \left(\sum_{l=1}^{n} \alpha \cdot kn\alpha \mid x - x_0 \mid \right),$

$$U_3(x) \ll \left(kn^2\alpha^2\ \frac{\mid x - x_0 \mid^2}{2!}\right),$$

and similarly for $l = 3, 4, \ldots,$

$$U_{l+1}(x) \ll \left(kn^l\alpha^l\ \frac{\mid x - x\cdot \mid^l}{l\,!}\right) \ll k\left(\frac{\{n\alpha\overline{b-a}\}^l}{l\,!}\right).$$

However the infinite series of matrices

$$\left(\sum_{\substack{l=0\\1}}^{\infty} \frac{\{\alpha n\overline{b-a}\}^l}{l\,!}\right)$$

converges to the matrix

$$k\ (e^{\alpha n\overline{b-a}} - 1).$$

Hence the series

$$Y_0 + \sum_{l=1}^{\infty} U_l(x)$$

converges uniformly throughout the interval $a \leqq x \leqq b$, and since its terms are matrices of continuous functions the matrix $Y(x)$ which it represents is necessarily one with elements continuous for $a \leqq x \leqq b$. Now

$$Y_m(x) = Y_0 + \int_{x_0}^{x} \{A(t) Y_{m-1}(t) + B(t)\}\, dt$$

by definition, and since the convergence of $Y_m(x)$ to $Y(x)$ is uniform

$$\lim_{m-\infty} Y_m(x) = Y_0 + \int_{x_0}^{x} \lim_{m=\infty} \{ A(t)\, Y_{m-1}(t) + B(t)\}\, dt,$$

namely,
$$Y(x) = Y_0 + \int_{x_0}^{x} \{A(t) Y(t) + B(t)\}\, dt.$$

Differentiating we see that the elements of $Y'(x)$ are continuous, $a \leqq x \leqq b$, while $Y'(x) = A(x)\, Y(x) + B(x)$. Hence the solution exists as stated.

To prove the solution unique suppose that both $Y(x)$ and $\overline{Y}(x)$ are solutions of equation (3) each reducing to the matrix Y_0 for $x = x_0$, i.e.

$$Y'(x) = A(x)\, Y(x) + B(x), \qquad Y(x_0) = Y_0,$$
$$\overline{Y}'(x) = A(x)\, \overline{Y}(x) + B(x), \qquad \overline{Y}(x_0) = Y_0.$$

Now $Y(x) - \overline{Y}(x) \equiv D(x)$ is a matrix whose elements are continuous, $a \leqq x \leqq b$, and which satisfies the relations

$$D'(x) = A(x)\, D(x), \quad D(x_0) = O.$$

Let any neighborhood of the point x_0 be chosen, say $| x - x_0 | \leqq \delta$. Then, if d denotes the largest numerical maximum of any element of $D(x)$ for any x of this neighborhood so that $D(x) << (d)$,

$$A(x)\, D(x) << (nad),$$

and
$$D'(x) << (nad).$$

Hence
$$D(x) << (nad\, | x - x_0 |\,) << (nad\delta).$$

But for some x of the interval at least one element of $D(x)$ is numerically equal to d (by the definition of d), and for this element it follows that $d \leqq nad\delta$, i.e. $d\, \{ 1 - na\delta\} \leqq 0$.

However, since α and n are fixed numbers and δ can be chosen so that $\delta < \frac{1}{n\alpha}$ this leads to a contradiction unless $d = 0$. It follows that there always exists a finite interval throughout which $d = 0$ and hence throughout which $D(x) \equiv 0$. Inasmuch as this implies that $D(x) \equiv 0$, $a \leqq x \leqq b$, the solutions $Y(x)$ and $\overline{Y}(x)$ must be identically the same. Q. E. D.

The solution of equation (3) can be easily expressed in terms of the solutions $Y_h(x)$ and $Z_h(x)$ of the homogeneous equations (1) and (2) respectively. Thus, multiplying both sides of equation (3) on the left by $Z_h(x)$ we have

$$Z_h \frac{dY}{dx} = Z_h A Y + Z_h B,$$

which, in view of equation (2), can be written

$$Z_h \frac{dY}{dx} = - \frac{dZ_h}{dx} Y + Z_n B,$$

i.e.

$$\frac{d}{dx} Z_h Y = Z_h B.$$

Integrating we obtain

$$Z_h Y = C + \int_a^x Z_h(t)\, B(t)\, dt,$$

while the multiplication of this equation on the left by $Y_h(x)$, the solution associated with $Z_h(x)$, yields

$$Y(x) = Y_h(x)\, C + \int_a^x Y_h(x)\, Z_h(t)\, B(t)\, dt. \tag{4}$$

This is the general solution of equation (3). In consequence any particular solution $\overline{Y}(x)$ may be written

$$\overline{Y}(x) = Y_h(x)\, \overline{C} + \int_a^x Y_h(x)\, Z_h(t)\, B(t)\, dt.$$

Subtracting this from equation (4), however, we obtain

$$Y(x) = \overline{Y}(x) + Y_h(x)\,\{ C - \overline{C}\},$$

from which it is seen that if $\overline{Y}(x)$ is any particular solution of equation (3) the general solution is given by

$$(5) \qquad Y(x) = \overline{Y}(x) + Y_h(x)\, C.$$

If in particular C is taken as $C = O$ in formula (4), the solution which is characterized by the fact that $Y(\alpha) = O$ is obtained. This is called the *principal solution* at $x = \alpha$. Since the choice of α as a limit of integration is unrestricted, the expression for the principal solution at any chosen point is at hand.

Section IV.

The differential system

$$Y'(x)\cdot = A(x)Y(x)\cdot + B(x)\cdot$$
$$W_aY(a)\cdot + W_bY(b)\cdot = O.$$

A matrix in which any row (or column) is precisely like every other row (or column) is called a *vector*. That a particular matrix is a vector is indicated by means of a dot suitably placed in relation to the letter designating the matrix in question, the dot preceding in case it is a vector in which the rows are the same and succeeding in case it is a vector of identical columns.

Thus

$$\cdot A = (a_j) = \left\{\begin{matrix} a_1 & a_2 & \ldots & a_n \\ a_1 & a_2 & \ldots & a_n \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ a_1 & a_2 & \ldots & a_n \end{matrix}\right\},$$

and

$$A\cdot = (a_i) = \left\{\begin{matrix} a_1 & a_1 & \ldots & a_1 \\ a_2 & a_2 & \ldots & a_2 \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ a_n & a_n & \ldots & a_n \end{matrix}\right\}.$$

That $\cdot AB = \cdot C$, and that $AB\cdot = C\cdot$, are facts readily established by direct reference to the rules for multiplication. A matrix (a) all of whose elements are the same is written $\cdot A \cdot$. From the preceding statement it is clear that $\cdot AB\cdot = \cdot C\cdot$.

Given W_a and W_b, any two constant matrices for which $|W_a| \neq 0$ and $|W_b| \neq 0$, then the two corresponding homogeneous vector differential systems

$$\left\{\begin{array}{l} Y'(x)\cdot = A(x)\, Y(x)\cdot \\ W_a Y(a)\cdot + W_b Y(b)\cdot = 0 \end{array}\right. \tag{6}$$

and

$$\left\{\begin{array}{l} \cdot Z'(x) = - \cdot Z(x)\, A(x) \\ \cdot Z(a)\, W_a^{-1} + \cdot Z(b)\, W_b^{-1} = 0 \end{array}\right. \tag{7}$$

are said to be *adjoint*.

THEOREM: The number of linearly independent solutions of system (6) is always equal to the number of linearly independent solutions of system (7).

Proof: It has been shown that if $\overline{Y}(x)$ is any matrix solution of the differential equation (1), the most general solution is $Y(x) = \overline{Y}(x)C$. From this, $C = \overline{Y}^{-1}(x)\, Y(x)$, and if the solution $Y(x)$ is a vector $Y(x)\cdot$ then C will be a vector $C\cdot$.

The general solution of the differential equation in system (6) is, therefore,

$$Y(x)\cdot = \overline{Y}(x)\, C\cdot,$$

and since the substitution of this in the boundary conditions gives

$$W_a\, \overline{Y}(a)\, C\cdot + W_b\, \overline{Y}(b)\, C\cdot = 0, \tag{8}$$

$Y(x)\cdot$ is seen to be a solution of system (6) when and only when $C\cdot$ is a solution of the equation (8). Moreover, it is readily seen that a necessary and sufficient condition that a set of solutions of system (6) be linearly independent is that the corresponding solutions of equation (8) be independent. Setting $W_a\, \overline{Y}(a) + W_b\, \overline{Y}(b) = (\rho_{ij})$, equation (8) can be written in the form

$$\sum_{k=1}^{n} \rho_{ik}\, c_k = 0, \quad i = 1, 2, \ldots n. \tag{9}$$

The number of linearly independent solutions of the linear algebraic system (9), (and hence of equation (8)) is, however, precisely equal to the difference between the number of equations, n, and the rank of the determinant $|\rho_{ij}|$. Thus when $|W_a\bar{Y}(a) + W_b\bar{Y}(b)| = 0$, and is of rank $(n-k)$, then there are precisely k linearly independent solutions $C_1\cdot, C_2\cdot, \ldots C_k\cdot$, of equation (8), and correspondingly just k linearly independent solutions $Y_i(x)\cdot = \bar{Y}(x)\,C_i\cdot \quad i = 1, 2, \ldots k$, of system (6). Conversely if there are k independent solutions of system (6) the rank of the determinant is $(n-k)$.

Suppose then that system (6) is known to have just k linearly independent solutions. This means that the determinant vanishes and is of rank $(n-k)$, and since $|X_1 U X_2|$ is of the same rank as $|U|$ when $|X_1| \neq 0$, and $|X_2| \neq 0$,[5] it follows that

$$|\bar{Z}(a)\,W_a^{-1}\,\{W_a\bar{Y}(a) + W_b\bar{Y}(b)\}\,\bar{Z}(b)\,W_b^{-1}| = 0$$

and is of rank $(n-k)$, $\bar{Z}(x)$ being any matrix solution of equation (2). If in particular $\bar{Z}(x)$ is chosen as the solution associated with $\bar{Y}(x)$ this reduces to the statement that $|\bar{Z}(a)\,W_a^{-1} + \bar{Z}(b)\,W_b^{-1}| = 0$; and is of rank $(n-k)$, which implies that system (7) also has just k linearly independent solutions. Thus the theorem is proved.

A system of the type (6) or (7) is said to be either *compatible* or *incompatible* according as it does or does not admit of a solution not identically O. Its compatibility is said to be k-fold when the number of its linearly independent solutions is k.

Consider the non-homogeneous system

$$\text{(10)} \qquad \begin{cases} Y'(x)\cdot = A(x)\,Y(x)\cdot + B(x)\cdot \\ W_a\,Y(a)\cdot + W_b\,Y(b)\cdot = O. \end{cases}$$

THEOREM: A necessary and sufficient condition that system (10) has a unique solution is that the corresponding homogeneous system (6) is incompatible.

Proof: It was shown (formula (5)) that the general solution of equation (3) is

$$Y(x) \equiv Y_h(x)\,C + \bar{Y}(x)$$

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[5] Cf. Bôcher, loc. cit., pp. 77–79.

where $\overline{Y}(x)$ is any particular solution and $Y_h(x)$ is a matrix solution of the homogeneous equation (1). Hence the general solution of the differential equation in (10) is

$$Y(x)\cdot \equiv Y_h(x)\, C\cdot + \overline{Y}(x)\cdot$$

The substitution of $Y(x)\cdot$ in the boundary conditions shows it to be also a solution of the system (10) provided only that $C\cdot$ satisfies

$$W_a\,\overline{Y}(a) + W_b\,\overline{Y}(b) + \{ W_a\, Y_h(a) + W_b\, Y_h(b)\}\, C\cdot = O.$$

But this relation can be solved for $C\cdot$, and uniquely determines $C\cdot$ when and only when $| W_a\, Y_h(a) + W_b\, Y_h(b) | \neq 0$, that is when system (6) is incompatible. Q. E. D.

Assuming then that system (6) is incompatible it is possible to obtain by the following procedure a solution of the equations (10) which is symmetrical with respect to the ends of the given interval $a \leqq x \leqq b$.

Let $Y_1(x)\cdot$ and $Y_2(x)\cdot$ be any pair of solutions of the differential equation

$$Y'(x)\cdot = A(x)\, Y(x)\cdot + \tfrac{1}{2}\, B(x)\cdot. \tag{11}$$

Then clearly their sum $Y_1(x)\cdot + Y_2(x)\cdot = \overline{Y}(x)\cdot$ is a solution of equation (10). But by formula (4), applied to equation (11), a particular choice of $Y_1(x)\cdot$ and $Y_2(x)\cdot$ is seen to be

$$Y_1(x)\cdot = \tfrac{1}{2} \int_a^x Y_h(x)\, Z_h(t)\, B(t)\cdot dt$$

$$Y_2(x)\cdot = \tfrac{1}{2} \int_b^x Y_h(x)\, Z_h(t)\, B(t)\cdot dt,$$

from which it follows that

$$\overline{Y}(x)\cdot = \tfrac{1}{2} \left\{ \int_a^x Y_h(x)\, Z_h(t)\, B(t)\cdot dt + \int_b^x Y_h(x)\, Z_h(t)\, B(t)\cdot dt \right\}$$

is a particular solution of the differential equation (10).

Defining $\bar{G}(x, t)$ by the relations

$$\bar{G}(x, t) = \begin{cases} \tfrac{1}{2}\, Y_h(x)\, Z_h(t) & \text{when } t < x \\ -\tfrac{1}{2}\, Y_h(x)\, Z_h(t) & \text{when } t > x, \end{cases}$$

we may write

$$\bar{Y}(x)\cdot = \int_a^b \bar{G}(x, t)\, B(t)\cdot\, dt.$$

In accordance with formula (5), therefore, the general solution of the differential equation (10) is given by the equation

$$Y(x)\cdot = \int_a^b \bar{G}(x, t)\, B\,(t)\cdot\, dt + Y_h(x) C\cdot\, .$$

The substitution of this form in the boundary conditions yields the equation

$$\int_a^b \{W_a \bar{G}(a, t) + W_b \bar{G}(b, t)\}\, B(t)\cdot dt + \{W_a\, Y_h(a) + W_b Y_h(b)\} C\cdot = 0$$

for the constant vector $C\cdot$. Multiplying by the inverse of the matrix

$$\Delta = \{W_a\, Y_h(a) + W_b\, Y_h(b)\},$$

(Δ^{-1} exists since system (6) is incompatible) we see that

$$C\cdot = -\Delta^{-1} \int_a^b \{W_a \bar{G}\,(a, t) + W_b \bar{G}\,(b, t)\}\; B(t)\cdot dt.$$

It follows that the general solution of system (10) is, in terms of a matrix $G(x, t)$ which is defined by the formula

$$G(x, t) = \bar{G}(x, t) - Y_h(x)\, \Delta^{-1}\, \{W_a\, \bar{G}(a, t) + W_b\, \bar{G}(b, t)\}, \tag{12}$$

given by the expression

$$Y(x) = \int_a^b G(x, t)\, B(t)\cdot\, dt. \tag{13}$$

The matrix of functions $G(x, t)$ is known as the *Green's function* for the homogeneous system (6). By a precisely similar method the Green's function $H(x, t)$ for the related system (7) may be derived, the general solution of the non-homogeneous system

$$(14) \qquad \begin{cases} \cdot Z'(x) = - \cdot Z(x)\, A(x) + \cdot D(x) \\ \cdot Z(a)\, W_a^{-1} + \cdot Z(b)\, W_b^{-1} = 0 \end{cases}$$

being given in terms of $H(x, t)$ by the formula

$$(15) \qquad \cdot Z(x) = \int_a^b \cdot D(x)\, H(x, t)\, dt.$$

THEOREM: If $G\,(x, t)$ and $H(x, t)$ are the Green's functions for systems (6) and (7) respectively, then

$$G(x, t) + H(t, x) \equiv 0, \quad t \neq x.$$

Proof:[6] Let $B\cdot$ and $\cdot D$ be any two vectors of the types indicated and consider the two systems

$$\begin{array}{ll} Y'\cdot = AY\cdot + B\cdot, & W_a\, Y(a)\cdot + W_b\, Y(b)\cdot = 0, \\ \cdot Z' = - \cdot ZA + \cdot D, & \cdot Z(a)\, W_a^{-1} + \cdot Z(b)\, W_b^{-1} = 0. \end{array}$$

Multiplying the differential equations respectively by $\cdot Z$ on the left and $Y\cdot$ on the right and adding we obtain

$$\cdot ZY\cdot' + \cdot Z'Y\cdot = \cdot ZB\cdot + \cdot DY\cdot,$$

an equality which upon integration yields

$$(16) \qquad \cdot ZY\cdot \Big|_a^b = \int_a^b \cdot Z(t)\, B(t)\cdot dt + \int_a^b \cdot D(x)\, Y(x)\cdot dx.$$

Now from the boundary conditions we have

$$\cdot Z(b) = - \cdot Z(a)\, W_a^{-1}\, W_b, \quad Y(b)\cdot = - W_b^{-1}\, W_a\, Y(a)\cdot,$$

whence

$$\cdot Z(b)\, Y(b)\cdot = \cdot Z(a)\, Y(a)\cdot,$$

i.e.

$$\cdot ZY\cdot \Big|_a^b = 0.$$

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[6] The proof by direct computation is not difficult though somewhat laborious.

But
$$\cdot Z(t) = \int_a^b \cdot D(x)\, H(t, x)\, dx$$

and
$$Y(x)\cdot = \int_a^b G(x, t)\, B(t)\cdot dt$$

from formulas (13) and (15).
Hence equation (16) is equivalent to

$$(17) \qquad \int_a^b\int_a^b \cdot D(x)\,\{H(t, x) + G(x, t)\}\, B(t)\cdot dx\, dt = 0.$$

This result, having been obtained without reference to the nature of $B\cdot$ and $\cdot D$ must, moreover, hold for all possible choices of these vectors. We shall proceed to choose a particular set apposite to the proof in hand.

Let any point (x_0, t_0) of the region $a \leqq x, t \leqq b, x \neq t$, be arbitrarily chosen and draw the surrounding small rectangle Δs whose sides are $t = t_0 \pm \Delta t$ and $x = x_0 \pm \Delta x$. Then choose $B\cdot$ and $\cdot D$ so that
$b_l(t) \equiv 0$, when $l \neq j_0$, $\quad b_{j_0}(t) \neq 0$ in Δs, $\quad b_{j_0}(t) \equiv 0$ outside of Δs,
$d_k(x) \equiv 0$, when $k \neq i_0$, $\quad d_{i_0}(x) \neq 0$ in Δs, $\quad d_{i_0}(x) \equiv 0$ outside of Δs.

For this choice equation (17) is equivalent to

$$(18) \qquad \int_{t_0-\Delta t}^{t_0+\Delta t}\int_{x_0-\Delta x}^{x_0+\Delta x} d_{i_0}(x)\,\{h_{i_0j_0}(t, x) + g_{i_0j_0}(x, t)\}\, b_{j_0}(t)\, dxdt = 0,$$

and inasmuch as $d_{i_0}(x)\, b_{j_0}(t) \neq 0$ in Δs, clearly $\{h_{i_0j_0}(t, x) + g_{i_0j_0}(x, t)\}$ changes sign, i.e. vanishes somewhere in this region. Now let Δx and Δt approach zero. Then in the limit we have

$$h_{i_0j_0}(t_0, x_0) + g_{i_0j_0}(x_0, t_0) = 0.$$

From this it is seen that $G(x_0, t_0) + H(t_0, x_0) = 0$, and since (x_0, t_0) was any point not on the diagonal it follows that

$$G(x, t) + H(t, x) \equiv 0, \quad x \neq t. \qquad \text{Q. E. D.}$$

By direct reference to formula (12) it is readily seen that the Green's function possesses the following characteristics:

i) The elements of $G(x, t)$ are continuous in x except for $x = t$. Along this line ($x = t$) there is a jump of unit magnitude in the elements of the principal diagonal, i.e.

$$G(x, x - 0) - G(x, x + 0) = I.$$

ii) For any given t, $G(x, t)$ satisfies equation (6) in x, except along the line $x = t$, i.e.

$$\frac{\partial G(x, t)}{\partial x} \equiv A(x)\, G(x, t), \quad x \neq t.$$

iii) For any given t, $G(x, t)$ satisfies the boundary conditions of system (6) in x, i.e.

$$W_a G(a, t) + W_b\, G(b, t) \equiv O.$$

Conversely we have the

THEOREM: The dependence of $G(x, t)$ upon the variables x, t is completely determined by the characteristics (i) to (iii) above.

Proof: Suppose $\bar{G}\,(x, t)$ is a matrix possessing the characteristics (i), (ii), and (iii). By (i), (ii), $J(x, t)$ defined by $J(x, t) \equiv \bar{G}\,(x, t) - G(x, t)$ is continuous for all x, $a \leqq x \leqq b$. Moreover, $J(x, t)$ satisfies (iii) and is therefore a solution of system (6) in x. But system (6) is incompatible by hypothesis. Hence $J(x, t) \equiv O$

and $$\bar{G}(x, t) \equiv G(x, t).$$ Q. E. D.

It is readily verified that a further set of three characteristics which completely determines the dependence $G(x, t)$ upon the variables may be obtained by interchanging x and t and replacing system (6) by system (7) in the discussion above.

The fact established by this theorem should be carefully noted. While the choice of the pair of associated solutions Y_h and Z_h on the right-hand side of equation (12) is not unique, yet the entire function $G(x, t)$ is independent of that choice.

SECTION V.

The formal solutions of the equation $Y'(x)\cdot = \{A(x)\lambda + B(x)\}Y(x)\cdot$.

Returning to the homogeneous equation let us consider the nature of the solutions when the matrix coefficient of $Y\cdot$ is made to depend linearly upon a parameter λ which is free to take on all values in the finite complex λ plane. To this end we shall study the equation

$$(19) \qquad Y'(x)\cdot = \{ A(x)\lambda + B(x) \} Y(x)\cdot,$$

where $A(x)$ and $B(x)$ are matrices of continuous functions, by making the assumption that the equation has a formal solution

$$(20) \qquad Y(x)\cdot \equiv e^{\lambda\int_\alpha^x \gamma(x)\,dx} \left\{ P_0(x)\cdot + \frac{1}{\lambda} P_1(x)\cdot + \frac{1}{\lambda^2} P_2(x)\cdot + \dots \right\}$$

where α is any chosen constant.

If $n = 1$, equation (19) can be directly integrated and is seen to have an actual solution of the form (20). The passage to formulas (45) can, therefore, be made directly in this case, and hence we shall assume in the intervening work that $n \geqq 2$.

Setting $\int_\alpha^x \gamma(x)\,dx = \Gamma(x)$, and substituting the form (20) in the equation (19), we obtain the formal identity

$$\lambda\gamma(x)e^{\lambda\Gamma(x)} \left\{ P_0(x)\cdot + \frac{1}{\lambda} P_1(x)\cdot + \dots \right\} + e^{\lambda\Gamma(x)} \left\{ P_0'(x)\cdot + \dots \right\}$$

$$\equiv \{A(x)\lambda + B(x)\}\, e^{\lambda\Gamma(x)} \left\{ P_0(x)\cdot + \frac{1}{\lambda} P_1(x)\cdot + \dots \right\},$$

from which it is seen, upon equating the coefficients of λ, that

$$\gamma(x)\, P_0(x)\cdot \equiv A(x)\, P_0(x)\cdot.$$

This is satisfied by a vector $P_0(x)\cdot$ not all of whose elements vanish, when and only when

$$(21) \qquad |\, a_{ij}(x) - \delta_{ij}\,\gamma(x) \,| = 0,$$

i.e. when $\gamma(x)$ satisfies equation (21). For any given x, say $x = x_0$, however, the left-hand side of (21) is a polynomial of degree n in $\gamma(x_0)$. Hence the equation is satisfied by n roots $\gamma_1(x_0), \gamma_2(x_0), \ldots \gamma_n(x_0)$. We shall assume that for the case in hand these roots can be grouped into the n functions $\gamma_1(x), \gamma_2(x), \ldots \gamma_n(x)$, which satisfy the three conditions

$$(22) \quad \begin{cases} \text{(i)} & \gamma_i(x) \text{ continuous}, \quad i = 1, 2, \ldots n, \quad a \leqq x \leqq b, \\ \text{(ii)} & \gamma_j(x) \neq \gamma_i(x) \text{ for } j \neq i, \\ \text{(iii)} & \gamma_i(x) \neq 0, \quad i = 1, 2, \ldots n. \end{cases}$$

Clearly the last condition can be fulfilled only if $|A| \neq 0$; we shall assume this to be the case.

Consider now a change of the dependent variable in equation (19). Setting $Y(x)\cdot = \Phi(x)\,\overline{Y}(x)\cdot$, where $\Phi(x)$ is a matrix whose elements are continuous as well as their first derivatives, $a \leqq x \leqq b$, and $|\Phi| \neq 0$, the equation becomes $\Phi'\overline{Y}\cdot + \Phi\overline{Y}\cdot' = \{A\lambda + B\}\,\Phi\overline{Y}\cdot$, that is $\overline{Y}'\cdot = \{\Phi^{-1}A\Phi\lambda + \Phi^{-1}B\Phi - \Phi^{-1}\Phi'\}\,\overline{Y}\cdot$ It is evident here that if the coefficient of λ is not originally I then no such change of variable has the effect of reducing it to I. In the subsequent work it is desirable to transform $A(x)$ into the matrix $R(x)$ given by $R(x) = (\delta_{ij}\gamma_j(x))$. By such a transformation the formal solutions (20) are carried into others with the same functions $\gamma_i(x)$ in the exponents.

Since the $\Phi(x)$ in question must satisfy $\Phi^{-1}A\Phi \equiv R$,[7] it must fulfill the conditions

$$\text{(i)} \qquad A\Phi \equiv \Phi R,$$

$$\text{(ii)} \qquad |\Phi(x)| \neq 0.$$

To satisfy condition (i) the elements of Φ must be solutions of the algebraic system

$$\sum_{k=1}^{n} a_{ik}\varphi_{kj} = \varphi_{ij}\gamma_j, \quad i, j, = 1, 2, \ldots n,$$

while the possibility of fulfilling this condition by means of a matrix $\Phi(x)$, none of whose columns contain only vanishing elements, depends upon the existence of a solution for each of the n linear systems

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[7] For fuller discussion see, for instance, Bôcher, loc. cit. Chap. XXI.

$$\sum_{k=1}^{n} \{a_{ik} - \delta_{ik}\gamma_{j_0}\}\,\varphi_{kj_0} = 0, \tag{23}$$

obtained by giving to j_0 successively the values $1, 2, \ldots n$. This means that each of the determinants $|a_{ik} - \delta_{ik}\gamma_{j_0}|$, $j_0 = 1, 2, \ldots n$ must vanish, and inasmuch as the functions $\gamma_j(x)$ were chosen as the roots of equation (22), which is precisely the condition that the determinants in question do vanish, the existence of a matrix $\Phi(x)$ of the type desired is established.

Condition (ii) requires further, however, that this matrix Φ be one for which $|\Phi(x)| \neq 0$.

Suppose $|\Phi(x)| = 0$. Then the system of linear equations $\sum_{k=1}^{n} \varphi_{ik}\, v_k = 0$ admits of solution by means of a set of quantities $v_1, v_2, \ldots v_n$, at least one of which differs from zero,[8] i.e. there exists a vector $V\cdot \not\equiv 0$, such that $\Phi V\cdot \equiv 0$. But then $A\Phi V\cdot \equiv 0$, and in view of the relation (i) it follows further that $\Phi R V\cdot \equiv 0$. By continued repetition of this reasoning it can be shown likewise that $\Phi R^k V\cdot \equiv 0$ for all values of k. Consequently $\Phi\{c_0 I + c_1 R + c_2 R^2 + \ldots + c_{n-1} R^{n-1}\} V\cdot \equiv 0$. But on the hypothesis that $\gamma_j \neq \gamma_i$ when $j \neq i$ it is well known that the determinant

$$\begin{vmatrix} 1 & \gamma_1 & \gamma_1^2 & \cdots & \gamma_1^{n-1} \\ 1 & \gamma_2 & \gamma_2^2 & \cdots & \gamma_2^{n-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \gamma_n & \gamma_n^2 & \cdots & \gamma_n^{n-1} \end{vmatrix} \neq 0.$$

Hence the system of equations

$$\begin{gathered} c_0 + c_1\gamma_1 + c_2\gamma_1^2 + \ldots + c_{n-1}\gamma_1^{n-1} = k_1 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ c_0 + c_1\gamma_n + c_2\gamma_n^2 + \ldots + c_{n-1}\gamma_1^{n-1} = k_n, \end{gathered}$$

admits of a solution in the c's not all zero for any choice of the quantities k_i not all zero. Some choice of the set $c_0, c_1 \ldots c_{n-1}$, can therefore always be made to satisfy the relation

$$\{c_0 I + c_1 R + \ldots + c_{n-1} R^{n-1}\} \equiv (\delta_{ij}\, k_j).$$

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[8] Cf. Bôcher, loc. cit., pp. 47.

Then
$$\Phi(\delta_{ij}\, k_j)\, V\cdot \equiv 0,$$
i.e. $\sum_{l=1}^{n} \varphi_{il}\, k_l\, v_l \equiv 0$, for all choices of the set $k_1, k_2, \ldots k_n$. But by construction some v, say v_{j_0} is not zero. If then the k's are chosen so that $k_i = 0,\ i \neq j_0,\ k_{j_0} \neq 0$, it follows that $\varphi_{ij_0} = 0$ for all i. Inasmuch as no column of Φ contains only vanishing elements this involves a contradiction. Hence the hypothesis $|\Phi(x)| = 0$ is not tenable, and the Φ in question fulfills condition (ii), i.e. $|\Phi(x)| \neq 0$.

By a change of variable, then, equation (19) can be given the form $\bar{Y}'\cdot = \{R\lambda + \bar{B}\}\, \bar{Y}\cdot$, where the matrix $\bar{B}$, being given by the relation $\bar{B} = \Phi^{-1} B\Phi - \Phi^{-1}\Phi'$, is a matrix of continuous functions. Supposing this to have been done we may drop the dashes and consider the equation in the form

$$Y'(x)\cdot = \{R(x)\lambda + B(x)\}\; Y(x)\cdot. \tag{24}$$

Since we have found that the $Y(x)\cdot$ of (20) may be any one of the n vectors $Y_i(x)\cdot$ obtained by replacing $\Gamma(x)$ by $\Gamma_i(x)$ the further discussion might well be carried through for each of these vectors individually. However, if the matrix $E(x)$ is defined by the relation

$$E(x) \equiv (\delta_{ij}\, e^{\lambda\Gamma_j(x)}),$$

it is readily seen that the matrices $P_i(x)$ can be chosen so that the j^{th} column of the matrix

$$Y(x) \equiv \left\{ P_0(x) + \frac{1}{\lambda}\, P_1(x) + \ldots \right\} E(x) \tag{25}$$

is precisely the general column of the vector $Y_j(x)\cdot$ Hence all cases are simultaneously treated by the consideration of those formal solutions of the matrix equation

$$Y'(x) = \{R(x)\lambda + B(x)\}\; Y(x) \tag{26}$$

which have the form (25).

Inasmuch as $E'(x) = (\lambda\gamma_j(x)\, \delta_{ij}\, e^{\lambda\Gamma_j(x)}) = \lambda\, R(x)\, E(x)$, the formal substitution of (25) in (26) gives the identity

$$(27)\quad \left\{P_0+\frac{1}{\lambda}P_1+\frac{1}{\lambda^2}P_2+\dots\right\}\lambda RE+\left\{P_0'+\frac{1}{\lambda}P_1'+\dots\right\}E\equiv$$
$$\left\{\lambda R+B\right\}\left\{P_0+\frac{1}{\lambda}P_1+\dots\right\}E.$$

Equating the coefficients of λ we obtain the relation

$$P_0R = RP_0,$$

i.e.

$$p_{ij}^{(0)}\gamma_j = \gamma_i p_{ij}^{(0)},$$

whence it is seen that $p_{ij}^{(0)} = 0$, when $j \neq i$. Equating the coefficients of λ^0 we obtain in similar manner the equation

$$P_1R + P_0' = RP_1 + BP_0,$$

i.e.

$$(28)\qquad p_{ij}^{(1)}\gamma_j + p_{ij}^{(0)\prime} = \gamma_i p_{ij}^{(1)} + \sum_{k=1}^{n} b_{ik}\,p_{kj}^{(0)},$$

whence it is seen, upon setting $i = j$ that

$$p_{jj}^{(0)\prime} = \sum_{k=1}^{n} b_{jk}\,p_{kj}^{(0)} = b_{jj}\,p_{jj}^{(0)},$$

namely, that $p_{jj}^{(0)} = f_j e^{\int_a^x b_{jj}dx}$.

Having obtained in this manner all the elements of the matrix P_0 we find further from the equation (28) for $i \neq j$ that

$$(29)\qquad p_{ij}^{(1)} = \frac{b_{ij}\,p_{jj}^{(0)}}{\gamma_j-\gamma_i}.$$

Moreover, equating the coefficients of $\frac{1}{\lambda}$ we see that

$$P_2R + P_1' = RP_2 + BP_1,$$

i.e.

$$p_{ij}^{(2)}\{\gamma_j-\gamma_i\} + p_{ij}^{(1)\prime} = \sum_{k=1}^{n} b_{ik}\,p_{kj}^{(1)},$$

whence it follows upon setting $i = j$ that

$$(30)\qquad p_{jj}^{(1)\prime} = \sum_{k=1}^{n} b_{jk}\,p_{kj}^{(1)}.$$

Since the other quantities under the sign of summation in this formula have all been determined, $p_{jj}^{(1)}$ may be found by means of a quadrature. When this has been done the matrix P_1 has been completely determined.

It should be observed that we have thus far required only that the matrices $R(x)$ and $B(x)$ be continuous. To determine the elements of P_2 however, we have first, for $i \neq j$, the equation

$$p_{ij}^{(2)} = \frac{-p_{ij}^{(1)\prime} + \sum_{k=1}^{n} b_{ik}\, p_{kj}^{(1)}}{\gamma_j - \gamma_i},$$

and since this formula implies the existence of $p_{ij}^{(1)\prime}$, whereas it is seen from (29) that $p_{ij}^{(1)}$ has a derivative only if this is true of the matrices $R(x)$ and $B(x)$, we cannot proceed to the determination of P_2 if $R(x)$ and $B(x)$ are merely continuous.

Let us suppose then that $R(x)$ and $B(x)$ both possess continuous derivatives up to and including those of order $k \geqq 1$, but that perhaps one of these matrices possesses no such derivative of order $(k+1)$. If, in particular $R(x)$ and $B(x)$ possess infinitely many derivatives we may take $k = \infty$. The derivatives of P_1 up to and including that of order k are now seen to exist from formulas (29) and (30).

Equating formally the coefficients of $\frac{1}{\lambda^{\mu-1}}$ in the identity (27) we have

$$P_\mu R + P'_{\mu-1} = RP_\mu + BP_{\mu-1},$$

i.e.

$$p_{ij}^{(\mu)} \{\gamma_j - \gamma_i\} = -p_{ij}^{(\mu-1)\prime} + \sum_{k=1}^{n} b_{ik}\, p_{kj}^{(\mu-1)},$$

whence $$p_{ij}^{(\mu)} = \frac{-p_{ij}^{(\mu-1)\prime} + \sum_{k=1}^{n} b_{ik}\, p_{kj}^{(\mu-1)}}{\gamma_j - \gamma_i} \quad \text{for } i \neq j,$$

and again, equating the coefficients of $\frac{1}{\lambda^{\mu}}$ we have

$$P_{\mu+1} R + P'_\mu = RP_{\mu+1} + BP_\mu$$

i.e.
$$p_{ij}^{(\mu+1)}\{\gamma_j-\gamma_i\}+p_{ij}^{(\mu)\prime}=\sum_{k=1}^{n} b_{ik}\,p_{kj}^{(\mu)},$$

whence it is seen, upon setting $i=j$ that

$$p_{jj}^{(\mu)\prime}=\sum_{k=1}^{n} b_{jk}p_{kj}^{(\mu)},$$

namely that $p_{jj}^{(\mu)}$ can be determined by means of a quadrature if the quantities on the right are known.

The determination of the elements of $p_{jj}^{(\mu)}$ by means of these formulas depends, therefore, only upon a knowledge of the elements of $P_{\mu-1}$ and upon the existence of $P'_{\mu-1}$. Moreover, it is seen that in general P_μ possesses one less derivative than $P_{\mu-1}$. Inasmuch as P_1 has already been determined, and was seen to possess k derivatives it is clear that the matrices P_μ for $\mu=0, 1,\ldots, (k+1)$, may be successively determined, and that in general the matrix P_{k+1} is merely continuous.

If k is finite, we can, therefore, determine a differentiable matrix $S(x)$, given by

$$(31)\qquad S(x)\equiv\left\{P_0(x)+\frac{1}{\lambda}P_1(x)+\ldots+\frac{1}{\lambda^k}P_k(x)\right\}E(x)$$
$$\equiv\left(e^{\lambda\Gamma_j(x)}\left\{p_{ij}^{(0)}+\frac{p_{ij}^{(1)}}{\lambda}+\ldots+\frac{p_{ij}^{(k)}}{\lambda^k}\right\}\right),$$

which will satisfy the equation

$$S'(x)=\{\lambda R+B\}\,S(x)+\frac{1}{\lambda^k}\{P'_k(x)-B(x)\,P_k(x)\}\,E(x),$$

i.e. an equation of the form

$$(32)\qquad S'(x)=\left\{\lambda R(x)+B(x)+\frac{1}{\lambda^k}L(x,\lambda)\right\}S(x),$$

where $L(x, \lambda)$ is a matrix each element of which is rational in λ, with coefficients continuous in x, given by power series in $\left(\frac{1}{\lambda}\right)$.

If on the other hand $k=\infty$, as many terms of the infinite series

$$p_{ij}=p_{ij}^{(0)}+\frac{p_{ij}^{(1)}}{\lambda}+\frac{p_{ij}^{(2)}}{\lambda^2}+\ldots.$$

as are desired may be used. These series are not in general convergent. Nevertheless the formal matrix

$$S(x) \equiv \left\{ P_0(x) + \frac{1}{\lambda} P_1(x) + \ldots\ldots \right\} E(x)$$
$$\equiv \left(e^{\lambda \Gamma_j(x)} \left\{ p_{ij}^{(0)} + \frac{p_{ij}^{(1)}}{\lambda} + \ldots \right\} \right)$$

which is found in this case, formally satisfies the equation (26).

With the convention that the term in $\frac{1}{\lambda^k}$ is to be omitted if $k = \infty$ formula (32) holds in all cases. $S(x)$ will be called a formal matrix solution of equation (26) regardless of whether k is, for the case in hand, finite or infinite.

It should be observed that each $p_{ij}^{(\mu)}$ is not wholly determined but contains a single arbitrary constant of integration, independent of x. This arbitrariness corresponds to the fact that any convergent power series c_j in $\left(\frac{1}{\lambda}\right)$ with constant coefficients may be multiplied into each column of $S(x)$ without thereby destroying its property of being a formal solution in the sense (32).

Using the notation $[\psi_0]_k$, or $[\psi_0]$, for an expression of the form

$$\psi_0 + \frac{\psi_1}{\lambda} + \ldots\ldots + \frac{\psi_k}{\lambda^k} + \frac{\psi(x,\lambda)}{\lambda^{k+1}}$$

where ψ is bounded for $|\lambda|$ large, we have

$$S(x) = \left(\left[\delta_{ij}\, c_j e^{\int_a^x b_{jj} dx} \right]_k \right) E(x).$$

It is clear that an alternative form is

$$S(x) = \left(c_j\, e^{\lambda \Gamma_j(x) + B_j(x)} \, [\delta_{ij}]_k \right), \tag{33}$$

where $B_j(x) = \int_a^x b_{jj}\, dx$.

By similar considerations the equation $Z' = -Z\{A\lambda + B\}$ may be transformed and a formal matrix solution $T(x)$ for the resulting equation

$$Z'(x) = -Z(x)\,\{R(x)\lambda + B(x)\} \tag{34}$$

be obtained. This $T(x)$ has the form

$$T(x) = \left(\bar{c}_i\, e^{-\lambda\Gamma_i(x) - B_i(x)}\,[\delta_{ij}]_k\right), \tag{35}$$

and satisfies an equation of the form

$$T'(x) = -T(x)\left\{\lambda R(x) + B(x) + \frac{1}{\lambda^k} M(x, \lambda)\right\}.$$

Moreover each row is a formal solution of the vector equation

$$\cdot Z'(x) = -\cdot Z(x)\,\{R(x)\,\lambda + B(x)\}. \tag{36}$$

Considering differentiation as merely a formal process defined by the usual rules it is readily seen that the differentiation of the formal matrices $S(x)$ and $T(x)$ is permissible. Hence we have

$$\begin{aligned} \frac{d}{dx} TS &= TS' + T'S \\ &= T\left\{\lambda R + B + \frac{1}{\lambda^k} L\right\} S - T\left\{\lambda R + B + \frac{1}{\lambda^k} M\right\} S \\ &= \frac{1}{\lambda^k}\, T\,\{L - M\} S\,. \end{aligned}$$

Since

$$\begin{aligned} TS &\equiv \left(\sum_{h=1}^{n} \bar{c}_i\, e^{-\lambda\Gamma_i(x) - B_i(x)}\,[\delta_{ih}]_k\, c_j\, e^{\lambda\Gamma_j(x) + B_j(x)}\,[\delta_{hj}]_k\right) \\ &\equiv \left(\bar{c}_i\, c_j\, e^{\lambda\{\Gamma_j(x) - \Gamma_i(x)\} + B_j(x) - B_i(x)}\left\{\delta_{ij} + \frac{\sigma_{ij}^{(1)}}{\lambda} + \ldots\ldots + \frac{\sigma^{(2k)}}{\lambda^{2k}}\right\}\right), \end{aligned}$$

we find upon differentiating, and removing the exponential factor,

$$\begin{aligned} \bar{c}_i\, c_j \Bigg\{ \{\lambda(\gamma_j - \gamma_i) + b_{jj} - b_{ii}\} \left\{\delta_{ij} + \frac{\sigma_{ij}^{(1)}}{\lambda} + \ldots + \frac{\sigma_{ij}^{(2k)}}{\lambda^{2k}}\right\} \\ + \left\{\frac{\sigma_{ij}^{(1)\prime}}{\lambda} + \ldots + \frac{\sigma_{ij}^{(2k)\prime}}{\lambda^{2k}}\right\} \Bigg\} \equiv \end{aligned}$$

$$\frac{1}{\lambda^k}\sum_{h,s=1}^{n}[\delta_{ih}]_k\,\{l_{hs}-m_{hs}\}\,[\delta_{sj}]_k.$$

Equating the coefficients of λ^0 we have

$$\bar{c}_i\,c_j\,\{\,(\gamma_j-\gamma_i)\,\sigma_{ij}^{(1)}+(b_{jj}-b_{ii})\,\delta_{ij}\}\equiv 0,$$

from which it follows that $\sigma_{ij}^{(1)}=0$ when $i\neq j$. Again, equating the coefficients of $\frac{1}{\lambda}$ we have

$$(\gamma_j-\gamma_i)\,\sigma_{ij}^{(2)}+(b_{jj}-b_{ii})\,\sigma_{ij}^{(1)}+\sigma_{ij}^{(1)\prime}\equiv 0,$$

from which it is seen, upon setting $i=j$ that $\sigma_{jj}^{(1)\prime}\equiv 0$, namely that $\sigma_{jj}^{(1)}\equiv$ constant. The relation shows on the other hand that $\sigma_{ij}^{(2)}=0$ when $i\neq j$. Equating to zero successively the coefficient of each individual power of $\frac{1}{\lambda}$ it is found in the same way that $\sigma_{ij}^{(\mu)}\equiv 0$ when $i\neq j$, $\sigma_{jj}^{(\mu)}\equiv$ constant for $\mu=1,2,\ldots,(k-1)$. It follows that

$$TS=\left(e^{\lambda\{\Gamma_j-\Gamma_i\}+B_j-B_i}\left\{\bar{c}_i\,c_j\,\delta_{ij}\left\{1+\frac{\sigma_{jj}^{(1)}}{\lambda}+\ldots+\frac{\sigma_{jj}^{(k-1)}}{\lambda^{k-1}}\right\}+\frac{\sigma_{ij}(x,\lambda)}{\lambda^k}\right\}\right)$$

$$=(\bar{c}_j\,c_j\,\delta_{ij}\,[1]_{k-1})+\frac{1}{\lambda^k}E^{-1}\,(\sigma_{ij})\,E\,,$$

where the coefficients of $[1]_{k-1}$ are constants.

Now for any choice of the set of series c_j it is clearly possible to choose a set $\bar{c}_j$ such that $\bar{c}_j\,c_j\,[1]_{k-1}=1$, $j=1,2,\ldots n$. The formal solutions $S(x)$ and $T(x)$ corresponding respectively to these values of c_j and $\bar{c}_j$ are closely related. They are called *associated formal* solutions and satisfy the relation $T(x)\,S(x)=I+\frac{1}{\lambda^k}E^{-1}\,(\sigma_{ij})\,E$.

Since in particular c_j may be chosen as $c_j=1$ it is seen that there exists a formal solution having the form

$$S(x)=(e^{\lambda\Gamma_j(x)+B_j(x)}\,[\delta_{ij}]). \tag{37}$$

In accordance with the definition above the associated solution is given by the set of $\bar{c}_j$'s which satisfy the relations $\bar{c}_j=1$. It is apparent, therefore, that the solution associated with (37) is of the form

$$T(x) = (e^{-\lambda \Gamma_i(x) - B_{ij}(x)} [\delta_{ij}]). \tag{38}$$

Section VI.

The relation of the formal solutions to the actual solutions.

It was observed in the preceding section that the formal matrix $S(x)$ either satisfies equation (26) only in an approximate sense (i.e. satisfies (32)) or, if $k = \infty$ satisfies it only formally, since the elements of $S(x)$ are in that case infinite series which are not in general convergent. $S(x)$ is, therefore, not a matrix solution of equation (26), and its significance requires further investigation.

Consider the actual matrix which is derived from $S(x)$ by retaining in the latter only the first $(m + 1)$ terms of its elements, where m is any positive integer not exceeding k. This matrix $\bar{S}(x)$ may be written, in accordance with formula (37),

$$\begin{aligned} \bar{S}(x) &= (e^{\lambda \Gamma_j(x) + B_j(x)} [\delta_{ij}]_m) \\ &= \bar{P}(x, \lambda)\, E(x), \end{aligned}$$

where
$$\bar{P}(x,\lambda) = (e^{B_j(x)} [\delta_{ij}]_m),$$

and is seen to be analytic in λ. Since $| e^{B_j(x)} \delta_{ij} | \neq 0$, it follows that $| \bar{S} | \neq 0$ for $| \lambda | > N$. If the formal solution $S(x)$ is written as $P(x, \lambda)\, E(x)$, it is apparent that we have the formulas

$$\bar{S}(x) = \left\{ I - \frac{1}{\lambda^{m+1}} [Q]\, P^{-1} \right\} S(x),$$

$$S(x) = \left\{ I + \frac{1}{\lambda^{m+1}} [Q]\, \bar{P}^{-1} \right\} \bar{S}(x),$$

from which we obtain upon differentiating and substituting from the equation (32), the relation

$$\overline{S}'(x) = \left\{ \left\{ I - \frac{1}{\lambda^{m+1}}[Q]\,P^{-1} \right\} \left\{ \lambda R + B + \frac{1}{\lambda^{k}} L \right\} - \frac{1}{\lambda^{m+1}}([Q]\,\overline{P}^{-1})' \right\} \left\{ I + \frac{1}{\lambda^{m+1}}\,[Q]\,\overline{P}^{-1} \right\} \overline{S}(x).$$

It is apparent from this that $\overline{S}(x)$ is a solution of a homogeneous differential matrix equation of the type

$$\overline{S}'(x) = \left\{ \lambda R + B + \frac{1}{\lambda^{m}}\,\Phi(x, \lambda) \right\} \overline{S}(x). \tag{39}$$

The elements of $\Phi(x, \lambda)$ are power series in $\left(\frac{1}{\lambda}\right)$ with coefficients which are continuous in x, and are seen to be convergent, since these elements are rational in λ.

But equation (26) can be written in the form

$$Y'(x) = \left\{ R(x)\lambda + B(x) + \frac{1}{\lambda^{m}}\Phi(x, \lambda) \right\} Y(x) - \frac{1}{\lambda^{m}}\Phi(x, \lambda)\, Y(x), \tag{40}$$

and considering this as a non-homogeneous equation we know, in virtue of the developments of page 62, that its solutions, i.e. the solutions of (26), are given by

$$Y(x) = \overline{S}(x)\, C + \int^{x} \overline{S}(x)\, \overline{T}(t) \left\{ -\frac{1}{\lambda^{m}}\,\Phi(t, \lambda)\; Y(t) \right\} dt,$$

i.e. by

$$Y(x) = \overline{S}(x)\, C - \frac{1}{\lambda^{m}} \int^{x} \overline{S}(x)\; \overline{T}(t)\, \Phi(t, \lambda)\; Y(t)\, dt, \tag{41}$$

where $\overline{T}(x) \equiv \overline{S}^{-1}(x)$ and where the lower limit of integration, which has been omitted, may be chosen at pleasure for each of as many parts of the integrand as desired.

Substituting for $Y(x)$ in equation (41) its equivalent as given by the form

$$Y(x) = U(x)\, \overline{S}(x), \tag{42}$$

we have further

$$(43)\quad U(x) = \bar{S}(x)\, C\bar{T}(x) - \frac{1}{\lambda^m}\int^x \bar{S}(x)\,\bar{T}(t)\,\Phi(t,\lambda)\,U(t)\bar{S}(t)\,\bar{T}(x)\,dt.$$

THEOREM: If the functions $\gamma_i(x)$, $i = 1, 2, \ldots n$, satisfy the relations $\arg\{\gamma_j(x) - \gamma_i(x)\} = h_{ij}$, $i, j, = 1, 2, \ldots n$, where each h_{ij} is a constant, then there corresponds to each sector bounded by two adjacent rays $R\{\lambda\{\gamma_j(x) - \gamma_i(x)\}\} = 0$[8] a choice of the lower limits of integration which is such that for λ within the sector and $|\lambda| > N$, and for any continuous matrix $U(t)$, each element of the matrix

$$\psi(x,\lambda) \equiv \int^x \bar{S}(x)\,\bar{T}(t)\,\Phi(t,\lambda)\,U(t)\,\bar{S}(t)\,\bar{T}(x)\,dt$$

is less numerically than KM, where M is the largest numerical maximum attained by any element of U.

Proof: Writing

$$\bar{S}(x) = (s_{ij}(x))\,E(x), \quad \bar{T}(x) = E^{-1}(x)\,(t_{ij}(x)),$$

we have

$$\psi(x,\lambda) = \Bigg(\sum_{h,l,p,q,r=1}^{n} \int^x s_{ih}(x) e^{\lambda\{\Gamma_h(x) - \Gamma_h(t)\}} t_{hl}(t)\,\varphi_{lp}(t,\lambda)\,u_{pq}(t)$$

$$s_{qr}(t)\,e^{\lambda\{\Gamma_r(t) - \Gamma_r(x)\}}\,t_{rj}(x)\,dt\Bigg)$$

$$= \sum_{h,r}^{n}\left(\int^x e^{\lambda\int_t^x \{\gamma_h(\xi) - \gamma_r(\xi)\}d\xi}\,\omega_{ij}^{(hr)}(t,x,\lambda)\,dt\right),$$

where, for $|\lambda| > N$, $|\omega_{ij}^{(hr)}| < \kappa^{(hr)}M$ for all i and j, $k^{(hr)}$ being a positive constant.

Consider a particular sector and any element

$$\int^x e^{\lambda\int_t^x \{\gamma_h(\xi) - \gamma_r(\xi)\}d\xi}\,\omega_{ij}^{(hr)}\,dt.$$

If $R\{\lambda\{\gamma_h(\xi) - \gamma_r(\xi)\}\} \leqq 0$ for λ within the sector and any ξ it is so for all ξ, and, provided that $t \leqq x$, the integral

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[8] The notation $R\{\varphi\}$ is used to indicate "the real part of φ."

$$\int_a^x e^{\lambda\int_t^x\{\gamma_h(\xi)-\gamma_r(\xi)\}d\xi}\,\omega_{ij}^{(hr)}\,dt$$

is less numerically than $k^{(hr)}M$. However, if $R\{\lambda\{\gamma_h(\xi)-\gamma_r(\xi)\}\}>0$ for λ within the sector and any ξ, it is so for all ξ, and, provided that $t\geqq x$, the integral

$$\int_b^x e^{\lambda\int_t^x\{\gamma_h(\xi)-\gamma_r(\xi)\}d\xi}\,\omega_{ij}^{(hr)}\,dt$$

is similarly bounded. Consequently if $\kappa_{hr}(\lambda)$ is defined by the relations

$$\kappa_{hr}=a \text{ if } R\{\lambda\{\gamma_h(\xi)-\gamma_r(\xi)\}\}\leqq 0,\ \kappa_{hr}=b \text{ otherwise,}$$

the numerical value of each element of the matrix

$$\psi(x,\lambda)=\left(\sum_{h,r=1}^{n}\int_{\kappa_{hr}}^{x} e^{\lambda\int_t^x\{\gamma_h(\xi)-\gamma_r(\xi)\}d\xi}\,\omega_{ij}^{(hr)}\,dt\right)$$

is clearly less than KM, where $K=(b-a)\sum_{h,r=1}^{n}k^{(hr)}$, for λ within the sector and $|\lambda|>N$. Q. E. D.

Assuming the limits chosen in the manner above, a $\psi(x,\lambda)$ corresponding to each sector and to each U is uniquely determined. Moreover, we have

$$U(x)=\overline{S}(x)\,C\overline{T}(x)-\frac{1}{\lambda^m}\psi(x,\lambda). \tag{44}$$

Consider now the particular solution, $Y_0(x)$, of equation (26) which satisfies the relation $Y_0(a)=\overline{S}(a)$. Since $\overline{S}(x)$ as well as the coefficients of equation (26) are analytic in λ, $Y_0(x)$ is likewise analytic in λ. Moreover we know from page 56 that every solution of the equation is of the form $Y(x)=Y_0(x)\,D$, where the elements of D are constants with respect to x. Substituting this form of $Y(x)$ into equation (41) and fixing x and solving for C we obtain the relation

$$C=\overline{T}(x_0)\left\{Y_0(x_0)+\frac{1}{\lambda^m}\int^{x_0}\overline{S}(x_0)\,\overline{T}(t)\,\Phi(t,\lambda)\,Y_0(t)\,dt\right\}D,$$

i.e.
$$c_{ij} = \sum_{h=1}^{n} \rho_{ih}\, d_{hj},$$

where the quantities ρ_{ih} are analytic in λ.

Inasmuch as $Y(x)$ is a non-identically zero solution if $D \neq 0$ there will exist such a solution for which $C = 0$ provided the determinant $|\rho_{ih}|$ vanishes. If on the other hand this determinant does not vanish then there exists a solution $Y(x) \not\equiv 0$ corresponding to every $C \neq 0$.

Suppose the determinant $|\rho_{ih}| = 0$. Then there corresponds to the choice $C = 0$ a solution $\overline{Y}(x)$ for which the matrix $\overline{U}(x)$ defined by $\overline{Y}(x) = \overline{U}(x)\,\overline{S}(x)$ satisfies the relation

$$\overline{U}(x) = -\frac{1}{\lambda^m}\left(\sum_{h,r=1}^{n}\int_{\kappa_{h,r}}^{x} e^{\lambda\int_t^x \{\gamma_h(\xi)-\gamma_r(\xi)\}d\xi}\,\omega_{ij}^{(hr)}\,dt\right)$$

But we know that for some x, say x_0, and for some i, j, say i_0, j_0, $u_{i_0j_0}(x_0) = M$. Then since

$$\left|\sum_{h,r=1}^{n}\int_{\kappa_{h,r}}^{x_0} e^{\lambda\int_t^{x_0}\{\gamma_h(\xi)-\gamma_r(\xi)\}d\xi}\,\omega_{i_0j_0}^{(hr)}(x_0, t)\,dt\right| < KM,$$

we have for the i_0, j_0 element, $M < \dfrac{KM}{\lambda^m}$,

i.e.
$$M\left(1 - \frac{K}{\lambda^m}\right) < 0.$$

Inasmuch as M is positive and λ can be taken arbitrarily large this involves a contradiction, and proves the hypothesis untenable. Hence $|\rho_{ih}| \neq 0$. Accordingly there corresponds to every choice of $C \neq 0$, and hence to the particular choice $C = I$, a unique $D \neq 0$, and hence a solution $Y(x)$. By equation (44) the $U(x)$ for this $Y(x)$ must satisfy

$$U(x) = I - \frac{1}{\lambda^m}\,\psi(x, \lambda).$$

We infer from this the inequality,

$$M < 1 + \frac{KM}{\lambda^m},$$

whence it follows that for λ sufficiently large, M is less that 2 and $\psi(x, \lambda)$ is uniformly bounded, for λ in the sector in question and further $|\lambda| > N$. Accordingly we have

$$Y(x) = \left(\delta_{ij} + \frac{\psi_{ij}(x, \lambda)}{\lambda^m}\right)(s_{ij}(x))\,E(x),$$

i.e.

$$Y(x) = \left(s_{ij}(x) + \frac{\nu_{ij}(x, \lambda)}{\lambda^m}\right) E(x).$$

Since the matrices $Y_0(x)$ and D are analytic in λ the same is true also of $Y(x)$. It is seen, moreover, that $|Y| \neq 0$. Hence, when the functions $\gamma_i(x)$ satisfy the conditions of the theorem on page 83 there exists in every sector of the type described above a matrix solution $Y(x)$ whose elements are continuous in x and analytic in λ, and are the same as those of $\bar{S}(x)$ to terms in $\frac{1}{\lambda^{m-1}}$. But by construction the elements of $\bar{S}(x)$ are the same as those of $S(x)$ to terms in $\frac{1}{\lambda^m}$. Hence we have the

THEOREM: Given any formal solution $S(x)$ of equation (26) in which $R(x)$ and $B(x)$ both possess derivatives up to and including those of order k, then in each sector of the complex plane within which none of the quantities $R\{\lambda\{\gamma_k(x) - \gamma_r(x)\}\}$ change sign there exists an actual matrix solution $Y(x)$ which is continuous with its first derivative in x and analytic in λ, and is throughout the sector identical with $S(x)$ to terms in $\frac{1}{\lambda^k}$.

In virtue of this there exists in any sector of the type described a pair of actual associated matrix solutions of the form

$$(45)\qquad \begin{cases} Y(x) = \left(e^{\lambda\Gamma_j(x)+B_j(x)}\,[\delta_{ij}]_{k-1}\right) \\ Z(x) = \left(e^{-\lambda\Gamma_i(x)-B_i(x)}\,[\delta_{ij}]_{k-1}\right), \end{cases}$$

provided only that the matrices $R(x)$ and $B(x)$ are differentiable k terms. In all cases the functions $[\delta_{ij}]_{k-1}$ which occur in the expressions for $Y(x)$ and $Z(x)$ are, of course, respectively identical, to terms in $1/\lambda^{k-1}$ with the series $[\delta_{ij}]$ in the formulas (37) and (38).

In the course of the deduction of forms (45) it was assumed throughout that $n \geqq 2$. Direct integration of the equation, however, shows that in the case $n = 1$ there exist solutions which are of this form over the entire plane. Accordingly formulas (45) may be used in every case. The explicit forms

$$Y(x) = \left(e^{\lambda \Gamma_j(x) + B_j(x)} \left\{ \delta_{ij} + \frac{\varphi_{ij}(x, \lambda)}{\lambda} \right\} \right),$$

$$Z(x) = \left(e^{-\lambda \Gamma_i(x) - B_i(x)} \left\{ \delta_{ij} + \frac{\psi_{ij}(x, \lambda)}{\lambda} \right\} \right),$$

with φ_{ij}, ψ_{ij} bounded, hold if $R(x)$, $B(x)$ are differentiable.

SECTION VII.

The characteristic values of the system

$$Y'(x) \cdot = \{ R(x)\lambda + B(x) \} \, Y(x) \cdot$$
$$W_a Y(a) \cdot + W_b Y(b) \cdot = 0 .$$

It was shown on page 65 that a necessary and sufficient condition that the vector system

$$\text{(46)} \qquad \begin{aligned} &Y' \cdot = \{R\lambda + B\} \, Y \cdot \\ &W_a \, Y(a) \cdot + W_b \, Y(b) \cdot = 0 \qquad (| W_a | \neq 0, \ | W_b | \neq 0), \end{aligned}$$

has a solution is that

$$\text{(47)} \qquad | W_a \, Y(a) + W_b \, Y(b) | = 0,$$

$Y(x)$ being any matrix solution of equation (26). Let us make the specific form of this condition in any sector of the λ plane apparent by substituting a $Y(x)$ which is analytic in λ for $| \lambda | > N$, and which has within the sector the form

$$\text{(48)} \qquad Y(x) = (e^{\lambda \Gamma_j(x) + B_j(x)} \, [\delta_{ij}]),$$

as determined in the preceding sections. Choosing the α of formula (20) as $\alpha = a$ it follows that $Y(a) = ([\delta_{ij}])$.

Substituting from (48) in the determinant on the left of equation (47), (call it $D(\lambda)$) we have

$$D(\lambda) = \left| W_a([\delta_{ij}]) + W_b(e^{\lambda\Gamma_j(b)+B_j(b)}[\delta_{ij}]) \right|,$$

i.e.

$$(49) \qquad D(\lambda) = \left| [w_{rc}^{(a)}] + [w_{rc}^{(b)}]\, e^{\lambda\Gamma_c(b)+B_c(b)} \right|.$$

In this as in subsequent formulas when the expressions in question are determinants the letters r and c rather than i and j are used to indicate row and column respectively.

Due to the fact that those and only those values of λ which satisfy equation (47) are characteristic values, i.e. values which yield solutions of system (46), equation (47) is known as the *characteristic equation* of the system in question.

The introduction at this point of quantities $\overset{*}{\delta}_{ij}$ and $\overset{**}{\delta}_{ij}$ will do much toward simplifying the further discussion. These quantities are defined by the relations

$$\overset{*}{\delta}_{ij} \begin{cases} = \delta_{ij} \text{ when } R\{\lambda\Gamma_j(b)\} \leqq 0 \\ = 0 \quad \text{when } R\{\lambda\Gamma_j(b)\} > 0, \end{cases} \qquad \overset{**}{\delta}_{ij} \begin{cases} = 0 \quad \text{when } R\{\lambda\Gamma_j(b)\} \leqq 0 \\ = \delta_{ij} \text{ when } R\{\lambda\Gamma_j(b)\} > 0. \end{cases}$$

It is to be noted that $\overset{*}{\delta}_{jj}$ and $\overset{**}{\delta}_{jj}$ are functions of $\arg \lambda$ alone.

Suppose now that the rays[9] $R\{\lambda\Gamma_i(b)\} = 0$, $i = 1, 2, \ldots n$, are drawn in the plane of the parameter λ and let one of the sectors bounded by a pair of adjacent rays of this kind, $R\{\lambda\Gamma_k(b)\} = 0$, and $R\{\lambda\Gamma_l(b)\} = 0$, be denoted by σ_{kl}. Then if $\overset{*}{\delta}{}^{(kl)}_{ij}$ and $\overset{**}{\delta}{}^{(kl)}_{ij}$ are respectively the $\overset{*}{\delta}_{ij}$ and $\overset{**}{\delta}_{ij}$ for some (any) particular value of λ within σ_{kl}, we have the identities $\overset{*}{\delta}_{ij} \equiv \overset{*}{\delta}{}^{(kl)}_{ij}$, $\overset{**}{\delta}_{ij} \equiv \overset{**}{\delta}{}^{(kl)}_{ij}$, for all λ in the interior of the sector in question. It becomes apparent in virtue of the relations

$$(50) \qquad [w_{rc}^{(a)}] + [w_{rc}^{(b)}]\, e^{\lambda\Gamma_c(b)+B_c(b)} = \begin{cases} [w_{rc}^{(a)}] \text{ when } R\{\lambda\Gamma_c(b)\} < 0 \\ [w_{rc}^{(b)}]\, e^{\lambda\Gamma_c(b)+B_c(b)} \\ \qquad\qquad \text{when } R\{\lambda\Gamma_c(b)\} > 0, \end{cases}$$

and the fact that $R\{\lambda\Gamma_i(b)\} \neq 0$, $i = 1, 2, \ldots n$, that $D(\lambda)$ takes, for any λ in the interior of such a sector the form

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[9] A half-line issuing from the origin of the λ plane will be referred to as a *ray*.

$$(51)\qquad D(\lambda) = \left| [w_{rc}^{(a)}]\,\overset{*}{\delta}_{cc} + [w_{rc}^{(b)}]\,\overset{**}{\delta}_{cc}\, e^{\lambda\Gamma_c(b)+B_c(b)} \right|.$$

Factoring from the determinant the exponential factors, any one of which occurs in each element of an entire column, the further alternative form

$$(52)\qquad D(\lambda) = \prod_{k=1}^{n} e^{\{\lambda\Gamma_k(b)+B_k(b)\}\overset{**}{\delta}_{kk}} \left| [w_{rc}^{(a)}]\,\overset{*}{\delta}_{cc} + [w_{rc}^{(b)}]\,\overset{**}{\delta}_{cc} \right|$$

is obtained. It becomes necessary at this point to differentiate between certain types of conditions which inherently characterize any particular system of type (46).

The conditions of the system will be said to be *regular* (i) if $n = 1$, or if, when $n \geqq 2$, $\arg \Gamma_j(b) \neq \arg \{\pm \Gamma_i(b)\}$ when $j \neq i$, and (ii) the boundary conditions are such that each of the determinants

$$(53)\qquad W^{(kl)} = \left| w_{rc}^{(a)}\,\overset{*}{\delta}{}_{cc}^{(kl)} + w_{rc}^{(b)}\,\overset{**}{\delta}{}_{cc}^{(kl)} \right|$$

differs from zero.

The conditions of the system will be called *irregular* if either of these conditions is not satisfied. In the further discussion we shall consider first the case of regular conditions and then the particular type of irregular case which results from dropping the regularity condition (i) above.

Case I. Regular Conditions. In this case the rays $R\{\lambda\Gamma_i(b)\} = 0$, $i = 1, 2, \ldots n$, are distinct and divide the plane into $2n$ sectors of type σ_{kl}. Let us fix the attention upon any one of these rays, say the ray $R\{\lambda\Gamma_\nu(b)\} = 0$, and let $\sigma_{\mu\nu}$ and $\sigma_{\nu\tau}$ denote the abutting sectors, the former being that within which $R\{\lambda\Gamma_\nu(b)\} < 0$. For λ on the ray $R\{\lambda\Gamma_\nu(b)\} = 0$ the ν^{th} column of $D(\lambda)$ consists of terms whose form cannot be abbreviated by means of relations (50). Retaining therefore the original expressions for the elements of this column we have

$$D(\lambda) = \left| [w_{rc}^{(a)}]\,\overset{*}{\delta}{}_{cc}^{(\mu\nu)} + [w_{rc}^{(b)}]\,\{\overset{**}{\delta}{}_{cc}^{(\mu\nu)} + \delta_{c\nu}\}\, e^{\lambda\Gamma_c(b)+B_c(b)} \right|.$$

For λ either within $\sigma_{\mu\nu}$ or within $\sigma_{\nu\tau}$, on the other hand, the form of $D(\lambda)$ is given by formula (51), the difference in the value of $D(\lambda)$ in these sectors being accounted for by the difference in the values of $\overset{*}{\delta}_{cc}$ and $\overset{**}{\delta}_{cc}$, $c = 1, 2, \ldots, n$.

But it is readily seen that

$$(54)\qquad \overset{*}{\delta}{}_{cc}^{(\nu\tau)} = \overset{*}{\delta}{}_{cc}^{(\mu\nu)} - \delta_{c\nu}, \qquad \overset{**}{\delta}{}_{cc}^{(\nu\tau)} = \overset{**}{\delta}{}_{cc}^{(\mu\nu)} + \delta_{c\nu}.$$

Hence if S is any sector which includes the ray $R\{\lambda\Gamma_\nu(b)\} = 0$ and overlaps a part of each of the sectors $\sigma_{\mu\nu}$ and $\sigma_{\nu\tau}$ we have, for a λ in S, the following expressions

$$D(\lambda) = \left| [w_{rc}^{(a)}]\, \overset{*}{\delta}{}_{cc}^{\mu\nu} + [w_{rc}^{(b)}]\, \overset{**}{\delta}{}_{cc}^{(\mu\nu)}\, e^{\lambda\Gamma_c(b)+B_c(b)} \right|, \text{ for } \lambda \text{ within } \sigma_{\mu\nu},$$

$$D(\lambda) = \left| [w_{rc}^{(a)}]\, \overset{*}{\delta}{}_{cc}^{(\mu\nu)} + [w_{rc}^{(b)}]\, \{\overset{**}{\delta}{}_{cc}^{(\mu\nu)} + \delta_{c\nu}\} e^{\lambda\Gamma_c(b)+B_c(b)} \right|, \text{ for } \lambda \text{ on the ray } R\{\lambda\Gamma_\nu(b)\} = 0,$$

$$D(\lambda) = \left| [w_{rc}^{(a)}]\, \{\overset{*}{\delta}{}_{cc}^{(\mu\nu)} - \delta_{c\nu}\} + [w_{rc}^{(b)}]\, \{\overset{**}{\delta}{}_{cc}^{(\mu\nu)} + \delta_{c\nu}\}\, e^{\lambda\Gamma_c(b)+B_c(b)} \right|, \text{ for } \lambda \text{ within } \sigma_{\nu\tau},$$

It is readily seen from this that $D(\lambda)$ is given for any λ in S by the formula

$$(55)\qquad D(\lambda) = \left| [w_{rc}^{(a)}]\, \overset{*}{\delta}{}_{cc}^{(\mu\nu)} + [w_{rc}^{(b)}] \{\overset{**}{\delta}{}_{cc}^{(\mu\nu)} + \delta_{c\nu}\} e^{\lambda\Gamma_c(b)+B_c(b)} \right|.$$

Factoring from this determinant the product $\prod\limits_{k-1}^{n} e^{\{\lambda\Gamma_k(b)+B_k(b)\}\, \overset{**}{\delta}{}_{kk}^{(\mu\nu)}}$ we have

$$(56)\qquad D(\lambda) = \prod_{k-1}^{n} e^{\lambda\{\Gamma_k(b)+B_k(b)\}\, \overset{**}{\delta}{}_{kk}^{(\mu\nu)}}\, \overline{D}(\lambda),$$

where

$$\overline{D}(\lambda) = \left| [w_{rc}^{(a)}]\, \overset{*}{\delta}{}_{cc}^{(\mu\nu)} + [w_{rc}^{(b)}] \{\overset{**}{\delta}{}_{cc}^{(\mu\nu)} + \delta_{c\nu}\}\, e^{\lambda\Gamma_o(b)+B_\nu(b)} \right|.$$

Now $\overline{D}(\lambda)$ is seen to have elements consisting of a single term in every column but the ν^{th}, the elements of that column being binomial. Consequently the expression of the determinant as the sum of two others is possible, i.e.

$$\overline{D}(\lambda) = \left| [w_{rc}^{(a)}]\, \overset{*}{\delta}{}_{cc}^{(\mu\nu)} + [w_{rc}^{(b)}]\, \overset{**}{\delta}{}_{cc}^{(\mu\nu)} \right| + \left| [w_{rc}^{(a)}] \{\overset{*}{\delta}{}_{cc}^{(\mu\nu)} - \delta_{c\nu}\} + [w_{rc}^{(b)}] \{\overset{**}{\delta}{}_{cc}^{(\mu\nu)} + \delta_{c\nu}\} \right| e^{\lambda\Gamma_\nu(b)+B_\nu(b)}$$

or, in view of relations (53) and (54),

$$(57)\qquad \overline{D}(\lambda) = [W^{\mu\nu}] + [W^{\nu\tau}]\, e^{\Gamma_\nu\lambda(b)+B_\nu(b)}.$$

Since the roots of $D(\lambda) = 0$ and those of $\overline{D}(\lambda) = 0$ are the same we have, therefore, as the characteristic equation

$$[W^{\cdot(\mu\nu)}] + [W^{\cdot(\nu\tau)}]\, e^{\lambda\Gamma_\nu(b)+B_\nu(b)} = 0. \tag{58}$$

This yields, since $W^{\cdot(\mu\nu)} \neq 0$, the equation

$$e^{\lambda\Gamma_\nu(b)+B_\nu(b)} = \left[-\frac{W^{\cdot(\mu\nu)}}{W^{\cdot(\nu\tau)}}\right] = \frac{-W^{\cdot(\mu\nu)}}{W^{\cdot(\nu\tau)}} + \epsilon,$$

where ϵ is here introduced as a generic symbol for functions which approach the limit zero uniformly as $|\lambda|$ increases beyond limit.

Solving for λ, and observing that $\log(K + \epsilon) = \log K + \epsilon$, we see, therefore, that every characteristic value which lies in sector S is of the form

$$\lambda_p = \frac{1}{\Gamma_\nu(b)}\left\{-B_\nu(b) + \log\frac{-W^{\cdot(\mu\nu)}}{W^{\cdot(\nu\tau)}} + \epsilon(\lambda_p) + 2p\pi i\right\}, \tag{59}$$

where p is a positive integer. Moreover, since $Y(x)$ is analytic in λ throughout the sector S, $|\lambda| > N$, the same is readily seen to be true of ϵ also.

Consider now a small circle of fixed radius r drawn about the point

$$\frac{1}{\Gamma_\nu(b)}\left\{-B_\nu(b) + \log\frac{-W^{\cdot(\mu\nu)}}{W^{\cdot(\nu\tau)}} + 2p\pi i\right\},$$

for any given p. Then for a proper choice of origin in the λ plane (see page 98) this circle lies entirely within the sector $\sigma_{\mu\nu}$ or $\sigma_{\nu\tau}$, and $\left|\frac{\epsilon}{\Gamma_\nu(b)}\right| < r$ for $|\lambda| > N$. Also if p is sufficiently large the point

$$\frac{1}{\Gamma_\nu(b)}\left\{-B_\nu(b) + \log\frac{-W^{\cdot(\mu\nu)}}{W^{\cdot(\nu\tau)}} + \epsilon + 2p\pi i\right\}$$

lies within the circle for all λ on the circumference. Consequently

$$\arg\left\{\lambda - \frac{1}{\Gamma_\nu(b)}\left\{-B_\nu(b) + \log\frac{-W^{\cdot(\mu\nu)}}{W^{\cdot(\nu\tau)}} + \epsilon + 2p\pi i\right\}\right\}$$

increases by 2π as λ describes this circumference, and just one root of equation (59) is accordingly seen to lie within the circle. Since it is readily verified that every such root is a characteristic value it is seen that (59) determines such a value for every p which is sufficiently large, i.e. that for large values of λ the characteristic values lie approximately along a line parallel to the ray $R\{\lambda\Gamma_\nu(b)\} = 0$, the distance between two adjacent ones approaching as a limit the finite length $2\pi/|\Gamma_\nu(b)|$, as $|\lambda|$ increases indefinitely. From the derivation of this result it is seen moreover, that a similar sequence of characteristic values lies near each ray $R\{\lambda\Gamma_i(b)\} = 0$, and that no further distribution of characteristic values exists.

Case II. A Type of Irregular Conditions.[10] Let us suppose now that the regularity condition (ii) is fulfilled, that $n \geqq 2$, but that

$$\arg \Gamma_i(b) = \arg \{\pm\Gamma_j(b)\} \tag{60}$$

for the pair of values $i = \nu_1, j = \nu_2$. We have then a case of irregular conditions, and while we shall restrict the discussion to the case when the relation (60) holds for only a single set of values i, j, the reasoning to be employed is typical and may be applied with equal success to the cases in which a greater number of the points $\Gamma_i(b)$ are collinear with the origin of the complex plane. It is only for the sake of brevity that the simplest, rather than the most general case which results from dropping the regularity condition (i) is treated.

A review of the discussion applied to the case of regular conditions readily shows that the methods employed there apply equally well to the case in hand and yield the same results in any sector of the λ plane which does not contain the line $R\{\lambda\Gamma_{\nu_1}(b)\} = 0$. It is, therefore, necessary to consider here only the distribution of characteristic values in a sector S containing this line. Along the line in question neither the expressions for the elements of the ν_1^{th} nor of the ν_2^{th} column of $D(\lambda)$ can be contracted by means of the relations (50). Accordingly it is found that $D(\lambda)$ takes the form

$$D(\lambda) = \left| [w_{rc}^{(a)}]\, \overset{*}{\delta}{}_{cc}^{(\nu)} + [w_{rc}^{(b)}]\, \{\overset{**}{\delta}{}_{cc}^{(\nu)} + \delta_{c\nu_1} + \delta_{c\nu_2}\}\, e^{\lambda\Gamma_c(b) + B_c(b)} \right| \tag{61}$$

throughout the entire sector S, $\overset{*}{\delta}{}_{cc}^{(\nu)}$ and $\overset{**}{\delta}{}_{cc}^{(\nu)}$ representing the quantities δ_{cc}^{*} and δ_{cc}^{**} for λ on the ray $R\{\lambda\Gamma_{\nu_1}(b)\} = 0$ which lies in S. Factoring from (61) the product $\prod\limits_{k=1}^{n} e^{\{\lambda\Gamma_k(b)+B_k(b)\}\,\overset{**}{\delta}{}^{(\nu)}_{kk}}$ we have

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[10] For the discussion of a differential system representing a different type of irregular conditions see Hopkins, loc. cit.

$$(62) \qquad D(\lambda) = \prod_{k=1}^{n} e^{\{\lambda\Gamma_k(b)+B_k(b)\}\overset{**}{\delta}{}^{(\nu)}_{kk}}\,\overline{\overline{D}}(\lambda),$$

where

$$\overline{\overline{D}}(\lambda) = \Big| [w^{(a)}_{rc}]\overset{*}{\delta}{}^{(\nu)}_{cc} + [w^{(b)}_{rc}]\Big\{\overset{**}{\delta}{}^{(\nu)}_{cc} + \delta_{c\nu_1}\, e^{\lambda\Gamma_{\nu_1}(b)+B_{\nu_1}(b)}$$
$$+ \delta_{c\nu_2}\, e^{\lambda\Gamma_{\nu_2}(b)+B_{\nu_2}(b)}\Big\}\Big|.$$

The determinant $\overline{\overline{D}}(\lambda)$ is seen, therefore, to have monomial elements in each column except the ν_1^{th} and the ν_2^{th}, the elements in these columns being binomials. It is clear, therefore, that $\overline{\overline{D}}(\lambda)$ can be expressed as the sum of four determinants each containing only monomial elements, i.e.

$$(63) \quad \overline{\overline{D}}(\lambda) = \Big| [w^{(a)}_{rc}]\,\overset{*}{\delta}{}^{(\nu)}_{cc} + [w^{(b)}_{rc}]\,\overset{**}{\delta}{}^{(\nu)}_{cc} \Big| + \Big| [w^{(a)}_{rc}]\{\overset{*}{\delta}{}^{(\nu)}_{cc} - \delta_{c\nu_1}\}$$
$$+ [w^{(b)}_{rc}]\{\overset{**}{\delta}{}^{(\nu)}_{cc} + \delta_{c\nu_1}\}\Big|\, e^{\lambda\Gamma_{\nu_1}(b)+B_{\nu_1}(b)} +$$
$$\Big| [w^{(a)}_{rc}]\{\overset{*}{\delta}{}^{(\nu)}_{cc} - \delta_{c\nu_2}\} + [w^{(b)}_{rc}]\{\overset{**}{\delta}{}^{(\nu)}_{cc} + \delta_{c\nu_2}\}\Big| e^{\lambda\Gamma_{\nu_2}(b)+B_{\nu_2}(b)} +$$
$$\Big| [w^{(a)}_{rc}]\{\overset{*}{\delta}{}^{(\nu)}_{cc} - \delta_{c\nu_1} - \delta_{c\nu_2}\}$$
$$+ [w^{(b)}_{rc}]\{\overset{**}{\delta}{}^{(\nu)}_{cc} + \delta_{c\nu_1} + \delta_{c\nu_2}\}\Big|\, e^{\lambda\{\Gamma_{\nu_1}(b)+\Gamma_{\nu_2}(b)\}+B_{\nu_1}(b)+B_{\nu_2}(b)}.$$

Let us again denote by $\sigma_{\mu\nu}$ and $\sigma_{\nu\tau}$ the sectors abutting on the ray $R\{\lambda\Gamma_{\nu_1}(b)\} = 0$ in which $R\{\lambda\Gamma_{\nu_1}(b)\} < 0$ and > 0 respectively. It is necessary to consider the case in which $\arg\Gamma_{\nu_1}(b) = \arg\Gamma_{\nu_2}(b)$ and that in which $\arg\Gamma_{\nu_1}(b) = \arg\{-\Gamma_{\nu_2}(b)\}$.

Sub-case A. $\qquad \arg\Gamma_{\nu_1}(b) = \arg\Gamma_{\nu_2}(b)$.

In this case the quantities $R\{\lambda\Gamma_{\nu_1}(b)\}$ and $R\{\lambda\Gamma_{\nu_2}(b)\}$ are of the same sign throughout the sector S, and the relations

$$(64) \qquad \begin{cases} \overset{*}{\delta}{}^{(\nu\tau)}_{cc} + \delta_{c\nu_1} + \delta_{c\nu_2} = \overset{*}{\delta}{}^{(\nu)}_{cc} = \overset{*}{\delta}{}^{(\mu\nu)}_{cc} \\ \overset{**}{\delta}{}^{(\nu\tau)}_{cc} - \delta_{c\nu_1} - \delta_{c\nu_2} = \overset{**}{\delta}{}^{(\nu)}_{cc} = \overset{**}{\delta}{}^{(\mu\nu)}_{cc} \end{cases}$$

are readily verified. Defining the determinants W_1 and W_2 by the formulas

$$\begin{aligned} W_1 &= \left| w_{rc}^{(a)} \{\overset{*}{\delta}{}_{cc}^{(\mu\nu)} - \delta_{c\nu_1}\} + w_{rc}^{(b)} \{\overset{**}{\delta}{}_{cc}^{(\mu\nu)} + \delta_{c\nu_1}\} \right| \\ W_2 &= \left| w_{rc}^{(a)} \{\overset{*}{\delta}{}_{cc}^{(\nu\tau)} + \delta_{c\nu_1}\} + w_{rc}^{(b)} \{\overset{**}{\delta}{}_{cc}^{(\nu\tau)} - \delta_{c\nu_1}\} \right| \end{aligned} \tag{65}$$

we have $\bar{\bar{D}}(\lambda) = \bar{D}(\lambda)$ where, in view of (64), $\bar{D}(\lambda)$ satisfies the equation (56) and is given by

$$\begin{aligned} \bar{D}(\lambda) = [W'^{(\mu\nu)}] + [W'_1]\, e^{\lambda\Gamma_{\nu_1}(b)+B_{\nu_1}(b)} + [W'_2]\, e^{\lambda\Gamma_{\nu_2}(b)+B_{\nu_2}(b)} \\ + [W'^{(\nu\tau)}]\, e^{\lambda\{\Gamma_{\nu_1}(b)+\Gamma_{\nu_2}(b)\}+B_{\nu_1}(b)+B_{\nu_2}(b)} \end{aligned} \tag{66}$$

Sub-case B. $\qquad \arg \Gamma_{\nu_1}(b) = \arg \{-\Gamma_{\nu_2}(b)\}$.

In this case $R\{\lambda\Gamma_{\nu_2}(b)\}$ has throughout S the sign opposite to that of $R\{\lambda\Gamma_{\nu_1}(b)\}$, and it is found that

$$\begin{aligned} \overset{*}{\delta}{}_{cc}^{(\nu\tau)} + \delta_{c\nu_1} &= \overset{*}{\delta}{}_{cc}^{(\nu)} = \overset{*}{\delta}{}_{cc}^{(\mu\nu)} + \delta_{c\nu_2} \\ \overset{**}{\delta}{}_{cc}^{(\nu\tau)} - \delta_{c\nu_1} &= \overset{**}{\delta}{}_{cc}^{(\nu)} = \overset{**}{\delta}{}_{cc}^{(\mu\nu)} - \delta_{c\nu_2}\,. \end{aligned} \tag{67}$$

Accordingly we have

$$\begin{aligned} \bar{\bar{D}}(\lambda) = [W'_2] + [W'^{(\nu\tau)}]e^{\lambda\Gamma_{\nu_1}(b)+B_{\nu_1}(b)} + [W'^{(\mu\nu)}]e^{\lambda\Gamma_{\nu_2}(b)+B_{\nu_2}(b)} \\ + [W'_1]e^{\lambda\{\Gamma_{\nu_1}(b)+\Gamma_{\nu_2}(b)\}+B_{\nu_1}(b)+B_{\nu_2}(b)}. \end{aligned}$$

If now we define $\bar{D}(\lambda)$ by the relation

$$\bar{\bar{D}}(\lambda)\, e^{-\lambda\Gamma_{\nu_2}(b)-B_{\nu_2}(b)} = \bar{D}(\lambda),$$

it is readily verified on the basis of formulas (67) that $\bar{D}(\lambda)$ again satisfies equation (56) while it is given in this case by

$$\begin{aligned} \bar{D}(\lambda) = [W'^{(\mu\nu)}] + [W'_1]\, e^{\lambda\Gamma_{\nu_1}(b)+B_{\nu_1}(b)} + [W'_2]e^{-\lambda\Gamma_{\nu_2}(b)-B_{\nu_2}(b)} \\ + [W'^{(\nu\tau)}]e^{\lambda\{\Gamma_{\nu_1}(b)-\Gamma_{\nu_2}(b)\}+B_{\nu_1}(b)-B_{\nu_2}(b)}. \end{aligned} \tag{68}$$

Let us suppose now that the notation has been so chosen that $|\Gamma_{\nu_2}(b)| \geqq |\Gamma_{\nu_1}(b)|$, and set $\left|\dfrac{\Gamma_{\nu_2}(b)}{\Gamma_{\nu_1}(b)}\right| = r$. Then upon dividing equations (66) and (68) through by $[W'^{(\mu\nu)}]$, (recall that $W'^{(\mu\nu)} \neq 0$ by hypothesis) it is seen that the characteristic equation, i.e. $\overline{D}(\lambda) = 0$, is in each case of the form

$$(69) \qquad 1 + [c_1]e^{\lambda\Gamma_{\nu_1}(b)} + [c_2]\, e^{r\lambda\Gamma_{\nu_1}(b)} + [c_3]\, e^{\{1+r\}\lambda\Gamma_{\nu_1}(b)} = 0,$$

where c_i, $i = 1, 2, 3$, are complex constants, $c_3 \neq 0$ by hypothesis, and r is a real constant $r \geqq 1$.

Wilder[11] has shown that the roots of this equation are asymptotically represented by those of the equation

$$(70) \qquad 1 + c_1\, e^{\lambda\Gamma_{\nu_1}(b)} + c_2\, e^{r\lambda\Gamma_{\nu_1}(b)} + c_3\, e^{\{1+r\}\lambda\Gamma_{\nu_1}(b)} = 0,$$

and has discussed the distribution of the roots of this equation. We shall proceed to this discussion, observing, however, first that when $|\,\Gamma_{\nu_2}(b)\,|$ and $|\,\Gamma_{\nu_1}(b)\,|$ are commensurable a far simpler treatment is possible. In that case $r = \frac{p}{q}$ where p and q are integers and the equation (70) is an algebraic equation of degree $(p + q)$ in $e^{\frac{\lambda\Gamma_{\nu_1}(b)}{q}}$. Accordingly it has $(p + q)$ roots, i.e.

$$e^{\frac{\lambda\Gamma_{\nu_1}(b)}{q}} = \alpha_j, \qquad j = 1, 2, \ldots, (p + q)$$

from which it follows that

$$(71) \qquad \lambda_k = \frac{q}{\Gamma_{\nu_1}(b)}\{\log \alpha_j + 2k\pi i\}\,.$$

In this case, therefore, the characteristic values which lie in sector S for $|\,\lambda\,| > N$ are asymptotically represented by a set of points which are spaced at intervals of length $\frac{2q\pi}{|\,\Gamma_{\nu_1}(b)\,|}$ on $(p + q)$ lines (not necessarily distinct) parallel to the line $R\{\lambda\Gamma_{\nu_1}(b)\} = 0$.

When $\Gamma_{\nu_2}(b)$ and $\Gamma_{\nu_1}(b)$ are incommensurable no such simple treatment is possible. The distribution of the characteristic values may be obtained, however, by Wilder's procedure,[12] which follows.

Setting $\lambda\Gamma_{\nu_1}(b) = z = x + iy$ we have as the equation (70)

$$f(z) = 1 + c_1e^z + c_2e^{rz} + c_3e^{\{1+r\}z} = 0,$$

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[11] Wilder, loc. cit., p. 423.

[12] Cf. Wilder, loc. cit., pp. 420–422.

and it is readily verified that there corresponds to each choice of an arbitrarily small positive constant χ, some value of x, say $x = X$, such that

$$(72)\qquad \begin{cases} \text{(i)} & |1 - f(-z)| < \chi, \\ \text{(ii)} & \left| c_3 - \dfrac{f(z)}{e^{\{1+r\}z}} \right| < \chi, \text{ for } x \leqq X. \end{cases}$$

We shall assume that χ is chosen sufficiently small to preclude the vanishing of $f(z)$ outside or on the boundary of the region $|x| \leqq X$.

Now $f(z)$ is analytic throughout the entire finite plane, and hence it is possible to find in any interval of the Y axis, however small, some point $y = y_0$ which is such that the line $y = y_0$ contains no zero of $f(z)$. Let $y = Y_1$ and $y = Y_2$ be any two such lines and consider the rectangle K bounded by them and the lines $x = X$ and $x = -X$. We shall determine the number of zeros of $f(z)$ within K by observing the increase in $\arg f(z)$ as z describes the perimeter.

We have $\arg f(z) = \sin^{-1} \dfrac{I\{f(z)\}}{|f(z)|}$, where $I\{f(z)\}$ denotes the coefficient of $\sqrt{-1}$ in the expression for $f(z)$. Moreover, for $y =$ constant $I\{f(z)\}$ has the form

$$I\{f(z)\} = d_1 e^x + d_2 e^{rx} + d_3 e^{\{1+r\}x},$$

where the coefficients, d_i, $i = 1, 2, 3$ are real constants. The finite zeros of $I\{f(z)\}$, and hence those of $\dfrac{I\{f(z)\}}{|f(z)|}$, being the roots of the equation

$$d_1 + d_2 e^{\{r-1\}x} + d_3 e^{rx} = 0,$$

are, however, separated by the finite zeros of the derivative of the left-hand member, namely by the roots of the equation

$$d_2\{r - 1\} + d_3 r e^x = 0.$$

Since this equation is satisfied by at most one value of x it follows that $\dfrac{I\{f(z)\}}{|f(z)|}$ vanishes at most twice, and consequently that $\arg f(z)$ changes by less than 3π as x varies between the limits $x = -X$ and $x = +X$ along a line $y =$ constant.

Because of the relation (72i), however, we know that for every z on line $x = -X$, $f(z)$ lies within a circle of radius χ about the point $z = 1$, and hence that as z moves along this line $\arg f(z)$ changes by less than $2 \sin^{-1} \chi$. Similarly relation (72ii) shows that $\arg\{f(z)/e^{(1+r)z}\}$ changes by less than $2 \sin^{-1} \chi$ as z moves along the line $x = X$. From the identity

$$\arg f(z) = \arg e^{\{1+r\}z} + \arg \frac{f(z)}{e^{\{1+r\}z}}$$

it follows, therefore, that the increase in $\arg f(z)$ as z moves along the line $x = X$ from $y = Y_1$ to $y = Y_2$ lies between $(1+r)\{Y_2 - Y_1\} + 2\sin^{-1}\chi$ and $(1+r)\{Y_2 - Y_1\} - 2\sin^{-1}\chi$. Consequently the increase in $\arg f(z)$ as z describes the perimeter of K lies between

$$(1+r)\{Y_2 - Y_1\} + 6\pi + 4\sin^{-1}\chi \text{ and } (1+r)\{Y_2 - Y_1\} - 6\pi - 4\sin^{-1}\chi,$$

and accordingly the number of zeros located in the interior of K must lie between

$$\frac{(1+r)\{Y_2 - Y_1\}}{2\pi} + 3 + \frac{2}{\pi}\sin^{-1}\chi$$

and

$$\frac{(1+r)\{Y_2 - Y_1\}}{2\pi} - 3 - \frac{2}{\pi}\sin^{-1}\chi.$$

Since χ can be chosen arbitrarily small, however, this means that the number of characteristic values between any two lines $y = c_1$ and $y = c_1 + l$ is at least $\dfrac{1+r}{2\pi} l - 3$, and cannot on the other hand exceed $\dfrac{1+r}{2\pi} l + 3$.

Summarizing the results it is seen, therefore, that the characteristic values in sector S lie in a strip bounded by two parallels to the line $R\{\lambda\Gamma_{r_1}(b)\} = 0$, and that they are so distributed throughout this strip that for $|\lambda| > N$ no more than three lie between any two lines which are at a distance $d < \dfrac{2\pi}{1+r}$ from each other and are perpendicular to the line $R\{\lambda\Gamma_{r1}(b)\} = 0$.

SECTION VIII.

The formal expansion of an arbitrary vector.

It was shown in the preceding section that both under regular conditions and under the type of irregular conditions discussed the characteristic values for system (46) are numerable and cluster about the point $\lambda = \infty$. Denoting these values by $\lambda_1, \lambda_2, \lambda_3, \ldots$ it is possible, therefore, to assign the subscripts in such manner that $|\lambda_m| \leqq |\lambda_{m+1}|$. Moreover then $\lim_{m=\infty} |\lambda_m| = \infty$.

Assuming that system (46) is simply compatible at the characteristic values there exists for each of these values just one solution of the system in question and just one solution of its adjoint system. These solutions for $\lambda = \lambda_k$ will be designated by $Y^{(k)}(x)\cdot$ and $\cdot Z^{(k)}(x)$ respectively.

Now if $\lambda = 0$ is a characteristic value for system (46) let the parameter be changed by setting $\lambda = \bar{\lambda} + c$, c being a constant. Equation (46) then becomes

$$Y'(x)\cdot = \{ R(x)\,\bar{\lambda} + \bar{B}(x)\}\; Y(x)\cdot,$$

where

$$\bar{B}(x) = c\,R(x) + B(x).$$

The characteristic values of the system thus modified are $\bar{\lambda} = \lambda_k - c$ and it is clearly always possible to choose c so that $\bar{\lambda} = 0$ is not a characteristic value. No loss of generality is entailed, therefore, by the assumption, which will be made, that $\lambda_k \neq 0$ for any k.

Writing the equation (46) for $\lambda = \lambda_k$ in the form

$$Y^{(k)\prime}(x)\cdot = B(x)\; Y^{(k)}(x)\cdot + \lambda_k\, R(x)\; Y^{(k)}\,(x)\cdot$$

and considering this as a non-homogeneous equation we have from page 67

$$Y^{(k)}(x)\cdot = \lambda_k \int_a^b G(x,t)\; R(t)\; Y^{(k)}(t)\cdot dt, \tag{73}$$

where $G(x, t)$ is the Green's function for the system

(74) $$\begin{aligned} Y'(x)\cdot &= B(x)\,Y(x)\cdot \\ W_a\,Y(a)\cdot + W_b\,Y(b)\cdot &= 0. \end{aligned}$$ [13]

In precisely similar manner we have from the adjoint system

$$\cdot Z^{(l)}(x) = -\lambda_l \int_a^b \cdot Z^{(l)}(t)\,R(t)\,H(x,t)\,dt,$$

or, substituting from the relation

$$G(t,x) \equiv -H(x,t).$$

(75) $$\cdot Z^{(l)}(t) = \lambda_l \int_a^b \cdot Z^{(l)}(x)\,R(x)\,G(x,t)\,dx.$$

Consider, now, the integral

$$J = \int_a^b \int_a^b \cdot Z^{(l)}(x)\,R(x)\,G(x,t)\,R(t)\,Y^{(k)}(t)\cdot dx\,dt.$$

In view of relation (73) we have

$$\lambda_k J = \int_a^b \cdot Z^{(l)}(x)\,R(x)\,Y^{(k)}(x)\cdot dx,$$

while it is seen from (75) that

$$\lambda_l\,J = \int_a^b \cdot Z^{(l)}(t)\,R(t)\,Y^{(k)}(t)\cdot dt.$$

By subtraction, then

$$(\lambda_k - \lambda_l)\,J = 0,$$

and it follows that $J = 0$ provided $k \neq l$. But then $\lambda_l J = 0$, i.e.

(76) $$\int_a^b \cdot Z^{(l)}(x)\,R(x)\,Y^{(k)}(x)\cdot dx = 0, \text{ for } k \neq l.$$

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[13] Since $\lambda = 0$ is not a characteristic value of system (46) system (74) is incompatible and the Green's function exists.

It can, moreover, be easily shown (see page 107) that when the system (46) is simply compatible at every characteristic value, this is not true for $k=l$, i.e.

$$\int_a^b \cdot Z^{(l)}(x)\, R(x)\, Y^{(l)}(x)\cdot dx \neq 0, \text{ for any } l.$$

Let us suppose now that an arbitrarily chosen vector $F(x)\cdot$ can be developed into a series of the form

$$F(x)\cdot = \sum_{k=1}^{\infty} c_k\, Y^{(k)}(x)\cdot\,. \tag{77}$$

Multiplying both sides of this equation on the left by the vector $\cdot Z^{(l)}(x)\, R(x)$ and integrating term by term we have formally

$$\int_a^b \cdot Z^{(l)}(x)\, R(x)\, F(x)\cdot dx = \sum_{k=1}^{\infty} c_k \int_a^b \cdot Z^{(l)}(x)\, R(x)\, Y^{(k)}(x)\cdot dx,$$

which in view of relation (76) reduces to

$$\int_a^b \cdot Z^{(l)}(x)\, R(x)\, F(x)\cdot dx = c_l \int_a^b \cdot Z^{(l)}(x)\, R(x)\, Y^{(l)}(x)\cdot dx. \tag{78}$$

Inasmuch as the matrix on each side of this equation is one all of whose elements are identical the equation may equally well be written

$$\left\{\int_a^b \sum_{k=1}^{n} z_k^{(l)}(x)\, \gamma_k(x)\, f_k(x)\, dx\right\}(1) = c_l \left\{\int_a^b \sum_{k=1}^{n} z_k^{(l)}(x)\, \gamma_k(x)\, y_k^{(l)}(x)\, dx\right\}(1),$$

whence

$$c_l = \frac{\displaystyle\int_a^b \sum_{k=1}^{n} z_k^{(l)}(x)\, \gamma_k(x)\, f_k(x)\, dx}{\displaystyle\int_a^b \sum_{k=1}^{n} z_k^{(l)}(x)\, \gamma_k(x)\, y_k^{(l)}(x)\, dx}.$$

Consequently we have

$$(79)\qquad F(x)\cdot = \sum_{k=1}^{\infty} \left\{ \frac{\int_a^b \sum_{h=1}^{n} z_h^{(k)}(x)\, \gamma_h(x)\, f_h(x)\, dx}{\int_a^b \sum_{h=1}^{n} z_h^{(k)}(x)\, \gamma_h(x)\, y_h^{(k)}(x)\, dx} \right\} Y^{(k)}(x)\cdot$$

If, therefore, $F(x)\cdot$ may be developed into a series of form (74) which converges in such a manner as to legitimatize the processes above, then (79) is a necessary form for the series in question.

Thus far it has been stipulated only that $Y^{(k)}(x)\cdot$ and $\cdot Z^{(k)}(x)$ be respectively solutions of system (46) and its adjoint for $\lambda = \lambda_k$. But each of these systems is homogeneous, and if $\bar{Y}^{(k)}(x)\cdot$ and $\cdot\bar{Z}^{(k)}(x)$ are any particular solutions then $c\,\bar{Y}^{(k)}(x)\cdot$ and $\bar{c}\cdot\bar{Z}^{(k)}(x)$ are also solutions. Having chosen a definite pair $\bar{Y}^{(k)}(x)\cdot$ and $\cdot\bar{Z}^{(k)}(x)$ we have then

$$F(x)\cdot = \sum_{k=1}^{\infty} \left\{ \frac{\int_a^b \sum_{h=1}^{n} \bar{c}\,\bar{z}_h^{(k)}(x)\, \gamma_h(x)\, f_h(x)\, dx}{\int_a^b \sum_{h=1}^{n} \bar{c}\,\bar{z}_h^{(k)}(x)\, \gamma_h(x)\, \bar{y}_h^{(k)}(x)\, dx} \right\} \bar{Y}^{(k)}(x)\cdot$$

If in particular $\bar{c}$ is chosen so that

$$\bar{c}\int_a^b \sum_{h=1}^{n} \bar{z}_h^{(k)}(x)\, \gamma_h(x)\, \bar{y}_h^{(k)}(x)\, dx = 1,$$

and the vector $\bar{c}\cdot\bar{Z}^{(k)}$, for this value of $\bar{c}$ is associated with $\bar{Y}^{(k)}\cdot$ so that the choice of one implies the choice of the other, we have, on dropping the bars over the letters,

$$(80)\qquad F(x)\cdot = \sum_{k=1}^{\infty} \left\{ \int_a^b \sum_{h=1}^{n} z_h^{(k)}(x)\, \gamma_h(x)\, f_h(x)\, dx \right\} Y^{(k)}(x)\cdot.$$

In using this formula it must be remembered that $\cdot Z^{(k)}(x)$ is determined as soon as the particular $Y^{(k)}(x)\cdot$ is chosen.

In a precisely analogous manner it may be found that a development of the type

$$\cdot\bar{F}(x) = \sum_{k=1}^{\infty} \bar{c}_k \cdot Z^{(k)}(x),$$

which converges in such manner that it may be integrated term by term after being multiplied by any of the vectors $R(x)\ Y^{(l)}(x)\cdot$, must necessarily coincide with the expansion

$$\cdot\bar{F}(x) = \sum_{k=1}^{\infty} \left\{ \int_a^b \sum_{h=1}^{n} \bar{f}_h(x)\, \gamma_h(x)\, y_h^{(k)}(x)\, dx \right\} \cdot Z^{(k)}(x).$$

Inasmuch as the matrix $G(x, t)$ is not a vector the methods outlined above do not apply directly to the problem of expanding the Green's function. We shall proceed, therefore, as follows.

Let $G_j(x, t)\cdot$ denote the vector each of whose columns is the j^{th} column of $G(x, t)$. From the relation (80), $F(x)$ being replaced by $G_j(x,t)\cdot$, it follows that for any value of t, $G_j(x, t)\cdot$ can be formally developed, the series obtained being

$$G_j(x, t)\cdot = \sum_{k=1}^{\infty} \left\{ \int_a^b \sum_{h=1}^{n} z_h^{(k)}(x)\, \gamma_h(x)\, g_{hj}(x, t)\, dx \right\} Y^{(k)}(x)\cdot. \tag{81}$$

In view of equation (75) written in the form

$$(z_j^{(l)}(t)) = \lambda_l \int_a^b \sum_{h=1}^{n} z_h^{(l)}(x)\, \gamma_h(x)\, g_{hj}(x, t)\, dx.$$

(81) reduces to

$$G_j(x, t)\cdot = \sum_{k=1}^{\infty} \frac{z_j^{(k)}(t)}{\lambda_k}\, Y^{(k)}(x)\cdot,$$

from which we obtain the relation

$$g_{ij}(x, t) = \sum_{k=1}^{\infty} \frac{z_j^{(k)}(t)\, y_i^{(k)}(x)}{\lambda_k}. \tag{82}$$

But since $\cdot Z^{(k)}(t)$ and $Y^{(k)}(x)\cdot$ are vectors

$$Y^{(k)}(x)\cdot\cdot Z^{(k)}(t) = \left(\sum_{l=1}^{n} y_i^{(k)}(x)\, z_j^{(k)}(t)\right) = n\,(y_i^{(k)}(x)\, z_j^{(k)}(t)).$$

Hence we have from (81)

$$G(x, t) = \sum_{k=1}^{\infty} \frac{Y^{(k)}(x)\cdot\cdot Z^{(k)}(t)}{n\lambda_k}. \tag{83}$$

Having obtained this formal development of the Green's function it is possible to state the

THEOREM: If the development (83) converges to $G(x, t)$ in such a manner that a uniformly convergent series is obtained by multiplying (83) on the right by any matrix of continuous functions and integrating term by term, then any vector $F(x)\cdot$ the elements of which are continuous and have continuous first derivatives, and which satisfies the boundary conditions

$$W_a F(a)\cdot + W_b F(b)\cdot = 0$$

is represented by a convergent development of the type (80).

Proof: The relation

$$F'(x)\cdot = B(x)\, F(x)\cdot + C(x)\cdot$$

defines the vector $C(x)\cdot$, and inasmuch as $F(x)\cdot$ is then a solution of the non-homogeneous system

$$Y'(x)\cdot = B(x)\, Y(x)\cdot + C(x)\cdot$$
$$W_a Y(a)\cdot + W_b Y(b)\cdot = 0,$$

it is given by the formula

$$F(x)\cdot = \int_a^b G(x, t)\, C(t)\cdot dt \qquad \text{(see page 20).}$$

Substituting for $G(x, t)$ its series development we have

$$F(x)\cdot = \int_a^b \sum_{k=1}^{\infty} \frac{Y^{(k)}(x)\cdot\cdot Z^{(k)}(t)\, C(t)\cdot}{n\lambda_k}\, dt,$$

or, integrating term by term,

$$(84) \qquad F(x)\cdot = \sum_{k=1}^{\infty} Y^{(k)}(x)\cdot \left\{ \frac{1}{n\lambda_k} \int_a^b \cdot Z^{(k)}(t)\, C(t)\cdot dt \right\} \cdot$$

Now given any matrix of type $D\cdot$, and another of type $\cdot A\cdot$, we have $D\cdot\cdot A\cdot = \left(\sum_{l=1}^{n} d_i\, a\right) = (nad_i)$, i.e.

$$(85) \qquad D\cdot\cdot A\cdot = n\, a\, D\cdot$$

Hence (84) reduces to

$$F(x)\cdot = \sum_{k=1}^{\infty} \left\{ \frac{\int_a^b \sum_{h=1}^{n} z_h^{(k)}(t)\, c_h(t)\, dt}{\lambda_k} \right\} Y^{(k)}(x)\cdot,$$

the series on the right converging uniformly to $F(x)\cdot$ Q. E. D.

In formula (12) the explicit expression for the Green's function $G(x, t)$ for an incompatible system of type (6) is given, Y_k and Z_k being respectively a solution of the differential matrix equation $Y'(x) = A(x)\, Y(x)$, and of its adjoint. Now for any λ, not a characteristic value, system (46) being incompatible is precisely of type (6). Consequently we have from (12)

$$(86) \qquad G(x, t, \lambda) = \bar{G}(x, t, \lambda) - Y(x, \lambda)\Delta^{-1}(\lambda)\{W_a\bar{G}(a, t, \lambda) + W'_b\bar{G}(b, t, \lambda)\},$$

for any λ not a characteristic value.

Let us investigate now the functional dependence of $G(x, t, \lambda)$ upon the parameter λ, choosing as the solutions $Y(x, \lambda)$ and $Z(t, \lambda)$ a pair which are analytic in λ. Recalling that when the determinant of a matrix A is denoted by A then $A^{-1} = \left(\frac{A_{ji}}{A}\right)$, where A_{ij} is the cofactor of the ij^{th} element of A, the explicit form of $\Delta^{-1}(\lambda)$ is seen to be

$$\Delta^{-1}(\lambda) = \left(\frac{D_{ji}(\lambda)}{D(\lambda)}\right).^{14}$$

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[14] This notation holds when $n = 1$ provided it is agreed that when A is a matrix of one element then $A_{11} = 1$.

Now it was seen (page 87) that at a characteristic value $D(\lambda)$ vanishes. Let us assume further that the characteristic values of system (46) are all simple so that $D(\lambda)$ vanishes at each of these values to only the first order. Due to the relation

$$|D_{ij}| = D^{n-1}, \tag{87}$$

which is familiar from the theory of determinants,[15] it follows, then, that $D_{ij}(\lambda_k)$ cannot vanish for all i and j, since in the alternative case the left-hand side of (87) would vanish to the order n, while the right-hand side vanishes only to the order $(n - 1)$. Hence $(D_{ji}(\lambda_k)) \neq 0$, and $\Delta^{-1}(\lambda)$ has a simple pole at each of the characteristic values. Moreover, system (46) is simply compatible at the characteristic values as was assumed above. It is easily verified (i), that the poles of $\Delta^{-1}(\lambda)$ actually persist in $G(x, t, \lambda)$, and (ii), that $G(x, t, \lambda)$ has no other singularities.

Denoting by $G^{(k)}(x, t)$ the residue of $G(x, t, \lambda)$ at $\lambda = \lambda_k$ we obtain from (86) by familiar methods

$$G^k(x, t) = \frac{Y^{(k)}(x)\,(D_{ji}(\lambda_k))\,\{W_a Y^{(k)}(a) - W_b Y^{(k)}(b)\} Z^k(t)}{\left.\dfrac{d}{d\lambda} D(\lambda)\right]_{\lambda=\lambda_k}},$$

which, inasmuch as $\left.\dfrac{d}{d\lambda} D(\lambda)\right]_{\lambda=\lambda_k} \neq 0$, is of the form

$$G^{(k)}(x, t) = Y^{(k)}(x)\, C^{(k)}\, Z^{(k)}(t). \tag{88}$$

From this it follows that for fixed t, $G^{(k)}(x, t)$ is a solution of the system

$$\left\{\begin{aligned} & Y'(x) = \{R(x)\,\lambda_k + B(x)\}\; Y(x) \\ & W_a Y(a) + W_b Y(b) = 0. \end{aligned}\right. \tag{89}$$

But if $\overline{Y}(x)$ is a particular solution of the differential equation of this system every solution is of the form

$$Y(x) = \overline{Y}(x)\, C.$$

Substituting in the boundary conditions as on page 64 and denoting by (d_{ij}) the Δ which results from $Y = \overline{Y}$ we have

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[15] Cf. Bôcher, loc. cit., p. 33.

$$(90) \qquad (d_{ij})\,(c_{ij}) = O,$$

or
$$\sum_{l=1}^{n} d_{il}\, c_{lj} = 0, \qquad i, j, = 1, 2, \ldots n.$$

Now it was shown that $(D_{ij}(\lambda_k)) \neq O$, i.e. that the rank of $D(\lambda_k)$ is $(n-1)$. It follows that the solution of the system of linear equations

$$(91) \qquad \sum_{l=1}^{n} d_{il}\, c_l = 0 \qquad i = 1, 2, \ldots n$$

is unique, namely that the c_{lj} of system (90) is the c_l of system (91) for all j. Consequently the solution of system (89) is unique and of the type $Y(x)C\cdot$. One solution of (89) is known, however, namely $Y^{(k)}(x)\cdot$. Inasmuch as $G^{(k)}(x, t)$ was also seen to be a solution in x it follows that each column of $G^{(k)}(x, t)$ must be the same as the general column of $Y^{(k)}(x)\cdot$ except possibly for a factor independent of x. Accordingly $g_{ij}^{(k)}(x, t) = c_j^{(k)}(t)\, y_i^{(k)}(x)$,

from which
$$G^{(k)}(x, t) = \frac{1}{n}\, Y^{(k)}(x)\cdot\cdot C^{(k)}(t).$$

But from (88) it is also seen that $G^{(k)}(x, t)$ is, for fixed x, a solution in t of the system adjoint to (89). Since this solution is again unique and $\cdot Z^{(k)}(t)$ is a solution we have

$$g_{ij}^{(k)}(x, t) = y_i^{(k)}(x)\, c_j^{(k)}(t) = c_i^{(k)}(x)\, z_j^{(k)}(t),$$

i.e.
$$c_j^{(k)}(t) = c^{(k)}\, z_j^{(k)}(t).$$

Hence
$$g_{ij}^{(k)}\,(x, t) = c^{(k)}\, y_i^{(k)}(x)\, z_j^{(k)}(t)$$

or

$$(92) \qquad G^{(k)}(x, t) = \frac{c^{(k)}}{n}\, Y^{(k)}(x)\cdot\cdot Z^{(k)}(t),$$

the value of $c^{(k)}$ being as yet undetermined.

Writing equation (46) in the form
$$Y^{(k)\prime}(x)\cdot = \{R(x)\,\lambda + B(x)\}\, Y^{(k)}(x)\cdot + \{\lambda_k - \lambda\}\, R(x)\, Y^{(k)}(x)\cdot,$$

and considering this as a non-homogeneous equation, we have

$$Y^{(k)}(x)\cdot = -\{\lambda-\lambda_k\}\int_a^b G(x, t, \lambda)\, R(t)\, Y^{(k)}(t)\cdot dt.$$

Inasmuch as

$$\lim_{\lambda\to\lambda_k}\{\lambda-\lambda_k\}\, G(x, t, \lambda) = G^{(k)}(x, t),$$

this yields

$$Y^{(k)}(x)\cdot = -\int_a^b G^{(k)}(x, t)\, R(t)\, Y^{(k)}(t)\cdot dt,$$

and substituting from (92) we see that

$$Y^{(k)}(x)\cdot = -\int_a^b \frac{c^{(k)}}{n}\, Y^{(k)}(x)\cdot\cdot Z^{(k)}(t)\, R(t)\, Y^{(k)}(t)\cdot dt,$$

But in view of the relation (85),

$$Y^{(k)}(x)\cdot\int_a^b \cdot Z^{(k)}(t)\, R(t)\, Y^{(k)}(t)\cdot dt = n\left\{\int_a^b \sum_{h=1}^{n} z_h^{(k)}(t)\, \gamma_h(t)\, y_h^{(k)}(t) dt\right\} Y^{(k)}(x).$$

It follows that

$$1 = -c^{(k)}\int_a^b \sum_{h=1}^{n} z_h^{(k)}(t)\, \gamma_h(t)\, y_h^{(k)}(t)\, dt,$$

or upon associating with $Y^{(k)}(x)\cdot$ the proper $\cdot Z^{(k)}(x)$ (see page 101) that $c^{(k)} = -1$.

Hence

(93) $$G^{(k)}(x, t) = -\frac{1}{n}\, Y^{(k)}(x)\cdot\cdot Z^{(k)}(t).$$

This result enables us to deduce an expression for the sum of m terms of the series development of an arbitrary vector $F(x)\cdot$ Thus we have from (77) and (78), ($\cdot Z^l$ and $Y^l\cdot$ being associated in the manner stated)

$$\int_a^b \cdot Z^{(l)}(x)\, R(x)\, F(x)\cdot dx = c_l(1).$$

Multiplying this equation by $Y^{(l)}(x)\cdot$ on the left it becomes

$$c_l Y^{(l)}(x)\cdot(1) = \int_a^b Y^{(l)}(x)\cdot\cdot Z^{(l)}(t)\, R(t)\, F(t)\cdot dt,$$

which in view of (85) and (89) reduces to

$$c_l Y^{(l)}(x)\cdot \;=\; -\int_a^b G^{(l)}(x, t)\, R(t)\, F(t)\cdot dt. \tag{94}$$

This expression for the l^{th} term of the formal development was deduced upon the hypothesis that λ_l is a simple characteristic value. If at λ_l a number of characteristic values $\lambda_{l_1}, \lambda_{l_2}, \ldots \lambda_{l_\mu}$, coincide, we shall define the term of the formal series which corresponds to this value of λ to be

$$-\int_a^b G^{(l)}(x, t)\, R(t)\, F(t)\cdot dt.$$

In every case then we have a formal series which is completely determined and of which the l^{th} term is given by formula (94). It is to be noted that $G^{(l)}(x, t)$ for the case in which λ_l is not a simple characteristic value is not given by the left-hand side of (94). It must be computed for every such case individually.

Now let ρ_l be any contour in the λ plane which surrounds λ_l but no other characteristic values. Then by use of the relation

$$\frac{1}{2\pi i}\int_{\rho_l} G(x, t, \lambda)\, d\lambda = G^{(l)}(x, t)$$

we have from (94)

$$c_l\, Y^{(l)}(x)\cdot \;=\; -\frac{1}{2\pi i}\int_a^b \int_{\rho_l} G(x, t, \lambda)\, d\lambda\, R(t)\, F(t)\cdot dt,$$

or, upon interchanging the order of integration and choosing a contour C_m which encloses the characteristic values $\lambda_1, \lambda_2, \ldots \lambda_m$ and no others,

$$\sum_{k=1}^{m} c_k\, Y^{(k)}(x)\cdot \;=\; -\frac{1}{2\pi i}\int_{C_m}\int_a^b G(x, t, \lambda)\, R(t)\, F(t)\cdot dt\, d\lambda. \tag{95}$$

Section IX.

The Convergence of the expansion.

From the distribution of characteristic values as found in section VII, for both the cases of regular and irregular conditions there discussed, it becomes readily apparent that given any constant N sufficiently large it is always possible to choose a circle C whose center is at $\lambda = 0$, and whose radius both exceeds the value N and is such that the distance from any point of C to a characteristic value is greater than some fixed constant $\bar{\delta} > 0$.

We shall consider the convergence of the contour integral in formula (95) as the contour of integration is taken successively as a larger and larger circle of the type C above. With the size of the circle the number of characteristic values which it includes and hence the number of terms of the series which are summed by the integration may be increased indefinitely, the limit of the integral for $|\lambda| = \infty$, being, if it exists, the sum of the corresponding series.

Let us recall the hypotheses already made concerning the functions $\gamma_i(x)$. We have

$$(96)\quad \left\{\begin{array}{lll} \text{(i)} & \gamma_i(x) \neq 0 & \\ \text{(ii)} & \gamma_i(x) \text{ continuous } a \leqq x \leqq b & \\ \text{(iii)} & \text{if n} \geqq 2 \quad \gamma_i(x) \neq \gamma_j(x) \text{ for } i \neq j & \\ \text{(iv)} & \text{if } n \geqq 2 \quad \arg\{\gamma_i(x) - \gamma_j(x)\} = h_{ij} & \end{array}\right. \begin{array}{l} \left.\begin{array}{l} \\ \\ \\ \end{array}\right\} \text{(page 72)} \\ \\ \text{(page 83)} \end{array}$$

To these we shall add

$$(97)\qquad \arg \gamma_i(x) = H_i \text{ (a constant)}, \quad i = 1, 2, \ldots n.$$

Concerning the vector to be expanded we shall assume

(98) that the elements $f_i(x)$, $i = 1, 2, \ldots n$, consist in the interval $a \leqq x \leqq b$, of only a finite number of pieces each of which is real, continuous, and has a continuous derivative.

It will readily be seen from (96) and (97) that we have restricted each $\gamma_j(x)$ to vary along a ray in the complex plane. Moreover, if $n \geqq 2$ the dependence of $\gamma_j(x)$ upon x must be such that the slope of the line joining any two of these points remains constant. This means that every n sided polygon with vertices at the points $\gamma_j(x_0)$, $j = 1, 2, \ldots n$, $a \leqq x_0 \leqq b$, at most expands or contracts about $\lambda = 0$ as x_0 is allowed to vary.

If the lines along which $\gamma_j(x)$, $j = 1, 2, \ldots n$, vary are all distinct, the conditions of the system are regular provided W_a and W_b are suitably chosen. If, however, one or more sides of any of the polygons mentioned lie on a line through the point $\lambda = 0$, we have the irregular case discussed in section VII. In particular all the sides may lie on such a line, (for instance the quantities $\gamma_j(x)$ may be all real) and it is upon this configuration that the irregular case in question bases its chief claim for interest. Since the condition (96 iv) is automatically fulfilled in this case the functional dependence of $\gamma_j(x)$ upon x is far less restricted than when the points $\gamma_j(x)$ form the vertices of actual polygons.

Substituting in formula (86) the value $\bar{G}(x, t) = \pm Y(x)\, Z(t)$ we have

$$G(x, t, \lambda) = Y(x) \left\{ \pm \tfrac{1}{2} I + \tfrac{1}{2} \Delta^{-1} \{ W_a Y(a) - W_b Y(b) \} \right\} Z(t),$$

the upper sign holding for $t < x$ and the lower sign for $t > x$. It is desirable in the following work to express $G(x, t, \lambda)$ in a somewhat different form.

Upon setting

$$+ \tfrac{1}{2} I + \tfrac{1}{2} \Delta^{-1} \{ W_a Y(a) - W_b Y(b) \} = (\delta^*_{ij}) + U,$$

multiplication by Δ yields

$$\tfrac{1}{2} W_a Y(a) + \tfrac{1}{2} W_b Y(b) + \tfrac{1}{2} W_a Y(a) - \tfrac{1}{2} W_b Y(b) = \{ W_a Y(a) + W_b Y(b) \} (\delta^*_{ij}) + \Delta\, U,$$

which, in view of the relation $(\delta^*_{ij}) + (\delta^{**}_{ij}) = I$, reduces to

$$W_a Y(a)\, (\delta^{**}_{ij}) - W_b Y(b)\, (\delta^*_{ij}) = \Delta\, U.$$

Hence

$$U = \Delta^{-1} W_a Y(a)\, (\delta^{**}_{ij}) - \Delta^{-1} W_b Y(b)\, (\delta^*_{ij}).$$

Now setting

$$- \tfrac{1}{2} I + \tfrac{1}{2} \Delta^{-1} \{ W_a Y(a) - W_b Y(b) \} = - (\delta^{**}_{ji}) + V,$$

it is found by precisely the same method that $V' = U'$. In consequence we have for the Green's function

$$(99)\quad G(x, t, \lambda) = Y(x) \left\{ \left\{ \begin{matrix} (\delta^{*}_{ji}) \\ \text{or} \\ -(\delta^{**}_{ij}) \end{matrix} \right\} + \Delta^{-1} W'_a Y(a)\,(\delta^{**}_{ij}) \right.$$
$$\left. - \Delta^{-1} W'_b Y(b)(\delta^{*}_{ij}) \right\} Z(t),$$

where the upper form is to be chosen when $t < x$, and the lower one for $t > x$.

By means of this formula $G(x, t, \lambda)$ may be explicitly represented by choosing as the solutions $Y(x)$ and $Z(t)$ a pair which are analytic in λ and have the forms

$$Y(x) = \left(e^{\lambda \Gamma_j(x) + B_j(x)} \left\{ \delta_{ij} + \frac{[\varphi_{ij}(x)]}{\lambda} \right\} \right)$$
$$Z(t) = \left(e^{-\lambda \Gamma_i(t) - B_i(t)} \left\{ \delta_{ij} + \frac{[\psi_{ij}(t)]}{\lambda} \right\} \right),$$

obtained in sections V and VI. It should be observed that $G(x, t, \lambda)$ is unique (see page 70), despite the fact that if $n \geqq 2$ the choice of $Y(x)$ and $Z(t)$ thus determined upon changes from any one to any other of the sectors within which no quantities $R\{\lambda\{\gamma_i(x) - \gamma_j(x)\}\}$ change sign, and despite the fact that the values of δ^{*}_{ij} and δ^{**}_{ij} change from any one to any other sector $\sigma_{\mu\nu}$.

From (95) we have, upon denoting by $S_m(x)\cdot$ the sum of those terms of the formal development of $F(x)\cdot$ which correspond to the characteristic values enclosed by the circle C in question,

$$(100)\qquad S_m(x)\cdot = \frac{-1}{2\pi i} \int_C \int_a^b G(x, t, \lambda)\, R(t)\, F(t)\cdot dt\, d\lambda.$$

Substituting in this the value of $G(x, t, \lambda)$ as found above in (99) it is seen that $S_m(x)\cdot = \sum_{i=1}^{4} S^{(i)}_m(x)\cdot$, where the quantities $S^{(i)}_m(x)\cdot$ are given by the relations

$$
(101)\quad \left\{\begin{aligned}
S_m^{(1)}(x)\cdot &= \sum_C \frac{-1}{2\pi i}\int_{C_{\mu\nu}}\int_a^x Y(x)(\delta_{ij}^{*})Z(t)R(t)F(t)\cdot dt\, d\lambda\\
S_m^{(2)}(x)\cdot &= \sum_C \frac{1}{2\pi i}\int_{C_{\mu\nu}}\int_x^b Y(x)(\delta_{ij}^{**})Z(t)R(t)F(t)\cdot dt\, d\lambda\\
S_m^{(3)}(x)\cdot &= \sum_C \frac{-1}{2\pi i}\int_{C_{\mu\nu}} Y(x)\Delta^{-1}\int_a^b W'_a Y(a)(\delta_{ij}^{**})Z(t)R(t)F(t)\cdot dt\,d\lambda\\
S_{mt}^{(4)}(x)\cdot &= \sum_C \frac{1}{2\pi i}\int_{C_{\mu\nu}} Y(x)\Delta^{-1}\int_a^b W'_b Y(b)\,(\delta_{ij}^{*})Z(t)\,R(t)F(t)\cdot dt d\lambda,
\end{aligned}\right.
$$

$C_{\mu\nu}$ denoting any arc of circle C which lies within the sector $\sigma_{\mu\nu}$ and, if $n \geqq 2$, upon which none of the quantities $R\,\{\lambda\{\gamma_i(x) - \gamma_j(x)\}$ change sign. If $C_{\mu\nu}$ abuts upon a ray bounding $\sigma_{\mu\nu}$ it shall either include or exclude the end point for which the quantity $R\,\{\lambda\Gamma_\mu(b)\}$ or the quantity $R\,\{\lambda\Gamma_\nu(b)\}$ vanishes according as the quantity in question is < 0 or > 0 within $\sigma_{\nu\mu}$. The arc $C_{\mu\nu}$ may, therefore, include one, both, or neither end point, and since the reasoning is precisely the same in each case (the sector can be split if both end points are included) we shall consider for the sake of concreteness that it includes one, namely that one which lies on the ray $R\,\{\lambda\Gamma_\nu(b)\} = 0$. The symbol $\sum\limits_c$ indicates that the sum of the integrals over all arcs composing the circle C is to be taken.

We shall proceed to evaluate each of the integrals above in turn, and in the course of this evaluation it will be convenient to refer to the facts established by the following lemmas. The notation $|\varphi(x)| < M$ will be used to indicate that the function in question is bounded. It is not to be understood that the M in any one case represents the same constant as in any other, but merely that there exists in each case some constant for which the relation is true.

Lemma 1: Given any function $\varphi_1(x, \lambda)$ which is such that

$$
(102)\quad \left\{\begin{aligned}
&\text{(i)}\quad |\varphi_1(x, \lambda)| < M \text{ for } \alpha \leqq x \leqq \beta,\ |\lambda| > N,\\
&\text{(ii)}\quad \lim_{\lambda=\infty} |\varphi_1(x, \lambda)| = 0 \text{ uniformly, for } \alpha \leqq x \leqq \beta - \chi,
\end{aligned}\right.
$$

where χ is an arbitrarily small positive constant, then

$$\lim_{|\lambda|=\infty} \int_{\alpha}^{\beta} \varphi_1(x, \lambda)\, dx = 0\,.$$

Proof: Under the hypotheses (102) we have

$$\lim_{|\lambda|=\infty} \int_{\alpha}^{\beta-\chi} \varphi_1(x, \lambda)\, dx = 0 \text{ uniformly},$$

and

$$\left| \int_{\beta-\chi}^{\beta} \varphi_1(x, \lambda)\, dx \right| < M\chi,$$

while it follows from these relations and the relation

$$\left| \int_{\alpha}^{\beta} \varphi_1(x, \lambda)\, dx \right| \leqq \left| \int_{\alpha}^{\beta-\chi} \varphi_1(x, \lambda)\, dx \right| + \left| \int_{\beta-\chi}^{\beta} \varphi_1(x, \lambda)\, dx \right|,$$

that

$$\left| \int_{\alpha}^{\beta} \varphi_1(x, \lambda)\, dx \right| < 2M\chi$$

for $|\lambda|$ sufficiently large. Since χ is arbitrary, however, this means that

$$\lim_{|\lambda|=\infty|} \int_{\alpha}^{\beta} \varphi_1(x, \lambda)\, dx = 0. \qquad \text{Q. E. D.}$$

Lemma 2: Given any function $\varphi_2(\lambda)$ which is such that

$$(103) \begin{cases} \text{(i)} & |\varphi_2(\lambda)| < M \text{ for } \theta_\alpha \leqq \arg \lambda \leqq \theta_\beta,\ |\lambda| > N \\ \text{(ii)} & \lim\limits_{|\lambda|=\infty} |\varphi_2(\lambda)| = 0 \text{ uniformly, for } \theta_\alpha \leqq \arg \lambda \leqq \theta_\beta - \theta_\chi, \end{cases}$$

where θ_χ is a constant arbitrarily small but positive, then

$$\lim_{|\lambda|=\infty} \frac{1}{2\pi i} \int_{C_{\alpha\beta}} \varphi_2(\lambda)\, \frac{d\lambda}{\lambda} = 0,$$

$C_{\alpha\beta}$ being that arc of the circle $|\lambda| = \rho$ which lies in the sector bounded by $\arg \lambda = \theta_\alpha$ and $\arg \lambda = \theta_\beta$.

Proof: Writing $\lambda = \rho e^{i\theta}$ we have

$$\varphi_2(\lambda) = \varphi_2(\rho e^{i\theta}) = \psi_2(\theta, \rho).$$

Hence

$$\lim_{|\lambda| = \infty} \frac{1}{2\pi i} \int_{C_{\alpha\beta}} \varphi_2(\lambda) \frac{d\lambda}{\lambda} = \lim_{\rho = \infty} \frac{1}{2\pi} \int_{\theta_\alpha}^{\theta_\beta} \psi_2(\theta, \rho)\, d\theta,$$

and since the limit of the integral on the right is zero by lemma 1,

$$\lim_{|\lambda| = \infty} \frac{1}{2\pi i} \int_{C_{\alpha\beta}} \varphi_2(\lambda) \frac{d\lambda}{\lambda} = 0. \qquad \text{Q. E. D.}$$

Lemma 3: Given any function $\varphi_3(x, \lambda)$ which is such that

$$(104) \left\{ \begin{array}{ll} \text{(i)} & |\varphi_3(x,\lambda)| < M \text{ for } \alpha \leqq x \leqq \beta, \theta_\alpha \leqq \arg\lambda \leqq \theta_\beta, |\lambda| > N, \\ \text{(ii)} & \lim\limits_{|\lambda|=\infty} |\varphi_3(x,\lambda)| = 0 \text{ uniformly for } \alpha \leqq x \leqq \beta - \chi, \\ & \qquad\qquad \theta_\alpha \leqq \arg\lambda \leqq \theta_\beta - \theta_\chi, \end{array} \right.$$

then $I(\lambda) = \int_\alpha^\beta \varphi_3(x, \lambda)\, dx$ is a function of the type $\varphi_2(\lambda)$ of lemma 2.

Proof: When λ is confined to the sector $\theta_\alpha \leqq \arg \lambda \leqq \theta_\beta - \theta_\chi$, $\lim\limits_{|\lambda| = \infty} I(\lambda) = 0$ uniformly by lemma 1, while for λ in the larger sector $\theta_\alpha \leqq \arg \lambda \leqq \theta_\beta$

$$|I(\lambda)| \leqq \int_\alpha^\beta M dx = M(\beta - \alpha).$$

Hence $I(\lambda)$ is of type $\varphi_2(\lambda)$ by definition. Q. E. D.

As a matter of convenience we shall use hereafter the symbols φ_1, φ_2, and φ_3, to designate any function of λ and other arguments which satisfies conditions similar to (102), (103) or (104) respectively. As heretofore functions which approach the limit zero uniformly as $|\lambda|$ increases indefinitely will be denoted by ϵ. Let us return now to the direct evaluation of $S_m(x)$.

I. $S_m^{(1)}(x)\cdot$.

Writing

$$J_1\cdot = -\int_a^x Y(x)\,(\delta_{ij}^*)\,Z(t)\,R(t)\,F(t)\cdot dt$$

we have from (98)

$$S_m^{(1)}(x)\cdot = \frac{1}{2\pi i}\int_C J_1\cdot d\lambda.$$

From this it is immediately seen that

$$(105) \qquad S_m^{(1)}(a)\cdot = 0.$$

Consequently it is necessary to consider further only the case $x \neq a$. We have

$$J_1\cdot = -\left(\int_a^x \sum_{k,m,p,q=1}^{n} e^{\lambda\Gamma_k(x)+B_k(x)}\{\delta_{ik}+1/\lambda[\varphi_{ik}(x)]\}\delta_{km}^* e^{-\lambda\Gamma_m(t)-B_m(t)} \{\delta_{mp}+1/\lambda[\psi_{mp}(t)]\}\delta_{pq}\gamma_q(t)\,f_q(t)\,dt\right),$$

i.e.

$$(106)\quad J_1\cdot = -\left(\int_a^x e^{\lambda\{\Gamma_i(x)-\Gamma_i(t)\}+B_i(x)-B_i(t)}\;\delta_{ii}^*\gamma_i(t)\,f_i(t)\,dt\right) + 1/\lambda\left(\int_a^x \sum_{k,p=1}^{n} e^{\lambda\{\Gamma_k(x)-\Gamma_k(t)\}+B_k(x)-B_k(t)}\;\delta_{kk}^*\,\gamma_p(t)\,f_p(t)[\delta_{kp}\,\varphi_{ik}(x) + \delta_{ik}\,\psi_{kp}(t)]\,dt\right).$$

For λ on any arc $C_{\mu\nu}$, however, each term under the sign of summation in the last matrix on the right of this expression is seen to be of the type $\varphi_3(t, \lambda)$. Their sum therefore is of the same type, and by lemma 2 the integral is of type $\varphi_2(\lambda)$. Consequently we have, upon integrating by parts the elements in the first matrix on the right of (106),[15]

$$J_1\cdot = \frac{1}{\lambda}\left(-\delta_{ii}^*\left\{-f_i(x-0)+e^{\lambda\Gamma_i(x)+B_i(x)}f_i(a+0) + \int_a^x e^{\lambda\{\Gamma_i(x)-\Gamma_i(t)}\frac{d}{dt}\left\{f_i(t)e^{B_i(x)-B_i(t)}\right\}dt\right\}\right)+\frac{(\varphi_2)}{\lambda}.$$

---

[15] Recall that $\Gamma_i(a) = B_i(a) = 0$.

Accordingly it is seen that $J_1\cdot$ can be considered as a sum of matrices of which the first three are discussed as follows.

For the first matrix we have directly

$$\frac{1}{\lambda}\,(\delta^*_{ii} f_i(x-0)) = \frac{1}{\lambda}\,(\delta^*_{ij})\,F(x-0)\cdot.$$

The second matrix

$$-\frac{1}{\lambda}\,(\delta^*_{ii}\,e^{\lambda\Gamma_i(x)+B_i(x)}\,f_i(a))$$

is one each of whose elements is seen either to vanish or to be of type $\varphi_2(\lambda)/\lambda$ on arc $C_{\mu\nu}$. Hence the entire matrix may be represented by $(\varphi_2)/\lambda$.

The third matrix

$$-\frac{1}{\lambda}\left(\delta^*_{ii}\int_a^x e^{\lambda\{\Gamma_i(x)-\Gamma_i(t)\}}\,\frac{d}{dt}\{f_i(t)\,e^{B_i(x)-B_i(t)}\}\,dt\right)$$

is likewise one each of whose elements either vanishes or is, on arc $C_{\mu\nu}$, of type φ_2/λ by lemma 3. This matrix is, therefore, also of the type $(\varphi_2)/\lambda$, and inasmuch as a sum of matrices of type (φ_2) is again a matrix (φ_2) we have

$$J_1\cdot = \frac{1}{\lambda}\,\{(\delta^*_{ij})\,F(x-0)\cdot + (\varphi_2)\}$$

The integration of $J_1\cdot$ over the arc $C_{\mu\nu}$ yields, therefore, by lemma 2

$$\frac{1}{2\pi i}\int_{C_{\mu\nu}}\left\{(\overset{*}{\delta}{}_{ij}^{(\mu\nu)})\,F(x-0)\cdot + (\varphi_2)\right\}\frac{d\lambda}{\lambda} = \frac{\omega_{\mu\nu}}{2\pi}\,(\overset{*}{\delta}{}_{ij}^{(\mu\nu)})\,F(x-0)\cdot + (\epsilon),$$

where $\omega_{\mu\nu}$ is the angle subtended by $C_{\mu\nu}$ at $\lambda = 0$.

A similar expression results from integration over each arc similar in character to $C_{\mu\nu}$. Hence we have

$$S^{(1)}_m(x)\cdot = \sum_C \frac{\omega_{\mu\nu}}{2\pi}\,(\overset{*}{\delta}{}_{ij}^{(\mu\nu)})\,F(x-0)\cdot + (\epsilon).$$

Let us consider in detail the sum

$$(107)\qquad \sum_C \frac{\omega_{\mu\nu}}{2\pi}\,(\overset{*}{\delta}{}_{ij}^{(\mu\nu)}).$$

Corresponding to each arc $C_{\mu\nu}$ there exists another arc $C_{\bar\mu\bar\nu}$ which is the reflection of $C_{\mu\nu}$ in the point $\lambda = 0$. Since $C_{\bar\mu\bar\nu}$ also subtends the angle $\omega_{\mu\nu}$ at the point $\lambda = 0$ we can write the sum (107) equally well in the form

$$(108) \qquad \sum_{\frac{1}{2}C} \frac{\omega_{\mu\nu}}{2\pi} \left(\overset{*}{\delta}{}_{ij}^{(\mu\nu)} + \overset{*}{\delta}{}_{ij}^{(\bar\mu\bar\nu)}\right),$$

the summation covering now the arc of only one (any) half of circle C. But it will readily be seen that $\overset{*}{\delta}{}_{ij}^{(\bar\mu\bar\nu)} = \overset{**}{\delta}{}_{ij}^{(\mu\nu)}$, for quantities which have a positive real part in $\sigma_{\mu\nu}$ have naturally a negative real part in $\sigma_{\bar\mu\bar\nu}$. Hence $\overset{*}{\delta}{}_{ij}^{(\mu\nu)} + \overset{*}{\delta}{}_{ij}^{(\bar\mu\bar\nu)} = \delta_{ij}$, and the sum (108) i.e. (107) reduces to

$$\sum_{\frac{1}{2}C} \frac{\omega_{\mu\nu}}{2\pi} I = \tfrac{1}{2} I.$$

It follows, therefore, that

$$(109) \qquad \sum_{C} \frac{\omega_{\mu\nu}}{2\pi} \left(\overset{*}{\delta}{}_{ij}^{(\mu\nu)}\right) = \tfrac{1}{2} I.$$

In consequence we have

$$S_m^{(1)}(x)\cdot = \tfrac{1}{2} F(x-0)\cdot + (\epsilon),$$

i.e.

$$(110) \qquad \lim_{m\to\infty} S_m^{(1)}(x)\cdot = \tfrac{1}{2} F(x-0)\cdot$$

II. $S_m^{(2)}(x)\cdot$.

The treatment of this expression is parallel to that of $S_m^{(1)}(x)\cdot$ and is as follows.

Writing

$$J_2\cdot = \int_x^b Y(x)\,(\overset{**}{\delta}_{ij}) Z(t)\, R(t)\, F(t)\cdot dt,$$

we have from (101) $\qquad S_m^{(2)}(x)\cdot = \dfrac{1}{2\pi i}\displaystyle\int_C J_2\cdot d\lambda.$

Hence

$$(111) \qquad S_m^{(2)}(b)\cdot = 0,$$

and it is necessary to consider further only the case $x \mp b$. We have

$$J_2\cdot = \left(\int_x^b \sum_{k,m,p,q=1}^{n} e^{\lambda\Gamma_k(x)+B_k(x)}\left\{\delta_{ik} + \frac{[\varphi_{ik}(x)]}{\lambda}\right\}\delta^{**}_{km}e^{-\lambda\Gamma_m(t)-B_m(t)}\left\{\delta_{m\rho} + \frac{[\psi_m(t)]}{\lambda}\right\}\delta_{pq}\gamma_q(t)\,f_q(t)\,dt\right)$$

$$= \left(\int_x^b e^{\lambda\{\Gamma_i(x)-\Gamma_i(t)\}+B_i(x)-B_i(t)}\,\delta^{**}_{ii}\;\gamma_i(t)\,f_i(t)\,dt\right) +$$

$$\frac{1}{\lambda}\left(\int_x^b \sum_{k,p=1}^{n} e^{\lambda\{\Gamma_k(x)-\Gamma_k(t)\}+B_k(x)-B_k(t)}\delta^{**}_{kk}\,\gamma_p(t)\,f_p(t)\,[\delta_{kp}\,\varphi_{ik}(x) + \delta_{ik}\,\psi_{kp}(t)]\,dt\right).$$

Each integrand in the last matrix of this equation is of the type $\varphi_1(t, \lambda)$ and accordingly the matrix is, by lemma 1 of type $(\epsilon)/\lambda$. Integrating by parts the elements of the first matrix on the right we have

$$J_2\cdot = \frac{1}{\lambda}\left(\delta^{**}_{ii}\left\{-\,e^{\lambda\{\Gamma_i(x)-\Gamma_i(b)\}+B_i(x)-B_i(b)}\,f_i(b-0) + f_i(x+0) + \int_x^b e^{\lambda\{\Gamma_i(x)-\Gamma_i(t)\}}\,\frac{d}{dt}\left\{f_i(t)\,e^{B_i(x)-B_i(t)}\right\}dt\right\}\right) + \frac{(\epsilon)}{\lambda}.$$

Therefore, $J_2\cdot$ can also be expressed as a sum of matrices, the first and third of which, namely

$$\frac{1}{\lambda}\left(-\,\delta^{**}_{ii}\,e^{\lambda\{\Gamma_i(x)-\Gamma_i(b)\}+B_i(x)-B_i(b)}f_i(b)\right)$$

and

$$\frac{1}{\lambda}\left(\delta^{**}_{ii}\int_x^b e^{\lambda\{\Gamma_i(x)-\Gamma_i(t)\}}\,\frac{d}{dt}\left\{f_i(t)\,e^{B_i(x)-B_i(t)}\right\}dt\right),$$

are of the type $(\epsilon)/\lambda$ on arc $C_{\mu\nu}$, for each element either vanishes or approaches the limit zero as $|\lambda| = \infty$. The second matrix of the sum is directly

$$\frac{1}{\lambda}\,(\overset{**}{\delta}_{ii}\, f_i(x+\dot{0})) = \frac{1}{\lambda}\,(\overset{**}{\delta}_{ij})\, F(x+0)\cdot.$$

Hence

$$J_2\cdot = \frac{1}{\lambda}\,\{(\overset{**}{\delta}_{ij})\, F(x+0)\cdot + (\epsilon)\},$$

which integrated over $C_{\mu\nu}$ gives

$$\frac{1}{2\pi i}\int_{C_{\mu\nu}}\left\{(\overset{**}{\delta}{}_{ij}^{(\mu\nu)})\, F(x+0)\cdot + (\epsilon)\right\}\frac{d\lambda}{\lambda} = \frac{\omega_{\mu\nu}}{2\pi}\,(\overset{**}{\delta}{}_{ij}^{(\mu\nu)})\, F(x+0)\cdot + (\epsilon).$$

By familiar reasoning this leads to the equation

$$S_m^{(2)}(x)\cdot = \sum_C \frac{\omega_{\mu\nu}}{2\pi}\,(\overset{**}{\delta}{}_{ij}^{(\mu\nu)})\, F(x+0)\cdot + (\epsilon),$$

and inasmuch as

(112) $$\sum_C \frac{\omega_{\mu\nu}}{2\pi}\,(\overset{**}{\delta}{}_{ij}^{(\mu\nu)}) = \tfrac{1}{2}\, I,$$

as may be shown by applying again the argument by means of which relation (109) was established, we have

$$S_m^{(2)}(x)\cdot = \tfrac{1}{2}\, F(x+0)\cdot + (\epsilon),$$

i.e.

(113) $$\lim_{m\to\infty} S_m^{(2)}(x)\cdot = \tfrac{1}{2}\, F(x+0)\cdot$$

III. $S_m^{(3)}(x)\cdot$.

Following the procedure used in the discussion of the preceding expressions, let $J_3\cdot$ be defined by the equation

$$J_3\cdot = -\, Y(x)\,\Delta^{-1}\int_a^b W^{\cdot}_a Y(a)\,(\overset{**}{\delta}_{ij})\, Z(t)\, R(t)\, F(t)\cdot dt.$$

Then $$S_m^{(3)}(x)\cdot = \frac{1}{2\pi i}\int_C J_3\cdot d\lambda.$$

Now

$$\Delta^{-1} = \left(\frac{D_{ji}(\lambda)}{D(\lambda)}\right)^{\text{[16]}},$$

---

[16] See note on page 104.

and substituting for $D(\lambda)$ its value as given by equation (56) we have

$$\Delta^{-1} = \left(\frac{\prod\limits_{\substack{k=1\\k\neq i}}^{n} e^{\{\lambda\Gamma_k(b)+B_k(b)\}\,\overset{**}{\delta}{}_{kk}^{(\mu\nu)}}\,\overline{D}_{ji}(\lambda)}{\prod\limits_{k=1}^{n} e^{\{\lambda\Gamma_k(b)+B_k(b)\}\,\overset{**}{\delta}{}_{kk}^{(\mu\nu)}}\,\overline{D}(\lambda)} \right)$$

$$= \left(e^{-\{\lambda\Gamma_i(b)+B_i(b)\}\,\overset{**}{\delta}{}_{ii}^{(\mu\nu)}}\,\frac{\overline{D}_{ji}(\lambda)}{\overline{D}(\lambda)} \right),$$

where $\overline{D}(\lambda)$ is given for the case of regular conditions by formula (57) and for the case of irregular conditions discussed in section VII by either formula (66) or formula (68). A glance at these formulas shows, however, that in each case the expression for $\overline{D}(\lambda)$ reduces, under the conditions for which the expression is valid to the form

$$\overline{D}(\lambda) = W'^{(\mu\nu)} + \varphi_2, \text{ on arc } C_{\mu\nu}.$$

Now every point of a circle C, and hence in particular of an arc $C_{\mu\nu}$ is at a distance which exceeds a fixed quantity from any characteristic value. However, by analogy with formula (59), for any point of $C_{\mu\nu}$ under regular conditions for which the function $\overline{D}$ is sufficiently small we have

$$\lambda = \frac{1}{\Gamma_\nu(b)} \left\{ -B_\nu(b) + \log \frac{-W'^{(\mu\nu)}}{W'^{(\nu\tau)}} + \epsilon + 2p\pi i \right\}.$$

In other words the point in question will necessarily lie near one of the characteristic values.

This stands in contradiction with the fundamental property of the circle C.

Inasmuch as a similar relation holds on every arc of type $C_{\mu\nu}$, it follows that for every point of all circles C, $\overline{D}(\lambda)$ exceeds, under regular conditions, some fixed positive constant δ_1, i.e.

(114) $$|\overline{D}(\lambda)| \geqq \delta_1 > 0 \quad \text{for } \lambda \text{ on any circle } C.$$

While we have thus proved the relation (114) only for the case of regular conditions it can be shown to hold equally well under the type of irregular conditions considered in section VII.[17] In either case, therefore, Δ^{-1} is uniformly bounded on all circles of type C.

Proceeding, we have

$$Y(x)\Delta^{-1} = \left[\sum_{k=1}^{n} e^{\lambda\Gamma_k(x)+B_k(x)} \left\{ \delta_{ik} + \frac{[\varphi_{ik}(x)]}{\lambda} \right\} e^{-\{\lambda\Gamma_k(b)+B_k(b)\}\overset{**}{\delta}_{kk}} \frac{\overline{D}_{jk}(\lambda)}{\overline{D}(\lambda)} \right]$$

$$= \left[e^{\lambda\{\Gamma_i(x)-\Gamma_i(b)\overset{**}{\delta}_{ii}\}+B_i(x)-B_i(b)\overset{**}{\delta}_{ii}} \left\{ \frac{\overline{D}_{ji}(\lambda)}{\overline{D}(\lambda)} \right\} \right]$$

$$+ \left[\sum_{k=1}^{n} e^{\lambda\{\Gamma_k(x)-\Gamma_k(b)\overset{**}{\delta}_{kk}\}+B_k(x)-B_k(b)\overset{**}{\delta}_{kk}} \frac{[\varphi_{ik}(x)]}{\lambda} \frac{\overline{D}_{jk}(\lambda)}{D(\lambda)} \right].$$

Since the last matrix of this expression is of type (ϵ) we have

$$Y(x)\Delta^{-1} = \left(e^{\lambda\{\Gamma_i(x)-\Gamma_i(b)\overset{**}{\delta}_{ii}\}+B_i(x)-B_i(b)\overset{**}{\delta}_{ii}} \left\{ \frac{\overline{D}_{ji}(\lambda)}{D(\lambda)} + \epsilon \right\} \right),$$

or, more briefly,

$$(115) \qquad Y(x)\Delta^{-1} = \left(e^{\lambda\{\Gamma_i(x)-\Gamma_i(b)\overset{**}{\delta}_{ii}\}+B_i(x)-B_i(b)\overset{**}{\delta}_{ii}} \tau_{ij}(\lambda) \right),$$

where $\qquad |\tau_{ij}(\lambda)| < M \quad \text{for} \quad |\lambda| > N.$

Inasmuch as $\overline{D}(\lambda)$ was seen to be of the form

$$\overline{D}(\lambda) = W'^{(\mu\nu)} + \varphi_2$$

for λ on arc $C_{\mu\nu}$, we have for any λ on this arc

$$(\tau_{ij}) = \left(\frac{W'^{(\mu\nu)}_{ji} + \varphi_2}{W'^{(\mu\nu)} + \varphi_2} + \epsilon \right) = \left(\frac{W'^{(\mu\nu)}_{ji}}{W'^{(\mu\nu)}} \right) + (\varphi_2),$$

or, denoting by $\Omega^{(\mu\nu)}$ the matrix whose determinant is $W'^{(\mu\nu)}$,

$$(116) \qquad (\tau_{ij}) = \overset{-1}{\Omega^{(\mu\nu)}} + (\varphi_2).$$

---

[17] Cf. Wilder, loc. cit. p. 422.

We have, further,

$$\int_a^b W_a Y(a)\,(\delta^{**}_{ij})\,Z(t)\,R(t)\,F(t)\cdot dt =$$

$$\left(\int_a^b \sum_{h,k,m,p,q=1}^{n} w^{(a)}_{ih}\ \left\{\delta_{hk} + \frac{[\varphi_{hk}(a)]}{\lambda}\right\}\ \delta^{**}_{km} e^{-\lambda\Gamma_m(t)-B_m(t)} \left\{\delta_{mp} + \frac{[\psi_{mp}(t)]}{\lambda}\right\} \delta_{pq}\,\gamma_q(t)\,f_q(t)\,dt\right) =$$

$$\left(\sum_{h=1}^{n} w^{(a)}_{ih}\delta^{**}_{hh}\int_a^b e^{-\lambda\Gamma_h(t)-B_h(t)}\,\gamma_h(t)\,f_h(t)\,dt\right) +$$

$$\frac{1}{\lambda}\left(\sum_{h,k,p=1}^{n} w^{(a)}_{ih}\delta^{**}_{kk}\int_a^b e^{-\lambda\Gamma_k(t)-B_k(t)}\,\gamma_p(t)\,f_p(t)\,[\delta_{hk}\psi_{kp}\,(t) + \delta_{kp}\,\varphi^{(a)}_{hk}]\,dt\right)$$

$$= \frac{1}{\lambda}\left(\sum_{h=1}^{n} w^{(a)}_{ih}\delta^{**}_{hh}\left\{-e^{-\lambda\Gamma_h(b)-B_h(b)}f_h\,(b-0) + f_h(a+0) + \int_a^b e^{-\lambda\Gamma_h(t)}\,\frac{d}{dt}\left\{f_h(t)\,e^{-B_h(t)}\right\}dt\right\}\right) + \frac{(\epsilon)}{\lambda}$$

the last form representing as heretofore the result of an integration by parts. The multiplication of this matrix by the matrix (115) yields the result

$$J_3\cdot = \frac{-1}{\lambda}\left(\sum_{k,h=1}^{n} e^{\lambda\{\Gamma_i(x)-\Gamma_i(b)\delta^{**}_{ii}\}+B_i(x)-B_i(b)\delta^{**}_{ii}}\tau_{ik}(\lambda)\,w^{(a)}_{kh}\delta^{**}_{hh}\right.$$

$$\left.\left\{f_h(a+0) - e^{-\lambda\Gamma_h(b)-B_h(b)}\,f_h(b-0) + \int_a^b e^{-\lambda\Gamma_h(t)}\,\frac{d}{dt}\,\{f_h(t)e^{-B_h(t)}\}dt\right\}\right)$$

$$+\frac{(\epsilon)}{\lambda},$$

the principal matrix on the right again breaking up into three component matrices. Of these the first,

$$\frac{1}{\lambda}\left(\sum_{k,h=1}^{n} e^{\lambda\{\Gamma_i(x)-\Gamma_i(b)\overset{**}{\delta}_{ii}\}+B_i(x)-B_i(b)\overset{**}{\delta}_{ii}}\,\tau_{ik}(\lambda)w^{(a)}_{kh}\overset{**}{\delta}_{hh}\,e^{\lambda\Gamma_h(b)-B_h(b)}\,f_h(b-0)\right),$$

and the third,

$$\frac{-1}{\lambda}\left(\sum_{k,h=1}^{n} e^{\lambda\{\Gamma_i(x)-\Gamma_i(b)\overset{**}{\delta}_{ii}\}+B_i(x)-B_i(b)\overset{**}{\delta}_{ii}}\right.$$
$$\left.\tau_{ik}(\lambda)\,w^{(a)}_{kh}\overset{**}{\delta}_{hh}\int_a^b e^{-\lambda\Gamma_h(t)}\frac{d}{dt}\{e^{-B_h(t)}f_h(t)\}dt\right),$$

are for all values of x clearly of the type $(\epsilon)/\lambda$. Only the second matrix of the sum, namely

$$\bar{J}_3(x)\cdot=\frac{-1}{\lambda}\left(\sum_{k,h=1}^{n} e^{\lambda\{\Gamma_i(x)-\Gamma_i(b)\overset{**}{\delta}_{ii}\}+B_i(x)-B_i(b)\overset{**}{\delta}_{ii}}\,\tau_{ik}(\lambda)w^{(a)}_{kh}\overset{**}{\delta}_{hh}\,f_h(a+0)\right),$$

remains, therefore, to be considered. It depends for its character upon the value of x.

For $x \mp a$, $x \mp b$, the matrix is clearly of type $(\varphi_2)/\lambda$. For $x = a$, on the other hand, the exponential factor common to the elements of the i^{th} row reduces to $e^{-\{\lambda\Gamma_i(b)+B_i(b)\}\overset{**}{\delta}_{ii}} = \overset{*}{\delta}_{ii} + \epsilon$, so that

$$\bar{J}_3(a)\cdot=\frac{-1}{\lambda}\left(\sum_{k,h=1}^{n} \overset{*}{\delta}_{ii}\,\tau_{ik}(\lambda)\,w^{(a)}_{kh}\,\overset{**}{\delta}_{hh}\,f_h(a+0)+\epsilon\right).$$

The fact that

$$\bar{J}_3(b)\cdot=\frac{-1}{\lambda}\left(\sum_{k,h=1}^{n} \overset{**}{\delta}_{ii}\,\tau_{ik}(\lambda)\,w^{(a)}_{kh}\,\overset{**}{\delta}_{hh}\;f_h(a+0)+\varphi_2\right),$$

may be established in similar manner by use of the relation

$$e^{\{\lambda\Gamma_i(b)+B_i(b)\}\overset{*}{\delta}_{ii}} = \overset{**}{\delta}_{ii} + \varphi_2.$$

Substituting for (τ_{ij}) its value as given by formula (115) it is readily seen, therefore, that on arc $C_{\mu\nu}$

$$\bar{J}_3(a)\cdot = \frac{-1}{\lambda}\overset{*}{(\delta_{ij}^{(\mu\nu)})}\,\overset{-1}{\Omega^{(\mu\nu)}}\,W_a\,\overset{**}{(\delta_{ij}^{(\mu\nu)})}\,F(a+0)\cdot + (\varphi_2)\}$$

$$\bar{J}_3(b)\cdot = \frac{-1}{\lambda}\{\overset{**}{(\delta_{ij}^{(\mu\nu)})}\,\overset{-1}{\Omega^{(\mu\nu)}}\,W_a\overset{**}{(\delta_{ij}^{(\mu\nu)})}\,F(a+0)\cdot + (\varphi_2)\}.$$

It follows from this that

$$S_m^{(3)}(x)\cdot\begin{cases} = \sum\limits_C \dfrac{1}{2\pi i}\displaystyle\int_{C_{\mu\nu}}(\epsilon)\frac{d\lambda}{\lambda} & \overset{\text{ha}}{\text{when}}\ x \neq a,\ x \neq b. \\ = \sum\limits_C \dfrac{1}{2\pi i}\displaystyle\int_{C_{\mu\nu}}\{-\overset{*}{(\delta_{ij}^{(\mu\nu)})}\,\overset{-1}{\Omega^{(\mu\nu)}}\,W_a\,\overset{**}{(\delta_{ij}^{(\mu\nu)})}\,F(a+0)\cdot + (\varphi_2)\}\frac{d\lambda}{\lambda} & \\ & \text{when } x = a \\ = \sum\limits_C \dfrac{1}{2\pi i}\displaystyle\int_{C_{\mu\nu}}\{-\overset{**}{(\delta_{ij}^{(\mu\nu)})}\,\overset{-1}{\Omega^{(\mu\nu)}}\,W_a\,\overset{**}{(\delta_{ij}^{(\mu\nu)})}\,F(a+0)\cdot + (\varphi_2)\}\frac{d\lambda}{\lambda} & \\ & \text{when } x = b, \end{cases}$$

or, upon defining the matrices $\overline{K}_3$ and $\overline{\overline{K}}_3$ by the equations

$$(117)\qquad \begin{cases} \overline{K}_3 = \sum\limits_C \left\{-\dfrac{w_{\mu\nu}}{2\pi}\overset{*}{(\delta_{ij}^{(\mu\nu)})}\,\overset{-1}{\Omega^{(\mu\nu)}}\,W_a\,\overset{**}{(\delta_{ij}^{(\mu\nu)})}\right\} \\ \overline{\overline{K}}_3 = \sum\limits_C \left\{-\dfrac{w_{\mu\nu}}{2\pi}\overset{**}{(\delta_{ij}^{(\mu\nu)})}\,\overset{-1}{\Omega^{(\mu\nu)}}\,W_a\,\overset{**}{(\delta_{ij}^{(\mu\nu)})}\right\}, \end{cases}$$

$$(118)\qquad \lim_{m=\infty} S_m^{(3)}(x)\cdot\begin{cases} = 0 & \text{when } x \neq a,\ x \neq b. \\ = \overline{K}_3\,F(a+0)\cdot & \text{when } x = a. \\ = \overline{\overline{K}}_3\,F(a+0)\cdot & \text{when } x = b. \end{cases}$$

It should be observed that the values of $\overline{K}_3$ and $\overline{\overline{K}}_3$ depend only upon the differential system whose characteristic functions form the terms of the expansion in question and are in particular entirely independent of the vector F$(x)\cdot$

IV. $S_m^{(4)}(x)\cdot$

Proceeding precisely as in the case of $S_m^{(3)}(x)\cdot$ we have upon defining $J_4\cdot$ by the relation

$$J_4 \cdot = Y(x)\,\Delta^{-1}\int_a^b W'_b Y(b)\,(\delta^*_{ij})\,Z(t)\,R(t)\,F(t)\cdot dt,$$

$$S_m^{(4)}(x)\cdot = \frac{1}{2\pi i}\int_c J_4 \cdot d\lambda.$$

Further

$$\int_a^b W'_b Y(b)\,(\delta^*_{ij})\,Z(t)\,R(t)\,F(t)\cdot dt =$$

$$\left(\sum_{h=1}^{n} w_{ih}^{(b)}\,\delta^*_{hh}\int_a^b e^{\lambda\{\Gamma_h(b)-\Gamma_h(t)\}+B_h(b)-B_h(t)}\,\gamma_h(t)\,f_h(t)\,dt\right)$$

$$+\frac{1}{\lambda}\left(\sum_{h,k,p=1}^{n} w_{ih}^{(b)}\,\delta^*_{kk}\int_a^b e^{\lambda\{\Gamma_k(b)-\Gamma_k(t)\}+B_k(b)-B_k(t)}\,\gamma_p(t)\,f_p(t)\,[\delta_{hk}\,\psi_{kp}(t)\right.$$
$$\left. + \delta_{kp}\,\varphi_{hk}(b)]\,dt\right)$$

$$=\frac{1}{\lambda}\left(\sum_{h=1}^{n} w_{ih}^{(b)}\,\delta^*_{hh}\left\{-f_h(b-0)+e^{\lambda\Gamma_h(b)+B_h(b)}f_h(a+0)+\right.\right.$$
$$\left.\left.\int_a^b e^{\lambda\{\Gamma_h(b)-\Gamma_h(t)\}}\frac{d}{dt}\left\{e^{B_h(b)-B_h(t)}f_h(t)\right\}dt\right\}\right)+\frac{(\varphi_2)}{\lambda}$$

and multiplying this matrix by the matrix $Y(x)\,\Delta^{-1}$, as given by formula (115) it follows that

$$J_4 \cdot = \frac{1}{\lambda}\left(\sum_{k,h=1}^{n} e^{\lambda\{\Gamma_i(x)-\Gamma_i(b)\overset{**}{\delta_{ii}}\}+B_i(x)-B_i(b)\overset{**}{\delta_{ii}}}\,\tau_{ik}(\lambda)\,w_{kh}^{(b)}\delta^*_{hh}\left\{-f_h(b-0)+\right.\right.$$
$$\left.\left.e^{\lambda\Gamma_h(b)+B_h(b)}f_h(a+0)+\int_a^b e^{\lambda\{\Gamma_h(b)-\Gamma_h(t)\}}\frac{d}{dt}\left\{f_h(t)e^{B_h(b)-B_h(t)}\right\}dt\right\}\right)+\frac{(\varphi_2)}{\lambda}.$$

The second matrix of the sum on the right, namely

$$\frac{1}{\lambda}\left(\sum_{k,h=1}^{n} e^{\lambda\{\Gamma_i(x)-\Gamma_i(b)\overset{**}{\delta}_{ii}\}+B_i(x)-B_i(b)\overset{**}{\delta}_{ii}}\tau_{ik}(\lambda)\,w_{kh}^{(b)}\overset{*}{\delta}_{hh}\,e^{\lambda\Gamma_h(b)+B_h(b)}f_h(a+0)\right),$$

is seen to be of the type $(\varphi_2)/\lambda$, whereas the third, i.e.

$$\frac{1}{\lambda}\left(\begin{array}{l}\displaystyle\sum_{k,h=1}^{n} e^{\lambda\{\Gamma_i(x)-\Gamma_i(b)\overset{**}{\delta}_{ii}\}+B_i(x)-B_i(b)\overset{**}{\delta}_{ii}} \\ \displaystyle\qquad\qquad\tau_{ik}(\lambda)w_{kh}^{(b)}\,\overset{*}{\delta}_{hh}\int_a^b e^{\lambda\{\Gamma_h(b)-\Gamma_h(t)\}}\,\frac{d}{dt}\,\{\,f_h(t)\,e^{B_h(b)-B_h(t)}\}\,dt\end{array}\right),$$

is also of this type, as is apparent from the fact that by lemma 3, each of the integrals with respect to t which actually occurs in any element of the matrix is of type φ_2, while the remaining factors are bounded.

The remaining matrix of importance in the expression for $J_4\cdot$, namely

$$\bar{J}_4(x)\cdot=\frac{-1}{\lambda}\left(\sum_{k,h=1}^{n} e^{\lambda\{\Gamma_i(x)-\Gamma_i(b)\overset{**}{\delta}_{ii}\}+B_i(x)-B_i(b)\overset{**}{\delta}_{ii}}\tau_{ik}(\lambda)w_{kh}^{(b)}\overset{*}{\delta}_{hh}f_h(b-0)\right),$$

again depends for its character upon the value of x, being of the type $(\varphi_2)/\lambda$ when and only when $x \neq a$ and $x \neq b$. By reasoning now familiar the results

$$\begin{aligned}\bar{J}_4(a)\cdot &= \frac{-1}{\lambda}\left(\sum_{k,h=1}^{n}\overset{*}{\delta}_{ii}\,\tau_{ik}(\lambda)\,w_{kh}^{(b)}\,\overset{*}{\delta}_{hh}\,f_h(b-0)+\varphi_2\right)\\ &= \frac{-1}{\lambda}\,(\overset{*}{\delta}_{ij})\,\Omega^{-1}\,W_b\,(\overset{*}{\delta}_{ij})\,F(b-0)\cdot+\frac{1}{\lambda}\,(\varphi_2),\end{aligned}$$

and

$$\begin{aligned}\bar{J}_4(b)\cdot &= \frac{-1}{\lambda}\left(\sum_{k,h=1}^{n}\overset{**}{\delta}_{ii}\,\tau_{ik}(\lambda)\,w_{kh}^{(b)}\,\overset{*}{\delta}_{hh}\,f_h\,(b-0)+\varphi_2\right)\\ &= \frac{-1}{\lambda}\,(\overset{**}{\delta}_{ij})\,\Omega^{-1}\,W_b\,(\overset{*}{\delta}_{ij})\,F(b-0)\cdot+\frac{1}{\lambda}\,(\varphi_2),\end{aligned}$$

may be deduced, whereupon it follows that

$$
\mathrm{S}_m^{(4)}(x)\cdot \begin{cases} = \sum\limits_C \dfrac{1}{2\pi i}\displaystyle\int_{C_{\mu\nu}} (\varphi_2)\, \dfrac{d\lambda}{\lambda} & \text{when } x \mp a,\ x \mp b, \\ = \sum\limits_C \dfrac{1}{2\pi i}\displaystyle\int_{C_{\mu\nu}} \{-(\overset{*}{\delta}{}_{ij}^{(\mu\nu)})\overset{-1}{\Omega^{(\mu\nu)}}\, W^{\cdot}_b\, (\overset{*}{\delta}{}_{ij}^{(\mu\nu)})\, F(b-0)\cdot + (\varphi_2)\}\, \dfrac{d\lambda}{\lambda} & \\ & \text{when } x = a. \\ = \sum\limits_C \dfrac{1}{2\pi i}\displaystyle\int_{C_{\mu\nu}} \{-(\overset{**}{\delta}{}_{ij}^{(\mu\nu)})\overset{-1}{\Omega^{(\mu\nu)}}\, W^{\cdot}_b (\overset{*}{\delta}{}_{ij}^{(\mu\nu)})\, F(b-0)\cdot + (\varphi_2)\}\, \dfrac{d\lambda}{\lambda} & \\ & \text{when } x = b. \end{cases}
$$

Defining the matrices $\overline{K}_4$ and $\overline{\overline{K}}_4$ by the equations

$$
(119) \qquad \begin{cases} \overline{K}_4 = \sum\limits_C \left\{ \dfrac{-w_{\mu\nu}}{2\pi}\, (\overset{*}{\delta}{}_{ij}^{(\mu\nu)})\, \overset{-1}{\Omega^{(\mu\nu)}}\, W^{\cdot}_b\, (\overset{*}{\delta}{}_{ij}^{(\mu\nu)}) \right\} \\ \overline{\overline{K}}_4 = \sum\limits_C \left\{ \dfrac{-w^{\mu\nu}}{2\pi}\, (\overset{**}{\delta}{}_{ij}^{(\mu\nu)})\, \overset{-1}{\Omega^{(\mu\nu)}}\, W^{\cdot}_b\, (\delta_{ij}^{(\mu\nu)}) \right\}, \end{cases}
$$

it follows, therefore, that

$$
(120) \qquad \lim_{m-\infty} S_m^{(4)}(x)\cdot \begin{cases} = 0 & \text{when } x \mp a,\ x \mp b \\ = \overline{K}_4\, F(b-0)\cdot & \text{when } x = a \\ = \overline{\overline{K}}_4\, F(b-0)\cdot & \text{when } x = b. \end{cases}
$$

A summary of the various results as contained in formulas (110), (113), (118) and (120) is seen, in virtue of (105) and (111), to yield

$$
\begin{aligned} &\lim_{m-\infty} S_m(x)\cdot = \tfrac{1}{2}\, F(x-0)\cdot + \tfrac{1}{2}\, F(x+0)\cdot \qquad \text{when } x \mp a,\ x \mp b, \\ &\lim_{m-\infty} \mathrm{S}_m(a)\cdot = \tfrac{1}{2}\, F(a+0)\cdot + \overline{K}_3 F(a+0)\cdot + \overline{K}_4\, F(b-0)\cdot, \\ &\lim_{m-\infty} S_m(b)\cdot = \tfrac{1}{2}\, F(b-0)\cdot + \overline{\overline{K}}_3\, F(a+0)\cdot + \overline{\overline{K}}_4\, F(b-0)\cdot, \end{aligned}
$$

In consequence we have the following

THEOREM: Given that L is any vector differential system of the type

$$
\begin{aligned}
\overline{Y}'(x)\cdot &= \{A(x)\lambda + B(x)\}\overline{Y}(x)\cdot,\\
\overline{W}_a\overline{Y}(a)\cdot + \overline{W}_b\overline{Y}(b)\cdot &= 0,
\end{aligned}
$$

which can be reduced by a change of the dependent variable to a system of type (46) for which (a), $R(x)$ and $B(x)$ are continuous together with their first derivatives (b), the functions $\gamma_j(x)$, $j = 1, 2, \ldots n$, satisfy the relations (96) and (97), and (c), the condition (ii) on page 89 is fulfilled. Then the development in characteristic functions of the system L which is associated with any vector $F(x)\cdot$ whose elements satisfy condition (98), converges to

$$
\begin{aligned}
\tfrac{1}{2}F(x-0)\cdot + \tfrac{1}{2}F(x+0)\cdot &\quad \text{when } x \neq a,\ x \neq b,\\
H_aF(a+0)\cdot + J_aF(b-0)\cdot &\quad \text{when } x = a,\\
H_bF(a+0)\cdot + J_bF(b-0)\cdot &\quad \text{when } x = b,
\end{aligned}
$$

the four matrices of constants H_a, J_a, H_b, and J_b being explicitly determined by the matrices $R(x)$, $A(x)$, $\overline{W}_a$ and $\overline{W}_b$ as stated.

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LICHENES IN INSULA TRINIDAD A PROFESSORE R. THAXTER COLLECTI.

By Edward A. Vainio.

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Note. The following enumeration, which Dr. Vainio has been so kind as to prepare for publication, includes a portion of a small collection of lichens made largely in the vicinity of Port of Spain, Trinidad, B. W. I., from January to April, 1913. (R. T.)

1. Pertusaria commutata Müll. Arg. (Vain. Et. Lich. Brés. I, 1890, p. 105).
Ad corticem arboris. Maraval Valley (No. 8). Ster.

2. Pertusaria torulosa Vain. Addit. Lich. Antill. (Ann. Ac. Scient. Fennicae, Sér. A, tom. VI, 1915), p. 31.
Ad corticem arboris. La Seiva Valley (No. 100).

3. Placodium ochraceum (Nyl.) Vain. *Lecanora aurantiaca* var. *ochracea* Nyl. Et. Lich. de l'Algér (1854), p. 325 (sec. Herb. Nyl.)
Ad saxa corallina in Gasparee Island (No. 62). Fert.

4. Sticta Weigelii (Ach.) Vain. (Et. Lich. Brés. I, p. 105) f. *Beauvoisii* (Del.) Stizenb. Grübchenflecht. (1895), p. 133.
Ad corticem arboris. Sangre Grande (No. 86). Apotheciis novellis.

5. Pannaria stylophora Vain. (Addit. Lich. Antill. p. 102) f. *disserpens* Vain. Lich. Ins. Philipp. III (Annal. Ac. Scient. Fenn., Ser. A, tom. XV, 1921), p. 9.
Ad corticem arboris (*Theobroma cacao*), Sangre Grande (No. 75). Fert.

6. Pannaria rubiginosa (Thunb.) Del. f. *caesiocinerea* Vain., l. c., p. 12.
Ad corticem arborum. Sangre Grande (Nos. 84, 87). Fert.

7. Coccocarpia ciliata (Bél.) Vain. Thallus parce isidiosus. Etiam in specim. orig. Collem. ciliati Bél. (Voy., p. 130), n. 30981 in herb. Nyl., thallus est parce isidiosus.
Ad folia exarida arboris. St. Ann's Valley (No. 49). Ster.

8. **Heppia Trinitatis** Vain. (sp. n.). Thallus foliaceus, dichotome aut partim subirregulariter crebre repetito-laciniatus, caespites circ. 10–25 mm. latos formans, laciniis 0.4–0.7 mm. latis, sublinearibus, superne leviter convexis, subtus canaliculato-concavis, superne et subtus laevigatus et cinereo-nigricans, glaber, rhizinis nullis distinctis, sed hyphis hypothallinis penicillato-confertis, impure albidis passim substrato affixus, crass. circ. 0.220 mm., heteromericus, in parte superiore et inferiore zonam gonidialem continens, zona medullari gonidiis destituta, strato corticali destitutus. Gonidia cyanophycea, caerulescentia aut glaucescentia, cellulis subglobosis, diam. 0.005–0.004 mm., concatenatis aut solitariis binisve, membrana tenuissima, vagina haud evoluta, heterocystis decoloratis, aeque crassis, membrana distincta. Habitu Phycias in memoriam revocans.

Ad rupem in rivulis in silva. St. Ann's Valley, Trinidad (No. 36). Ster.

9. Physma byrsinum (Ach.) Müll. Arg. var. *hypomelaena* Nyl. (Vain. Lich. Ins. Philipp., III, p. 45)

Ad corticem arboris. Sangre Grande (No. 85). Fert.

10. Leptogium azureum (Sw.) Nyl. var. *laevior* Vain. l. c. p. 37.

Ad corticem arboris. Sangre Grande (No. 85). Fert.

11. Cladonia pityrea (Floerk.) Fr. f. *squamulifera* Vain. Mon. Clad. Univ. III, p. 255.

Ad plantas destructas. Four Roads (No. 35). Ster.

12. **Bilimbia Maravalensis** Vain. (sp. n.). Thallus crustaceus, sat tenuis, subverruculoso-inaequalis, sat continuus, cinereo-glaucescens, esorediatus, hypothallo indistincto. Apothecia dispersa, aut partim aggregata, lat. 0.8–0.4 mm. late adnata, basi leviter constricta, disco plano aut vulgo demum convexo, carneo-pallido, nudo, leviter nitido, margine concolore, tenui, integro, demum vulgo excluso. Perithecium hyphis radiantibus, sat leptodermaticis conglutinatis, cavitatibus cellularum 0.002–0.003 mm. latis. Hypothecium decoloratum, hyphis irregulariter contextis, sat pachydermaticis, conglutinatis, in parte superiore cellulis depressis, in seriebus horizontalibus dispositis. Hymenium 0.060–0.070 mm. crassum, jodo sat dilute caerulescens. Epithecium fere decoloratum. Paraphyses arcte conglutinatae, tenues, apice vix incrassatae. Sporae 8:nae, decolores, oblongae aut elongatae, apicibus obtusis aut rotundatis, rectae aut leviter curvatae, 3-septatae, long. 0.012–0.017, crass. 0.0025–0.003

mm. Gonidia pleurococcoidea, globosa, diam. 0.005–0.006 mm., simplicia aut glomerulosa, membrana sat tenui.

Subsimilis *Bilimbiae sphaeroidi,* sed sporis tenuioribus, thallo minus evoluto et reactione hymenii ab ea differt, ad Bacidias accedens.

Ad corticem arboris. Maraval Valley (No. 28).

13. **Bilimbia rufopunctata** Vain. (sp. nov.). Thallus crustaceus, sat tenuis aut sat crassus, continuus, verruculoso-inaequalis et majore parte verruculis minutissimis (circ. 0.050 mm. latis) et granulis soredioideis, crebris subcontiguisque instructus, glaucescens vel stramineo-virescens, hypothallo indistincto. Apothecia numerosissima crebraque, lat. 0.4–0.3 mm., basi leviter constricta, disco plano aut demum convexo, rufo aut obscure rufescente, nudo, opaco, margine tenuissimo, subconcolore aut paullo pallidiore, integro, demum excluso, KOH solutionem rubescentem vel subroseam effundentia. Hypothecium dilute fulvo-rubescens. Epithecium fulvo-rubescens, hymenium ceterum totum fulvescens, circ 0.040 mm. crassum, jodo persistenter caerulescens. Paraphyses arcte cohaerentes, simplices aut parce ramoso-connexae furcataeve, membranis leviter gelatinosis cavitatibus 0.0012 mm. crassis, apice saepe leviter incrassatis. Sporae 8:nae, decolores, oblongae aut cuneato-oblongae, rectae, apicibus obtusis, 3–1-septatae, long. 0.009–0.014, crass. 0.002–0.0035 mm. Gonidia pleurococcoidea, globosa, diam. 0.006–0.009 mm., simplicia aut glomerulosa, membrana modice incrassata aut gelatinosa.

Facie externa *Bacidiam inundatam* in memoriam revocans.

Ad truncum putridum palmae. Four Roads (No. 48).

14. **Sporopodium Thaxteri** Vain. (sp. n.). Thallus crustaceus, tenuis, continuus, verruculis minutissimis (circ. 0.050–0.080 mm. latis), intus albis crebre adspersus, stramineo- aut albido-glaucescens, sat opacus, KOH flavescens, hypothallo albido, parum distincto. Apothecia lat. 0.8–0.4 mm., dispersa, basi constricta, disco plano, tenuiter pruinoso, partim demum leviter denudato et livido-rufescente, margine sat tenui, haud prominente, albido, opaco, persistente, glabro, plus minus distincte verruculoso. Hypothecium superne rufescens, inferne dilutius coloratum. Parathecium rufescens, modice incrassatum, extus strato albido aeque crasso obductum. Hymenium 0.065–0.110 mm. crassum, totum decoloratum, jodo persistenter caerulescens. Paraphyses arcte cohaerentes, ramoso-connexae. Sporae singulae, decoloratae, oblongae, apicibus rotundatis, murali-divisae, cellulis numerosissimis, long. 0.046–0.066, crass. 0.015–0.018 mm., membrana modice incrassata, haud gelatinosa.

Proximum *Sp. cupulifero* Nyl. et *Sp. mastophoro* Vain. (Lich. Ins. Philipp. III, p. 91), quae sporis majoribus et disco obscuriore et thallo KOH non reagente ab hac specie differunt.

Ad folia arborum. St. Ann's Valley (No. 56). La Seiva Valley (No. 60). Supra thallum parce etiam observavi Cyphellam (conf. Vain. Etud. Lich. Brés. II, p. 27).

15. **Sporopodium glaucophaeopsis** Vain. (sp. n.). Thallus crustaceus, tenuissimus, continuus aut subdispersus, laevigatus, stramineo-glaucescens, sat opacus, KOH flavescens, hypothallo indistincto. Apothecia lat. 0.5–0.4 mm., dispersa aut subaggregata, basi constricta, disco plano, pallido, pruinoso, margine sat tenui, haud aut leviter prominente, parum verruculoso, stramineo-albido, opaco, persistente. Hypothecium rufescens. Hymenium circ. 0.080 mm. crassum, totum decoloratum aut epithecio pallido, jodo persistenter caerulescens. Paraphyses ramoso-connexae. Sporae singulae, decoloratae, demum pallidae, oblongae, apicibus rotundatis, muralidivisae, cellulis numerosissimis, long. 0.056–0.060(–0.030), crass. 0.018 (–0.012) mm. Conidangia (pycnidia) prominentia, depresso-subglobosa, albida. Conidia (stylosporae) decolorata, ovoideo-oblonga, altero apice rotundato, altero obtuso, recta, simplicia, long. 0.007, crass. 0.002 mm. Gonidia globosa, diam. 0.006–0.008 mm., tantum simplicia visa, membrana tenui, forsan pleurococcoidea.

Sporopodium glaucophaeodes (Nyl. Lich. Guin. p. 19) sporis majoribus margine apotheciorum laevigato et disco carneo-fuscescente, haud pruinoso ab hac specie differt.

Ad folia arboris. Maraval Valley (No. 70).

16. LECIDEA CORONULANS (Nyl.) Vain. (*Lecanora granifera* var. *coronulans* Nyl. in Fl. 1874, p. 72), var. **Gaspareina** Vain. (var. n.) Sporis subglobosis aut breviter ellipsoideis, long. 0.008–0.011, crass. 0.005–0.007 mm. a *Lec. coronulante* (Nyl.) differt. Apothecia disco cinereo fuscescente aut fusconigro, margine verruculoso, verruculis albidis, gonidia continentibus, ceterum perithecium intus fuscofuligineum, hyphis conglutinatis. Hypothecium fuscofuligineum, hyphis conglutinatis. Thallus albidus aut albido-glaucescens, verruculis minutis, intus albis inspersus, KOH flavescens, verruculis fulvescentibus (in No. 44 reactio minus distincta).

Ad hanc speciem non pertinet *Lecanora coronulans* Nyl. (Fl. 1876, p. 510).

Ad corticem arborum. Maraval Valley (No. 44). Gasparee Island (No. 38).

17. LECIDEA FUSCORUBESCENS Nyl., Not. Lich. Port. Natal (1868) p. 8 (*L. Natalensis* Nyl. in Cromb. Lich. Challeng. Exp. 1878, p. 216). Sporae long. 0.007–0.013, crass. 0.007–0.008 mm. in No. 64; long. 0.007–0.0010, crass. 0.006–0.007 mm. in No. 51.— *L. caliginosa* Stirt. est autonoma species, disco metalloideo-caerulescente differens etiam e speciminibus nomine *L. fuscorubescentis* in Nyl. Lich. Ins. Andam. (1873) p. 10 descriptis.

Ad corticem arborum. St. Ann's Valley (No. 51). La Seiva Valley (No. 64).

18. **Lecidea canoumbrina** Vain. (sp. n.). Thallus crustaceus, tenuis, subcontinuus, subgranuloso-inaequalis, glauco-virescens, opacus, hypothallo albido vel indistincto. Apothecia dispersa, lat. 1.2–0.7 mm., basi late adnata, parum constricta, disco depresso-convexo, cinereo-rufescente aut rufo-fuscescente subnitido, margine tenui, haud prominente, fuscescente, integro, parum conspicuo, demum excluso. Hypothecium purpureum aut purpureo-fuligineum, KOH non reagens, superne ex hyphis erectis, conglutinatis formatum, inferne irregulariter contextis conglutinatisque. Perithecium intus purpureum, extus dilute coloratum pallidumve. Hymenium circ. 0.050–0.060 mm. crassum, totum pallidum, jodo persistenter caerulescens. Paraphyses arcte cohaerentes, tubulo tenuissimo. Sporae 8-nae, distichae, decoloratae, fusiformes-oblongae, apicibus obtusis, simplices, long. 0.007–0.010 crass. 0.0025–0.003 mm.

Proxima est *L. canorubellae* Vain. Lich. Bras. Exs. (1892), No. 32, quae sporis crassioribus et hypothecio testaceo ab ea differt.

Ad corticem arboris. Maraval Valley (No. 19).

19. **Lecidea cinereopallida** Vain. (sp. n.). Thallus crustaceus, tenuis continuus, laevigatus, glaucescens, hypothallo indistincto. Apothecia dispersa, lat. 1.2–0.8 mm., basi constricta, sat late adnata, disco plano aut demum leviter convexo, cinereo-pallescente pallidove, parum nitido, haud pruinoso, margine sat tenui, leviter aut haud prominente, impure albido aut raro subcinerascente, persistente. Hypothecium inferius, albidum, ex hyphis irregulariter contextis, conglutinatis formatum, pars superior intense fuscescens, KOH non reagens. Perithecium albidum, ex hyphis subradiantibus aut irregulariter contextis, sat leptodermaticis, conglutinatis, cavitatibus cellularum oblongis aut ellipsoideis, 0.003–0.0015 mm. latis. Hymenium circ. 0.100–0.110 mm. crassum, jodo persistente caerulescens. Paraphyses arcte cohaerentes, simplices, membrana leviter gelatinosa,

cavitatibus tenuibus. Asci clavati. Sporae 8-nae, distichae, decolores, fusiformes-oblongae aut ellipsoideae aut rarius subglobosae, long. 0.009–0.014, crass. 0.006–0.008 mm. Gonidia pleurococcoidea, globosa, diam. 0.006–0.004 mm.

Proxima *L. declinii* Nyl. (haud *Bilimbia declinis* (Tuck.)) sive *L. declinandae* Nyl. (Lich. Nov. Zel. p. 146), quae thallo crassiore, verruculoso-rugoso ab eo differt.

Ad corticem arboris. Maraval Valley (No. 42).

20. **Pilocarpon glabrum** Vain. (sp. n.). Thallus crustaceus, sat tenuis, areolatus, areolis 0.3–0.2 (0.35–0.15) mm. latis, planis aut depresso convexis, contiguis aut partim dispersis, albido-glaucescentibus, leviter nitidis, nec KOH, nec $HCaCl_2O_2$ reagentibus, hypothallo indistincto. Apothecia vulgo dispersa, lat. 0.7–0.5 mm., basi leviter constricta, late adnata, disco plano, nigro nudoque aut vulgo tenuiter cinereo-pruinoso, opaco, margine sat tenui, primum diu leviter prominente, persistente, glabro, integro, laevigato, albo, opaco. Hypothecium fusco-fuligineum, basi usque ad substratum conice productum, KOH non reagens. Parathecium tenue, fusco-fuligineum. Perithecium albidum, ex hyphis crebre contextis, haud conglutinatis formatum, gonidiis destitutum. Hymenium 0.120–0.130 mm. crassum, jodo persistenter caerulescens. Epithecium olivaceo-obscuratum. Paraphyses ramoso-connexae, apicibus irregulariter contextis. Sporae 8:nae, decolores, fusiformes, leviter curvatae, apicibus sat obtusis aut altero apice rotundato, 5–7 septatae, loculis cylindricis, long. 0.052–0.080, crass. 0.008–0.013 mm. Gonidia globosa, tantum simplicia visa, diam. 0.006–0.010 mm., membrana tenui aut sat tenui, pleurococcoidea, ut videtur.

Supra folia arboris. La Seiva Valley (No. 59).

21. **Gyalecta pachyspora** Vain. (sp. n.). Thallus crustaceus, tenuissimus, continuus, laevigatus, glaucescens, hypothallo indistincto. Apothecia lat. 0.4–0.3 mm., basi constricta, disco plano aut primum subconcavo, fulvescenti-carneo aut carneo-pallido, nudo, margine concolore aut pallidiore, persistente, primum diu prominente, sat tenui, integro. Perithecium albidum, plectenparenchymaticum leviter pachydermaticum, gonidiis destitutum. Hypothecium decoloratum. Hymenium 0.055 mm. crassum, jodo non reagens. Asci subcylindrici. Sporae 8:nae, distichae, ovoideae aut ellipsoideae, vulgo altero apice rotundato, altero obtuso, aut rarius apicibus ambobus obtusis, 1-septatae, long. 0.008–0.0017, crass. 0.004–0.008 mm. Goni-

dia ad Heterothallum pertinentia, irregulariter ramosa, cellulis irregulariter oblongis aut difformibus, 0.004–0.005 mm. crassis.

Sporis majoribus differt a *G. epiphylla* (Müll. Arg.) Vain. Lich. Ins. Philipp. III, p. 148.

Supra folia arboris. Maraval Valley (No. 69).

22. COENOGONIUM INTERPLEXUM Nyl. Obs. Coenog. p. 92; Vain. Addit. Lich. Antill. p. 131. Sporae long. 0.010–0.011, crass. 0.002–0.0015 mm., 1-septatae. Gonidia cellulis 0.030–0.036 mm. longis, 0.018–0.020 mm. crassis.

Ad ramos arbusculorum. Aripo Savanna, Cumuto (No. 32). Fert.

23. **Tricharia Amazonum** Vain. (sp. n.). Thallus tenuis aut tenuissimus, continuus, glaucescens, leviter nitidus, laevigatus aut demum leviter inaequalis, pilis ornatus nigris, dispersis, apicem versus sensim attenuatis, basi in bulbum nigrum, vulgo haud prominentem, incrassatis, hypothallo indistincto. Apothecia numerosa et sat crebra aut sat dispersa, orbicularia, 0.25–0.2 mm. lata, lecideoidea, prominentia, basi tota adnata, nec constricta, disco leviter concavo aut planiusculo, rufescente aut pallido-rufescente, nudo, margine tenui, prominente, integro, fuscescente, persistente. Hypothecium albidum. Perithecium margine dilute fuscescens, ceterum pallidum. Hymenium jodo non reagens; paraphyses ramoso-connexae, tubulis 0.0005 mm. crassis, gelatinam hymenialem percurrentes. Sporae 8:nae aut pauciores, etiam solitariae, decoloratae, oblongae, apicibus rotundatis, murali-divisae, cellulis numerosissimis, long. 0.065–0.066, crass. 0.016–0.026 mm., membrana leviter gelatinoso-incrassata. Gonidia pleurococcoidea, globosa, tantum simplicia visa, diam. 0.005–0.007 mm.

Tricharia melanothrix (Vain. Lich. Ins. Philipp. III, p. 159) huic speciei proxime est affinis, praesertim sporis minoribus, tantum solitariis ab ea differens. Pilis crebris, bulbisque prominentibus planta in Fée. Ess. Crypt. Écorc. tab. III, f. 18 a et c delineata magis speciei Philippinae, quam Trinidadensi, similis est.

Supra folia coriacea arboris, La Seiva Valley (No. 50). Fert. Huc verisimiliter etiam pertinet No. 47 in Maraval Valley sterilis collectus.

24. **Thelotrema** (OCELLULARIA) **sublilacinum** (Ellis) Vain. (comb. n.). *Karstenia sublilacina* Ellis in Smith Centr. Amer. Fungi, No. 49 (sec. Thaxter in litt.) Sacc. Syll. Fung. XIV (1899) p. 810. Thallus modice incrassatus, laevigatus aut partim rugosus vel verrucoso-inaequalis, glaucescens vel stramineo-glaucescens, KOH demum leviter rubescens, gonidia trentepohliacea continens, hypothallo indis-

tincto. Apothecia vulgo dispersa, disco 4–1 mm. lato, immerso, sublilacino-carneo aut carneo-pallido, tenuissime, pruinoso, plano aut concavo, orbiculari aut leviter irregulari, margine prominente, discum et thallum superante, intus albido, leviter lacerato aut subintegro, extus strato thallino plus minus lacerato cincto. Parathecium albidum. Hypothecium albidum, subtus linea fuscescente limitatum. Paraphyses simplices. Sporae 8:nae–4:nae, fusiformi-elongatae, apicibus obtusis, septis transversis 16–19 (14–23 sec. Sacc. l.c.), decoloratae, jodo violascentes, long. 0.060–0.094 crass. 0.010–0.012 mm. loculis lenticularibus.

Ad corticem arboris. St. Ann's Valley (No. 36).

25. **Thelotrema** (OCELLULARIA) **platycarpella** Vain. (sp. n.). Thallus sat tenuis, leviter subverruculoso-inaequalis, olivaceo-virescens, leviter nitidus, hypothallo indistincto. Apothecia partim nonnulla aggregata vel confluentia, disco 0.8–0.3 mm. lato, immerso, orbiculari aut leviter irregulari, plano, bene pruinoso, caesio, margine prominente, discum et thallum superante, tenui, albo, minute lacerato. Parathecium et hypothecium albida. Hymenium circ. 0.045 mm. crassum, columella nulla. Epithecium albidum. Paraphyses simplices. Sporae 8:nae, distichae, altero apice rotundato obtusove, altero acuto, decoloratae, jodo non reagentes, septis transversis 4–5, long. 0.012–0.014, crass. 0.005 mm., membrana haud gelatinosa.

Affinis speciei praecedenti, sed omnibus partibus minoribus.

Ad corticem arboris. Verdant Vale, Arima (No. 57).

26. THELOTREMA (PHAEOTREMA) DIFFORME (Tuck.) Vain. Lich. Ins. Philipp. III, p. 194 (*Graphis subnivescens* Nyl. Fl. 1886, p. 174, Lich. Guin., 1889, p. 27). Apothecia aggregata aut confluentia, orbicularia aut difformia, margine albo, disco aperto, obscure cinerescente, tenuiter pruinoso. Perithecium album. Paraphyses simplices. Sporae 8:nae, monostichae, fumoso-fuscescentes, oblongae, apicibus obtusis, 3-septatae, loculis lenticularibus, long. 0.0011–0.013, crass. 0.005 mm.

Thelotrema platycarpoidi Tuck. affine, et ad species inter Thelotrema et Graphidem intermedias pertinet.

Ad corticem Mangiferae. La Seiva Valley (No. 31).

27. GRAPHIS (PHAEOGRAPHINA) CHRYSOCARPA (Raddi) Eschw.

Ad corticem arborum. La Seiva Valley (No. 52). Maraval Valley (No. 12).

28. GRAPHIS DISSIMILIS (Nyl.) Vain. (*Gr. sculpturata* f. *dissimilis* Nyl. Prodr. Fl. Novo-Granat. Lich. Addit., p. 564). Thallus KOH

rubescens. Perithecium tenue, latere fuligineum, basi albidum. Sporae binae aut raro solitariae, murali-divisae, obscuratae, long. 0.045–0.084, crass. 0.018–0.025 mm. Sporae saepe binae etiam in specim. orig. sec. annot. Nyl.

Ad corticem arborum. Maraval Valley (Nos. 13 and 15).

29. GRAPHIS (PHAEOGRAPHIS) HAEMATITES Fée.

Ad corticem arboris. La Seiva Valley (No. 71).

30. **Graphis Sangrensis** Vain. (sp. n.). Thallus bene evolutus, modice incrassatus aut sat tenuis, laevigatus, stramineo-glaucescens, leviter nitidus, KOH demum intense ferrugineo-rubescens, partim linea hypothallina nigricante limitatus. Apothecia elongata, long. circ. 4–7, lat. 0.3–0.4 mm., flexuosa curvatave, increbre dichotome vel varie ramosa, subsolitaria aut parce aggregata, leviter elevata aut subimmersa, apicibus obtusis aut attenuatis. Discus dilatatus, planus aut leviter concavus, cinereus, pruinosus, margine albo, discum vulgo superante, cinctus. Perithecium tenue, subpallidum et partim fuscescens, aut evanescens. Hymenium circ. 0.090 mm. crassum. Sporae 4:nae, cylindricae, apicibus obtusis aut rotundatis, fuscescentes, septis transversis 5, long. circ. 0.027, crass. 0.009 mm.

Affinis *Gr. medusaeformi* Kremplh. (Vain. Etud. Lich. Brés. II, p. 115). *Gr. inusta* Ach. sec. herb. Ach. thallo tenuissimo, KOH haud distincte reagente, apotheciis brevioribus, tenuiter pruinosis et sporis 5-septatis, 8:nis ab his distinguiter.

Ad corticem arboris. Sangre Grande (No. 88).

31. GRAPHIS TRICOSA Ach.

Ad corticem arboris. Maraval Valley (No. 93).

32. GRAPHIS LABYRINTHICA (Ach.) Vain. var. QUATUORSEPTATA Vain. Lich. Ins. Philipp. III, p. 230. Sporae in eodem apothecio 3–4-septatae, long. 0.015–0.017, crass. 0.007 mm., fumoso-obscuratae.

Ad corticem arboris (*Theobroma cacao*). Sangre Grande (No. 78).

33. GRAPHIS FÉEI (Messn.) Vain. l. c. p. 229. Sporae 3-septatae, long. 0.014–0.017, crass. 0.007 mm., fumoso-fuscescentes.

Ad corticem arborum (*Theobroma cacao* etc.). Sangre Grande (No. 82 pro parte). Maraval Valley (No. 24).

34. GRAPHIS DIFFORMIS Vain. l. c. p. 233. Vix nisi sporis 5-septatis a *Gr. heteroclitica* (Mont.) Vain. differt. Thallus, sicut etiam in specim. orig. KOH demum rubescens.

Ad corticem arboris (*Theobroma cacao*). La Seiva Valley (No. 53).

35. **Graphis regressa** Vain. (sp. n.). Thallus modice incrassatus, glaucescens, KOH lutescens deindeque ferrugineo-rubescens. Sporae 3-septatae, long. 0.016–0.017, crass. 0.006–0.007 mm.

Gr. difformis et *Gr. regressa* forsan sunt variationes *Gr. heterocliticae*, cui habitu similes sunt.

Ad corticem arborum (*Theobroma cacao*). Sangre Grande (No. 79).

36. **Graphis dilatescens** Vain. (sp. n.). Thallus modice incrassatus, sat laevigatus, glaucescens, leviter nitidus, KOH lutescens deindeque ferrugineo-rubescens. Apothecia long. 18–1 mm. lat. 1.5–0.7 mm., simplicia aut dichotome vel irregulariter ramosa vulgo leviter flexuosa, bene aut modice elevata, alt. 0.4–0.2 mm., apicibus vulgo rotundatis, lateribus praeruptis aut demum constrictis, margine sat crasso, albido, thallino, sicut thallus reagente, discum vulgo superante, saepe etiam zeorino, margine interiore tenui, albo, rima e margine thallino sejuncta. Discus apertus, planus, caesio-cinerascens, tenuiter pruinosus, transversim et longitudinaliter rimis et marginibus tenuibus albisque saepe tantum defecte et plus minus increbre divisus. Hypothecium crassum, fuscofuligineum. Parathecium fuscum, tenue aut partim evanescens aut ad superficiem apothecii deficiens. Epithecium fuscofuligineum. Sporae 8:nae, distichae, fumoso-fuscescentes, oblongae, subcylindricae, apicibus rotundatis obtusisve, septis transversis 5, long. 0.018–0.022, crass. 0.007–0.009 mm.

Proxima est *Gr. Labuanae* Nyl. (Vain. Lich. Ins. Philipp. III, p. 224), thallo crassiore et disco latiore ab ea differens.

Ad corticem arborum (*Theobroma cacao*). La Seiva Valley (No. 54). Sangre Grande (No. 80). No. 54 pertinet ad f. *constrictam* Vain. apotheciis magis elevatis, basi constrictis dignotam.

37. **Graphis** (Graphina) **Maravalensis** Vain. (sp. n.). Thallus tenuis, laevigatus, cinerascenti-albidus, parum nitidus, KOH non reagens, linea hypothallina nigricante limitatus. Apothecia long. circ. 4–1, lat. 0.45–0.25 mm., sat dispersa, curvata flexuosave, simplicia aut parce furcata, leviter prominentia, margine modice incrassato, KOH lutescente deindeque ferrugineo-rubescente. Perithecium album, labiis apertis, amphithecio thallino obductis, latere haud praerupto. Discus apertus, leviter impressus, pallidus, sat tenuiter pruinosus. Hymenium jodo dilute caerulescens. Epithecium albidum. Sporae 4–2:nae decoloratae, jodo violascentes, murali-divisae, cellulis numerosis, long. 0.040–0.060, crass. 0.014–0.021 mm., strato gelatinoso indutae.

Proxima est *Gr. subobtectae* Nyl. (Lich. Ins. Andaman, p. 18), quae apotheciis crebris, epithecio obscurato et sporis majoribus ab ea recedit. *Gr. affinissima* et *Gr. tetraphora* Nyl. habitu ab ea magis differunt.

Ad corticem arboris. Maraval Valley (No. 94).

38. **Graphis exsolvens** Vain. (sp. n.). Thallus modice incrassatus aut sat tenuis, sat laevigatus, glaucescens aut ad ambitum partim subroseus, opacus, nec KOH nec jodo reagens, fragilis, inferne strato hypothallino tenui, nigro crebre contexto, totus obductus, demum substrato saltem passim laxe adhaerens. Apothecia subdendroideo-aggregata aut dispersa, elongata aut brevia, long. 7–0.5, lat. 0.55–0.3 mm., dichotome ramosa aut simplicia, leviter prominentia, margine modice incrassato, KOH demum leviter subfulvescente. Perithecium album, labiis apertis, amphithecio thallino obductis, latere praerupto. Discus apertus, leviter impressus, planus, livido-pallescens, pruinosus. Hymenium jodo dilutissime caerulescens. Epithecium dilute olivaceum. Sporae solitariae, decoloratae, jodo caerulescentes, murali-divisae, cellulis numerosissimis, long. circ. 0.120, crass. 0.040–0.050 mm., membrana haud gelatinosa.

Habitu subsimilis *Gr. fissurinoideae* (Nyl.) Vain. (Fl. Koh Chang, p. 360), quae thallo jodo caerulescente ab ea differt.

Ad ramos arborum. Maraval Valley (No. 14).

39. **Graphis collosporella** Vain. (sp. n.). Thallus tenuis, laevigatus, albido-glaucescens aut cinerascenti-albidus, KOH lutescens deindeque fulvescens aut aurantiaco-subfulvescens arcte adnatus, linea hypothallina nigricante limitatus. Apothecia sat crebra, long. 2.2–0.4, lat. 0.25–0.3(–0.5) mm., leviter curvata fluxuosave, simplicia aut raro furcata, leviter prominentia, margine sat tenui, KOH lutescente deindeque aurantiaco-subrubescente. Perithecium album, labiis demum apertis, amphithecio thallino obductis, basim versus sensim dilatatis aut demum sat praeruptis. Discus demum apertus, angustus, lat. 0.15–0.2 mm., planus, leviter impressus, livido-pallescens, tenuiter pruinosus. Hymenium jodo dilute caerulescens. Epithecium impure pallidum. Sporae 4:nae aut binae, decoloratae, jodo caerulescentes, murali-divisae, cellulis numerosissimis, long. 0.050–0.062, crass. 0.014–0.020 mm., strato gelatinoso tenui aut sat crasso vulgo indutae.

Affinis *G. collosporae* Vain. (Addit. Lich. Antill. p. 153).

Ad corticem arborum. Maraval Valley (Nos. 9–10). La Seiva Valley (No. 97).

40. GRAPHIS (SCOLAECOSPORA) RUFULA Mont. (Vain., Lich. Ins. Philipp. III, p. 259).

Ad corticem arboris. St. Ann's Valley (No. 33).

41. **Graphis timida** Vain. (sp. n.). Thallus hypophloeodes, macula olivaceo-pallida indicatus, hypothallo nigricante partim limitatus. Apothecia sat crebra, long. 0.4–1.2 (–0.3) mm., lat. 0.3–0.4 mm., substrato erumpentia, vulgo recta, labiis subalbidis, conniventibus, fragmenta substrati continentibus aut substrato obductis, clausis aut demum rima angusta disjunctis, basim versus sensim dilatatis. Discus rimaeformis. Perithecium album. Sporae 8:nae, distichae, decoloratae, altero apice rotundato, altero obtuso, septis transversis 3, long. circ. 0.016, crass. 0.006 mm., jodo non reagentes (sed tantum novellae visae).

Gr. timidulam Nyl. (Flor. 1886, p. 174) facie externa in memoriam revocans.

Ad corticem arboris. Maraval Valley (No. 30).

42. GRAPHIS ANGUILLIFORMIS Tayl. (Vain. Addit. Lich. Antill. p. 156) var. **infecunda** Vain. (nov. v.) Sporis binis aut solitariis, long. 0.082–0.98, crass. 0.013–0.016 mm., 17–19-septatis a f. typica differt, ceterum ei etiam habitu omnino similis.

Ad corticem arboris. Maraval Valley (No. 92).

43. GRAPHIS SUBCAESIA (Nyl.) Vain. Lich. Ins. Philipp. III, p. 254. Thallus subcaesio-albidus, opacus, KOH non reagens (similis in specim. orig. n. 7163 in herb. Nyl.). Perithecium fuligineum, dimidiatum. Sporae 8:nae, decoloratae, septis transversis 10–6, long. 0.017–0.038, crass. 0.007–0.010 mm., jodo caerulescentes.

Ad corticem arborum. Maraval Valley (Nos. 11 and 17). Sangre Grande (No. 83).

44. CHIODECTON (STIGMATIDIOPSIS) SERIALE Ach.

Ad Corticem arboris. St. Ann's Valley (No. 55).

45. CHIODECTON (BYSSOPHORUM) CINERITIUM (Ach.) Vain. var. *corallina* Vain. Lich. Antill. Elliott (Journ. of Botany 1896), p. 29.

Ad corticem arboris. Verdant Vale, Arima (No. 61).

46. **Arthonia** (ARTHOTHELIUM) **candida** (Krempelh.) Vain. (comb. n.). *Myriostigma candidum* Krempelh. Lich. foliic. (1874), p. 22, Lich. Beccari Born. (N. Giorn. Bot. Ital. II, 1875), p. 45. Arthothelium Müll. Arg. Lich. Beitr. (Fl. 1890) no. 1544.

var. **hypocreoides** (Ferd. & Winge) Vain. *Myxotheca hypocreoides* Ferd. & Winge in Bot. Tidskr. XXX (1910), p. 212 (sec. Thaxter); Sacc. Syll. Fung. XXII (1913) p. 582. Thallus tenuis, continuus, albido-glaucescens, opacus, laevigatus, hypothallo albido partim anguste limitatus. Apothecia sat crebra, difformia aut subrotundata, lat. 1.5–0.5 mm., leviter prominentia, valde depresse subconvexa, alba, opaca, immarginata, submembranacea. Hypothecium albidum, tenue. Hymenium totum albidum, impellucidum, jodo caerulescens. Paraphyses ramoso-connexae ramosaeque, haud gelatinosae. Asci vulgo subglobosi, diam. circ. 0.064–0.080 mm., primum membrana bene incrassata. Sporae 8:nae aut 4:nae, decoloratae, oblongae, apicibus rotundatis aut subobtusis, saepe curvatae, murali-divisae, cellulis numerosissimis, long. 0.051–0.064, crass. 0.018 mm. Sec. Ferd. & Winge 0.064–0.074, crass. 0.018–0.020 mm. Gonidia ad Trentepohliam pertinentia, cellulis concatenatis aut pro parte simplicibus, globosis aut subellipsoideis, 0.008–0.010 mm. crassis, membrana sat tenui.

Ab *A. candida* (Krempelh.) et *A. cardinali* (Krempelh.) apotheciis minoribus differt et forsan est autonoma species.

In pagina superiore et inferiore Hymenophyllacearum et foliorum variarum arborum etc. Maraval Valley (No. 91) et vulgatim per insulam.

47. **Arthonia** (Euarthonia) **thamnocarpa** Vain. (sp. n.) Thallus sat tenuis, continuus, laevigatus aut parum distincte verruculosus, albidus, leviter nitidus, KOH lutescens, hypothallo indistincto. Apothecia long. 10–3 mm., dichotome repetito-ramosa, disco immerso, nigro, nudo, vix 0.1 mm. lato, immarginato. Hypothecium albidum. Hymenium praesertim latere jodo caerulescens, epithecium subolivaceo-fuscescens. Paraphyses apice incrassatae. Asci clavati. Sporae 8:nae, distichae, decoloratae, ovoideo-oblongae, apicibus rotundatis, 3-septatae, loculis sat aequalibus aut loculo apicis crassioris reliquis duplo longiore, membrana haud gelatinosa, long. 0.016–0.017, crass. 0.005–0.006 mm., jodo non reagentes.

Ab *A. dispartibili* Nyl. apotheciis multo longioribus et sporis minoribus differt.

Ad corticem arboris. Sangre Grande (No. 89).

48. Astrothelium conicum Eschw. (in Mart. Fl. Bras. Lich., 1833, p. 163) var. *pallida* Müll. Arg. (Pyr. Cub. 1885, p. 382).

Ad ramos arboris. Aripo Savanna, Cumuto (No. 1).

49. THELENELLA (PHYLLOBATHELIUM) EPIPHYLLA (Müll. Arg.) Vain. Etud. Lich. Brés. II (1890) p. 216.

Supra folia arboris. La Seiva Valley (No. 45).

50. **Thelenella Thaxteri** Vain. (sp. n.). Thallus cinereo glaucescens, nitidus, tenuis, primordiis pycnidiorum crebre inspersus, hypothallo nigricante partim limitatus. Apothecia sat crebra, verrucas hemisphaericas, basi haud constrictas, thallo obductas, formantia, saepe 2–3 confluentia aut simplicia. Perithecium fuscofuligineum, basi plus minus late albidum, KOH solutionem flavo-virescentem effundens. Nucleus depresso-subglobosus. Paraphyses breves, crass. 0.002 mm., simplices, periphysibus lichenum similes. Sporae 8:nae, murali-divisae, cellulis numerosis, demum pallidae, jodo non reagentes, long. 0.064–0.084, crass. 0.020–0.022 mm., membrana haud gelatinosa. Gonidia ramosa, cellulis 0.003–0.004 mm. crassis, ellipsoideis aut irregulariter oblongis.

Sporis majoribus et verrucis apotheciorum basi nec constrictis nec praeruptis a *Th. epiphylla* differt, sed forsan non est autonoma species.

Supra folia coriacea arboris. La Seiva Valley (No. 46).

Var. **heterogena** Vain. (var. n.). Ad corticem arboris crescens, vix nisi statione a *Th. Thaxteri* differt. Perithecium basi albidum, ceterum fuscofuligineum, KOH solutionem flavo-virescentem effundens. Paraphyses breves, periphysibus similes, simplices. Sporae in ascis numero variabiles, decoloratae, murali-divisae, cellulis numerosissimis, long. 0.078–0.102, crass. 0.018–0.03 mm., membrana haud gelatinosa, jodo non reagentes. Gonidia trentepohliacea, ramosa, cellulis globosis aut ellipsoideis aut irregularibus concatenatis, 0.003 (–0.006) mm. crassis, membrana sat tenui.

Ad corticem arboris. La Seiva Valley (No. 66).

51. **Thelenella** (MICROGLAENA) **elaeophthalma** Vain. (sp. n.). Thallus sat tenuis, continuus verruculis crebre inspersus, cinereo-glaucescens, nitidus, hypothallo nigricante limitatus. Apothecia sat crebra, verrucas formantia hemisphaericas, 0.8–0.6 mm. latas, basi sat praeruptas, haud constrictas, sat laevigatas, olivaceo- aut cinereo-glaucescentes, vertice circiter 0.2 mm. lato fusco-nigricante, haud prominente, instructas, ostiolo minutissimo. Perithecium ceterum albidum, tenue, strato thallino obductum. Paraphyses simplices crass. 0.0015 mm., gelatinam abundantem percurrentes. Sporae 8:nae distichae, decoloratae, jodo non reagentes, murali-divisae, cellulis numerosissimis, long. 0.065–0.090, crass. 0.024–0.026 mm., membrana 0.0025–0.006 mm. crass.

Facie externa Porinis similis.
Ad corticem arboris. La Seiva Valley (No. 65).

52. PSEUDOPYRENULA (TRYPETHELIUM) ANNULARIS (Fée) Müll. Arg. (Vain. Addit. Lich. Antill. p. 196).
Ad corticem arborum. Aripo Savanna, Cumuto, (Nos. 6 and 90).

53. PSEUDOPYRENULA (BATHELIUM) TROPICA (Ach.) Müll. Arg. (Vain. Etud. Lich. Brés. II, p. 210).
Ad corticem arboris. Maraval Valley (No. 27).

54. PSEUDOPYRENULA (POLYMERIA) CALOSPORA Müll. Arg. (Pyr. Cub. p. 409) var. **rhodocheila** Vain. (var. n.). Thallus sat crassus, glaucescens, leviter inaequalis. Apothecia immersa, margine ostiolari saepe annulum 0.2 mm. latum, haud aut leviter prominentem, roseum formante. Perithecium fuligineum. Paraphyses ramoso-connexae. Sporae decoloratae, elongatae, septis transversis 14–18 divisae, apices versus loculis diminutis, long. 0.090–0.160, crass. 0.018–0.024 mm.
Ad corticem arboris. La Seiva Valley (No. 63).

55. PYRENULA (MELANOTHECA) AGGREGATA Fée (Müll. Arg. Pyr. Feean. p. 18).
Ad corticem arboris. Maraval Valley (No. 26).

56. **Pyrenula** (EUPYRENULA) **Maravalensis** Vain. (sp. n.). Thallus hypophloeodes, macula olivaceo-pallida indicatus, hypothallo nigricante limitatus. Apothecia sat dispersa, verrucas formantia subconoideo- aut hemisphaerico-depressas, 0.6–0.5 mm. latas, nigras, nudas, leviter nitidas. Perithecium subhemisphaericum dimidiatim fuligineum, latere extus acutato, haud membranaceo dilatato. Sporae 8:nae, monostichae, fuscescentes, subfusiformes, apicibus obtusis, 3-septatis, loculis apicalibus valde minutis, long. 0.021–0.025, crass. 0.010 mm.
Ad corticem ramorum arboris. Maraval Valley (No. 20).

57. **Pyrenula novemseptata** Vain. (sp. n.). Thallus epiphloeodes, sat tenuis, sat laevigatus, substramineo-glaucescens, leviter nitidus, KOH subfulvescens deindeque rubescens, hypothallo nigricante partim limitatus. Apothecia sat crebra, 1–0.7 mm. lata, verrucas formantia conoideo-hemisphaericas, nudas, parum nitidas, margine ostiolari leviter conoideo-prominente. Perithecium conoideo-hemisphaericum, fusco-fuligineum, basi tenue. Sporae fuscescentes, fusi-

formes, apicibus anguste obtusis, septis transversis 7–9, apices versus loculis diminutis, long. 0.036–0.060, crass. 0.016–0.020 mm.
Ad ramos arborum. Aripo Savanna, Cumuto (No. 3).

58. PYRENULA ASPISTEA Ach. (Vain. Fl. Koh Chang Lich. p. 379).
Ad corticem arboris. Maraval Valley (No. 16).

59. PYRENULA SUBCONFLUENS Vain. (Etud. Lich. Brés. II, 1890, p. 202; Lich. Bras. Exsic. no. 470). Apothecia pro parte confluentia, strato thallino tenuiter velata. Perithecium integrum, hemisphaericum. Sporae 3-septatae, long. 0.021–0.024, crass. 0.010–0.011 mm., loculis apicalibus paulo minoribus.
Ad corticem arboris (*Theobroma cacao*). Sangre Grande (No. 76).

60. **Strigula difformis** Vain. (sp. n.). Thallus plagulas 8–27 mm. latas formans, laciniatus, laciniis glaucescenti-albidis, leviter nitidis, glabris, leviter verruculoso-inaequalibus, bene discretis, iteratim dichotome aut digitatim ramosis, leviter convexis, haud sulcatis, majore parte sublinearibus et 0.1–0.15 mm. latis, passim apicibus subspathulatis vel cuneato-dilatatis et 0.2–1.3 mm. latis, hypothallo indistincto aut dilute fumoso-obscurato tenuissimoque. Apothecia dispersa, neque numerosa, diam. 0.5–0.6 mm., verrucas subhemisphaericas formantia, thallino-velata, apice dilute obscurata. Perithecium dimidiatim fusco-fuligineum. Asci cylindrici. Sporae 4:nae, monostichae, oblongae aut fusiformi-oblongae, rectae aut leviter curvatae, apicibus obtusis aut subrotundatis, 1-septatae, constrictae, decoloratae, long. 0.012–0.015, crass. 0.0045–0.006 mm. Gonidia phycopeltidea, glabra, cellulis elongatis, 0.002 mm. latis aut angustioribus, in membranam connatis.

Str. argyronema Müll. Arg. (Pyrenocarp. Cub. p. 379) sporis majoribus et laciniis subsulcatis ab hac specie differt.
Supra folia coriacea arboris. Maraval Valley (No. 21).
Var. **Arimensis** Vain. (var. n.). Sporae 8:nae, monostichae, subfusiformes, apicibus obtusis aut sat acutis, rectae, saepe leviter constrictae, long. 0.014–0.015, crass. 0.0035–0.0045 mm. Non sit autonoma species.
Supra folia coriacea arboris. Verdant Vale, Arima (No. 58).

61. **Porina granulifera** Vain. (sp. n.). Thallus modice incrassatus, continuus, hypothallo nigricante limitatus, cinereo-glaucescens, nitidus, leviter verruculoso-inaequalis, demum verruculis minutissimis, subsorediosis, crebre inspersus, hypothallo nigricante limitatus.

Apothecia sat crebra, verrucas formantia mammaeformes, 0.7–0.4 mm. latas, demum basi praeruptas aut levissime constrictas, amphithecio thallino, thallo concolore, laevigato aut leviter verruculoso obducta, margine ostiolari nigro, maculam aut verruculam leviter prominentem, 0.2–0.15 mm. latam formante. Perithecium apice fuscescens, ceterum pallidum. Sporae fusiformes, apicibus acutis, decoloratae, septis transversis 3–7 divisae, loculis sat aequalibus, long. 0.032–0.036, crass. 0.005–0.006 mm., membrana haud gelatinosa.

Proxime affinis *P. isidiophorae* Vain. (Addit. Lich. Antill. p. 203) et *P. mastoideae* (Ach.) Vain. (Etud. Lich. Brés. II, p. 222).

Ad corticem arboris. Maraval Valley (No. 29).

62. PORINA EPIPHYLLA Fée var. PRAESTANS (Nyl.) Vain. (Lich. Ins. Philipp. III, p. 365).

Supra folia coriacea arborum. Maraval Valley (Nos. 22–23).

63. ARTHROPYRENIA (ANISOMERIDIUM) INFERNALIS (Mont.) Müll. Arg. (Lich. Beitr. in Fl. 1884, No. 879) var. **rhynchostoma** Vain. (var. n.). Margine ostiolari demum bene excrescente denudatoque a forma typica hujus speciei differens. Thallus modice incrassatus, cinereo-glaucescens. Perithecium fuligineum, integrum. Paraphyses ramoso-connexae. Asci clavati, long. circ. 0.200, crass. 0.040 mm. Sporae 8:nae, distichae, decoloratae, ovoideae, 1-septatae, cellula inferiore multo breviore, long. 0.039–0.050, crass. 0.018–0.020 mm.

Ad corticem arboris. Sangre Grande (No. 73).

64. **Didymosphaeria megalospora** Vain. (sp. n.). Apothecia vulgo dispersa, verrucas depresso-hemisphaericas demum formantia. Perithecium hemisphaericum, integre fuligineum, lat. circ. 0.6–0.8 mm., basi saepe anguste membranaceo-dilatato. Paraphyses crass. circ. 0.0007 mm., ramoso-connexae. Sporae binae, dilute fumoso nigricantes, oblongae aut ovoideae, rectae aut leviter curvatae, apicibus obtusis aut rotundatis, 1-septatae, medio saepe constrictae, long. 0.058–0.066, crass. 0.020–0.026 mm., membrana leviter incrassata.

Ad fungos pertinat, gonidiis destituta, sed passim parce Trentepohliae in cellulis substrati observantur.

Ad corticem laevigatum arboris. St. Ann's Valley (Nos. 99 and 68).

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THE EFFECT OF PRESSURE ON THE ELECTRICAL RESISTANCE OF COBALT, ALUMINUM, NICKEL, URANIUM, AND CAESIUM.

By P. W. Bridgman.

THE EFFECT OF PRESSURE ON THE ELECTRICAL RESISTANCE OF COBALT, ALUMINUM, NICKEL, URANIUM, AND CAESIUM.

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Introduction.

In former numbers of the Proceedings of the American Academy [1] I have given data for the effect of pressure on the resistance of 39 of the elements. It is my intention to keep this work as up-to-date as possible by the revision of the previous data or the inclusion of new substances; this paper is in the nature of such a supplement to the former work. The first three of the metals above have been previously investigated, but I have now been able to obtain them in a state of considerably higher purity than formerly, so that a redetermination of the pressure effect was worth while. The pressure coefficients now found for these purer samples are of the order of 10% higher than the previous values. Measurements on uranium have not been made before; the specimen I had was presumably not of very great purity, but since even the sign of the coefficient was not known, and since as a general rule impurity does not greatly affect the pressure coefficient, it was of interest to make and record the measurements. The work on ceasium is not yet complete, but enough has been done to establish the existence of surprising phenomena for this element, and since it may be some time before I shall be able to complete the work, it seemed that announcement of the chief fact should not be delayed. I have found that caesium has a new polymorphic modification under high pressure, of smaller volume than the ordinary modification, but that the resistance of this new modification increases instead of decreases under pressure. Since caesium is the most compressible of the metals, this result may be of significance in suggesting what may happen to all metals at pressures sufficiently extreme.

Detailed Data.

Cobalt. The former sample of cobalt was obtained from Dr. Herbert T. Kalmus who had prepared it in the course of an investigation of the properties of metallic cobalt for the Canadian Government. This was one of the impurer of the samples which he prepared, but since it was the only one he had left, I was glad to have the chance to make the measurements. I have now obtained indirectly, through the kindness of Professor C. C. Bidwell of Cornell University, one of the purest of Kalmus's specimens. Since the preparation of cobalt in the form of wire is a matter of some difficulty, involving the swaging instead of drawing of the metal in the early stages, and since there seems to be no commercial use for pure cobalt, it is not likely that material other than that of Kalmus will be available for some time.

The analysis of this sample was: Co 99.73, Fe 0.14, Ni 0.00, S 0.019, Si 0.02, and C 0.09.

The total impurity is seen to be 0.27%, against 1.30% of the former sample.

The diameter of this piece was 0.10 cm. and the length about 6 cm. Its compressibility had been previously measured; for this purpose it had been annealed and straightened by rolling between red hot iron plates. For these measurements of resistance, current and potential terminals were soft soldered near the ends, and measurements made with the potentiometer. The technique was exactly like that used with those metals formerly measured with the potentiometer, and has been fully described in the previous paper. Because of the low resistance of this sample and the smallness of the pressure coefficient, the individual readings did not have as great regularity as those on the previous sample, and I did not think it worth while to try for the temperature coefficient of the pressure coefficient by making readings at different temperatures.

Measurements were made at 30° over the pressure range of 12000 kg/cm^2. The accuracy was not high enough to detect departure from linearity with pressure. Discarding one point, the average deviation of the observed points from a straight line was 2.0% of the maximum pressure effect. The average pressure coefficient over the range of 12000 kg. was -0.0_6934, against -0.0_6865 of the former sample. The purer sample has the numerically larger coefficient. This seems to be true in the majority of cases, although there is no general rule here as in the case of the temperature coefficient.

The resistance was measured at 30° and 75° at atmospheric pressure. Assuming the relation between temperature and resistance to be linear, as it was for the former sample, the temperature coefficient between 0° and 100° is 0.00439, against 0.00365 for the other sample. The highest value which I find listed for cobalt is 0.0033.[2] The higher coefficient of the new sample is what would be expected because of its greater purity.

Aluminum. This was a specimen of quite unusual purity, of the following analysis: Si 0.014, Fe 0.007, Cu 0.003, Al (by diff.) 99.976. I extruded it from a diameter of 1.27 cm. to 0.081 cm., from this diameter drew it through steel dies to a diameter of 0.051 cm., annealed it at a temperature of 350°, wound it non-inductively on one of the bone cores used in previous work, made connections to the current and potential leads with spring clips at the two ends, and made the measurements in the regular way on the potentiometer. The length was such as to give an initial resistance of about 0.2 ohms.

The regular procedure in making the measurements was followed. Several preliminary seasoning applications of pressure were made, and runs were made at 0°, 30°, 50°, and 95°. The run at 75° failed because of a minor accident. At 0° the average arithmetical deviation of the observed points from a line (no discards) was 0.23% of the maximum pressure effect, at 30° (one discard) 0.13%, at 50° (two discards) 0.6%, and at 95° (one discard) 0.17%. The deviation from linearity was symmetrical about the mean pressure within the limits of error, so that the changes of resistance can be represented by a second degree expression in the pressure. In addition to these pressure measurements, the temperature coefficient at atmospheric pressure was determined by measurements at 0°, 30°, 50°, 75°, and 95° in this order, and then the measurements at 50° and 0° were repeated. The repeated readings agreed with the original ones within the limits of setting the slider of the bridge wire, showing no temperature hysteresis.

TABLE I.

ALUMINUM.

Temp. °C.	Resistance	Pressure Coefficient			Maximum Deviation from Linearity	Pressure of Maximum Deviation
		At 0 kg.	At 12000 kg.	Average 0–12000 kg.		
0	1.0000	$-.0_54489$	$-.0_53954$	$-.0_54128$	−.00111	6000
25	1.1167	4418	4043	4135	95	6000
50	1.2334	4364	4087	4129	86	6000
75	1.3501	4336	4029	4088	103	6000
100	1.4668	4278	3879	3990	126	6000

The results are collected in Table I and Figure 1. The method of presenting these results is the same as that of the previous papers. The average pressure coefficient between 0 and 12000 kg. is that number which multiplied by 12000 gives the change of resistance produced by 12000 kg. pressure as a fractional part of the resistance at atmospheric pressure and the temperature in question. The instantaneous coefficients at 0 and 12000 kg. are $\frac{1}{R}\left(\frac{\partial R}{\partial p}\right)_\tau$, where R is the resistance at the pressure and temperature in question. The maximum deviation from linearity is in fractional parts of the resistance at 0° and atmospheric pressure. As an example, suppose that it is required to find the resistance of aluminum at 50° at 6000 kg. in terms of its resistance at 0° and atmospheric pressure as unity. The average coefficient at 50° to 12000 is -0.0_54129, and the initial resistance at 50° is 1.2334. If the change of resistance with pressure were linear, the decrease of resistance under 6000 kg. would be $1.2334 \times 6000 \times 0.0_54129$ or 0.03056. But the change of resistance is not linear, but as the sixth column shows, there is a deviation from linearity at 6000 of 0.00086, giving for the total decrease of resistance under 6000 0.03142, and for the actual resistance at 50° under 6000, 1.2020.

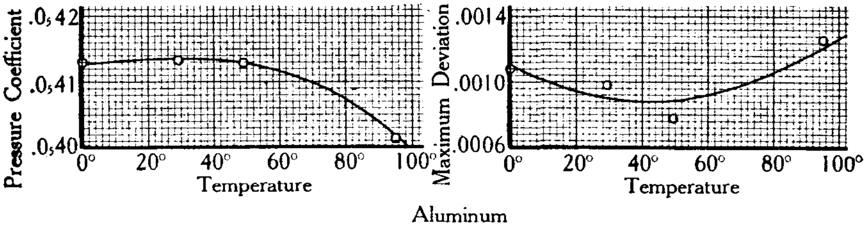

FIGURE 1. Aluminum, results for the measured resistance. The deviations from linearity are given as fractions of the resistance at 0 kg. and 0° C. The pressure coefficient is the average coefficient between 0 and 12000 kg.

Compared with the results for the previous sample, the pressure coefficient of this is in general higher by eight or nine per cent. As a function of temperature, the pressure coefficient of this new sample shows a very flat maximum near the lower end of the temperature range, and from here on decreases. The pressure coefficient of the other sample decreased linearly over the entire temperature range. A decrease of the coefficient with increasing temperature is not what one might at first expect, but its reality seems vouched for by independent measurement on two different samples. The maximum deviation from linearity of this new pure sample is not as simple as that of the less pure one; the former decreased linearly with rising temperature, whereas this new sample shows a minimum.

The temperature behavior at atmospheric pressure is worth comment. In the first place, the strict linearity with temperature is not usual. In the second place, the high value of the temperature coefficient of this piece as compared with the former sample, or the values of other observers, is to be noticed. The establishment of this high value removes aluminum from an apparently unique position with regard to its temperature coefficient. It is well known that practically all metals have a higher temperature coefficient in a condition of higher purity. Aluminum appeared to be the only exception, since the published values for samples of increasing purity have apparently become smaller with increasing purity.[3] But the purest of these previous samples had about 0.1% impurity. It is now seen that the apparent anomaly disappears in the region beyond 0.1%, the initial trend being reversed, and at sufficiently high purities the temperature coefficient increases with increasing purity, as it does for all other metals, and furthermore, the coefficient of the purest metal is higher than that of any of the impurer specimens.

Nickel. This was of exceptionally high purity, and I owe it to the kindness of Mr. I. B. Smith, of the Research Laboratory of Leeds and Northrup Co. I have already published data for the effect of tension on resistance,[4] and pressure on thermal conductivity [5] of this same nickel. Data for the compressibility are to be published shortly in These Proceedings. Hitherto it has not been possible to obtain in this country nickel of purity higher than that corresponding to a temperature coefficient of resistance of 0.0049. The coefficient of this was 0.00634, and is higher than any other published value except 0.00683 by Niccolai.[6] Except for the value of the temperature coefficient, I have no chemical analysis to indicate the purity.

The material was drawn by Leeds and Northrup to wire 0.0127 cm. in diameter, was annealed to redness, and then was double covered with silk insulation by the New England Electrical Works. It was wound non-inductively on a small glass core, seasoned at 135° for six hours, and subjected to three preliminary applications of 12000 kg. The initial resistance at 0° was about 115 ohms. The measurements were made on the Carey Foster bridge regularly used in measuring the changes of resistance of samples with high resistance.

The regular series of pressure measurements was made, at 0°, 25°, 50°, 75°, and 99°, and at the same temperature intervals there were also made two sets of readings of resistance as a function of temperature at atmospheric pressure, which agreed within the sensitiveness of setting the slider of the bridge. The accuracy of the pressure readings was as follows: at 0° the average arithmetical departure from a smooth curve (no discards) was 0.41% of the maximum pressure effect, at 25° (no discards) 0.32%, at 50° (one discard) 0.16%, at 75° (no discards) 0.34%, and at 99° (no discards) 0.24%. The average departure from linearity at the maximum was 0.9% of the maximum pressure effect.

TABLE II.

NICKEL.

Temp. °C	Resistance	Pressure Coefficient			Maximum Deviation from Linearity	Pressure of Maximum Deviation
		At 0 kg.	At 12000 kg.	Average 0−12000 kg.		
0	1.0000	$-.0_51880$	$-.0_51819$	$-.0_51830$	−.000149	6000
25	1.1443	1899	1824	1843	198	6000
50	1.2975	1915	1827	1851	248	6000
75	1.4607	1925	1830	1858	297	6000
100	1.6345	1934	1832	1859	347	6000

The numerical results of the measurements are reproduced in Table II and Figure 2. The method of computation and presentation is the same as that used in the preceding papers.

Compared with the previous results on less pure nickel, the pressure coefficient of this is on the average about 15% higher, again verifying the observation that in most cases impurity depresses the pressure coefficient, but by a less amount than the temperature coefficient. The temperature coefficient of this piece is 0.00634, against 0.00487 of the previous sample, or an increase of 30%. The pressure coefficient of this new sample increases with rising temperature, as did that of the other sample, but the increase is much less rapid, and is not linear, becoming less rapid at the higher temperatures. The deviation from linearity of this new sample increases linearly with rising temperature, whereas that of the less pure sample at first passed through a flat minimum, and then increased at the higher temperatures. The departure from linearity of this new specimen is furthermore symmetrical about the mean pressure, so that it can be represented by a parabolic formula, whereas that of the former sample was unsymmetrical, and there was a progressive shift of the pressure of maximum deviation with rising temperature.

Uranium. I am indebted to the kindness of Dr. A. W. Hull of the General Electric Company for a sample of this metal. This had been prepared with considerable difficulty for an investigation of its properties with a view to a possible commercial use. It turned out that its properties were not promising, so that no more will be prepared. This sample is probably unique, and I was most fortunate to obtain it. It was furnished in the form of rolled sheet, about 0.05 cm. thick, 0.7 cm. wide, and 4.5 cm. long. In this form the compressibility was measured, and I am reporting the results in another place. There were flaws in the specimen, which did not affect the compressibility measurements, since it was exposed to pressure all over, but which would have interfered with measurements of its electrical resistance. A small homogeneous sliver was cut from the sheet about 0.79 cm. wide and 2.2 cm. long. Uranium is very difficult to work, and the sliver had to be cut with a steel disc charged with diamond powder; I am indebted to the skill of Mr. David Mann for preparing the specimen.

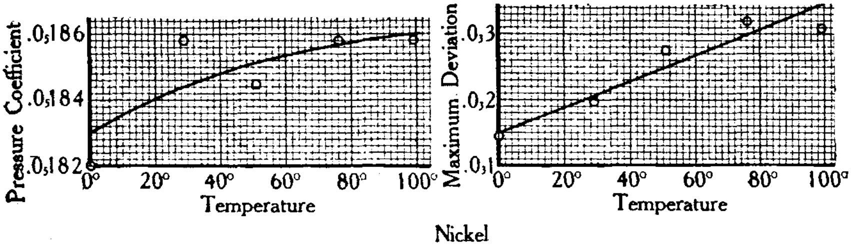

FIGURE 2. Nickel. Results for the measured resistance. The deviations from linearity are given as fractions of the resistance at 0 kg. and 0° C. The pressure coefficient is the average coefficient between 0 and 12000 kg.

The specimen was mounted for measurement with the potentiometer. It is not possible to solder leads to it. Fine grooves were filed around each end of the specimen, and a special spring clamp arrangement made by which the current and potential leads were pressed tightly into the grooves. The potential terminals were 1.43 cm. apart, and the current terminals 2.0 cm. The small resistance of the specimen and the nature of the connections prevented measurements of the highest accuracy. Furthermore, it is probable that this specimen was not of high purity, because its temperature coefficient of resistance between 0° and 100° was found to be only 0.00230. In view of the impurity and somewhat low accuracy I made no attempt to find the temperature coefficient of the pressure coefficient, but made the pressure run at a single temperature, 30°. The resistance was found to decrease under pressure, as is normal for most metals. The sign of the pressure coefficient was the most important fact to be established by the measurement. The position of uranium as the heaviest of the elements at the end of the periodic table would have given particular interest to a possible positive pressure coefficient of resistance. The average arithmetical deviation of the observed points from a smooth line (no discards) was 1.6% of the maximum pressure effect. It was not possible to detect any deviation from the linear relation between pressure and resistance, and the average pressure coefficient over the range 0 to 12000 kg. was found to be $-.0_5436$.

The specific resistance of uranium seems not to be recorded in the literature. The specific resistance of this sample at 0° was 76.0×10^{-6}, which is high for a metal, being of the order of magnitude of the resistance of liquid mercury or bismuth.

Caesium. As already mentioned in the introduction, the results on caesium are preliminary, but because of their interest it seems worth while to briefly describe them. The preparation of pure caesium and its manipulation requires some practise, and I have not yet achieved final success. My original purpose in measuring the resistance of caesium was to search for a more pronounced drop in the temperature coefficient of resistance at high pressures than was found in the case of potassium. It would be expected that there would be such a phenomenon here because of the chemical similarity of caesium and potassium, and because of the much greater compressibility of caesium. This search failed, however, because of the entrance of a new polymorphic form at high pressures.

The material for the measurements was obtained from the Foote Mineral Co. of Philadelphia. I am also indebted to the kindness of Professor Baxter and Professor G. N. Lewis for other samples, but I was not successful in the manipulation of these. The two samples from the Foote Mineral Co. were provided in glass tubes sealed under oil. One of the samples was apparently somewhat purer than the other; its melting point appeared to be sharper and its yellow color not so pronounced. These samples were transferred to capillaries of thin glass provided with four platinum terminals sealed through the glass, two for current and two for potential leads. The measurements were made in the regular way on the potentiometer. The details of the method of transferring to the capillary need not be described; it is possible to improve it.

Measurements were made on the first and purer sample with a view to establishing the temperature coefficient over the range 0 to 12000 kg. In order to avoid melting at atmospheric pressure, the temperature range of the measurements was low, from 0° to 16°. Over this range of pressure and temperature the most puzzling results were found. At low pressures the resistance apparently decreased greatly with increasing pressure, which was the result expected, but at higher pressures the resistance increased again. After a day's run, the apparatus was taken apart in the search for trouble, and the glass capillary was found cracked, so that air got at the caesium and destroyed it. The crack in the capillary had no effect on the measurements, however. The temperature coefficient of this first and purer sample between 0° and 16° was found to be 0.0054, which is high, and evidence of good purity.

The second sample, as already stated, seemed to be less pure initially, and during the manipulation received further impurity because of the accidental access of a small quantity of air to the inside of the apparatus during the transfer to the capillary. This heightened the yellow color and dirtied the metal so that it left a yellow scum on the glass, whereas the first sample had run as cleanly through the glass as clean mercury. However, the impurities introduced in this way were obviously non-metallic, and there is no reason to expect that this impurity introduced any essential change in the phase relations of the different polymorphic forms.

With the information received from the first sample it was possible to direct the measurements on the second sample much more intelligently, and to definitely establish the more important features. In the first place it was established that caesium has a new modification at high pressures, that the transition point is sharp, as it should be for a true transition, and that the transition pressure varies with the temperature. Two measurements were made of the transition pressure at 0°. The first of these established that the equilibrium point was contained between two pressure limits 140 kg. apart, and that the mean point was 1960 kg. The second measurement shut the transition within limits 100 kg. apart, at a mean pressure of 1990 kg. We take as the most probable transition pressure at 0°, 1980 kg. At 17° the transition was shut between two limits 400 kg. apart, mean 3260 kg. The transition curve is therefore of the normal type, rising to higher temperatures at higher pressures, and the phase stable at the higher temperature has the greater volume. Unless the transition line has exceptionally great curvature, it should be possible to find the new modification at atmospheric pressure at some temperature below 0°; linear extrapolation gives −26° as the atmospheric transition temperature.

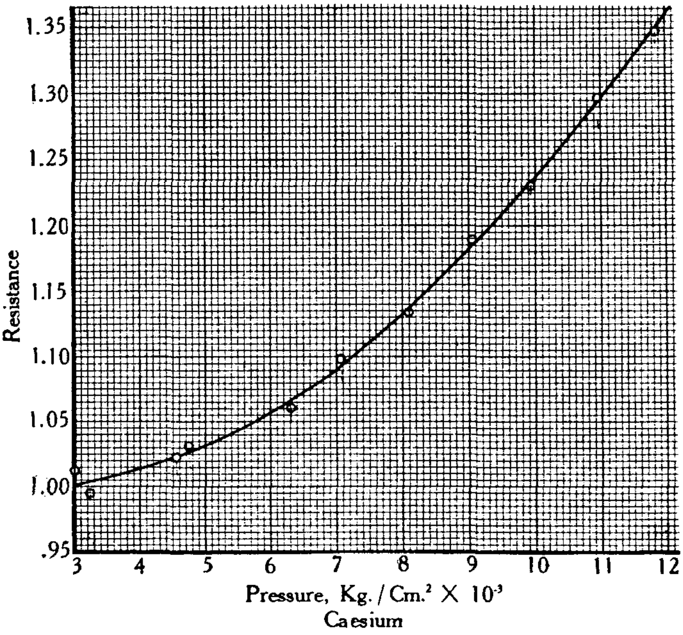

FIGURE 3. Caesium. The resistance at 0° C. of the new high pressure modification of caesium as a function of pressure in terms of its resistance at 0° C and 3000 kg/cm² as unity.

The electrical properties of the high pressure modification were found to be abnormal in that the resistance increases with increasing pressure. This makes the sixth metal now known with this property, the others being bismuth, antimony, calcium, lithium, and strontium.

It was possible to make fairly satisfactory measurements of the resistance of the new modification at 0° from the transition pressure out to 12000 kg. On the first application of pressure the results were irregular. This is due to the cracking of the glass capillary under pressure and the consequent change in the geometrical configuration of the caesium due to its extreme softness, but after the seasoning produced by the initial application of pressure the results were regular and would repeat. (The results obtained with the first sample were irregular for the same reason, but no results were found with the first sample that were inconsistent with those given by the second.) The experimental results are reproduced in Figure 3 plotting the measured resistances in terms of the resistance at 3000 kg. as unity. The curve shown was obtained with increasing and decreasing pressure; the points alternately are those with increasing or decreasing pressure. The curve is the same in character as that obtained for the five other abnormal metals in that the curvature is upward, or the resistance increases at a continually increasing rate at the higher pressures. This is most important as suggesting what the mechanism of conduction may be, and is just what would be expected on the basis of theoretical considerations which I have already described. The magnitude of the pressure coefficient is high; between 11000 and 12000 kg., the average pressure coefficient of resistance is 0.000493, which is about the same as that of strontium, which has the highest coefficient of the abnormal metals hitherto measured.

At the transition point there is a discontinuity in resistance in the normal direction, that is, the high pressure phase, or the phase with the smaller volume, has the smaller resistance. I know of no exception to this rule. At 0° the resistance of the high pressure modification at the equilibrium pressure is 0.407 that of the low pressure form.

I am indebted to my assistant Mr. I. M. Kerney, for help in making many of the readings.

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THE COMPRESSIBILITY OF THIRTY METALS AS A FUNCTION OF PRESSURE AND TEMPERATURE.

BY P. W. BRIDGMAN.

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THE COMPRESSIBILITY OF THIRTY METALS AS A FUNCTION OF PRESSURE AND TEMPERATURE.

By P. W. Bridgman.

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INTRODUCTION.

Our present knowledge of the compressibility of metals is mostly due to the work of Richards[1] and his collaborators, and to Adams Williamson and Johnston.[2] (I shall abbreviate reference to these authors by A. W. J.) Richards has measured the compressibility of a large number of metals over a pressure range of 500 kg. at room temperature. A. W. J. have measured the compressibility of a smaller number of metals over the much wider pressure range of 10000 kg/cm^2, but again only at room temperature. The work of Richards, therefore, does not enable us to find either the pressure or the temperature variation of compressibility, and although the work of A. W. J. gives valuable information as to the variation of compressibility with pressure, they have themselves recognized that the pressure variations so found are not accurate.

Recent theoretical work, in particular that of Born[3] is now bringing within the reach of the possibility of computation the compressibility of substances in terms of their crystalline structure. Born's theory of the compressibility of substances of the type of sodium chloride is far enough advanced to give an expression for the variation of compressibility with pressure. It seems therefore that the time is ripe for a more careful experimental examination of the question of the compressibility of the metals, although we may not have as yet a satisfactory theory of the metallic state itself.

The experimental work of this paper consists of a determination of the compressibility of 30 metals over a pressure range of 12000 kg/cm^2 at 30° and 75°. This range is sufficient to give the pressure and temperature coefficients of the compressibility.

The question of experimental accuracy is an important one here. It is well known that compressibility is one of the harder quantities to measure experimentally, so that all the more is it difficult to measure the pressure or the temperature coefficient of compressibility. The use of high pressure is indispensable here, for by increasing the magnitude of the effects to be measured it is possible to attain the necessary accuracy. This is particularly true with respect to the pressure coefficient of compressibility; the accuracy with which this may be determined increases as the square of the pressure range. Other things being equal, therefore, it should be possible with the pressure range of this work to determine the pressure coefficient 576 times as accurately as possible over a pressure range of 500 kg.

A new method of measuring compressibility has been developed for this work, which has very considerable advantages, both of speed and accuracy, over previous methods. As compared with previous methods, the accuracy of this is much increased by the fact that all the corrections are very much less. The method essentially involves the measurement of the difference of linear compressibility between the substance in question and pure iron. If the absolute linear compressibility of iron is known, we have at once the linear compressibility of the substance in a definite direction, and if equal compressibility in all directions is assumed, the true volume compressibility may be computed. It is to be noticed that over the pressure range used here, the compressibility cannot be found by simple multiplication by the factor three, but a correction has to be applied, which in some cases may rise to the order of ten per cent.

The assumption that the substance is equally compressible in all directions is applicable only when the material is amorphous or when it belongs to that one of the crystalline systems enjoying this property, namely the cubic. This assumption is true for the majority of metals, but it is definitely not true for several, which may have very materially different compressibilities in different directions. For these substances it is necessary to measure the linear compressibility in several directions in order to get the true cubic compressibility. This is at once a disadvantage and an advantage; the determination of the average cubic compressibility becomes more complicated than by the methods employed by Richards or A. W. J., but on the other hand the average cubic compressibility is not a datum of much significance in these cases, whereas it is possible by the use of the present method to obtain a complete description of the behavior under pressure. For substances of this type it is necessary to have single crystals. The growth of such crystals involves a technique in itself; in this paper only the beginnings of the attack on this subject are made. I have shown for a number of non-cubic crystals that there may be very great differences of linear compressibility in different directions, and have made a promising beginning at a method of obtaining metals with a uniform crystalline orientation throughout the entire mass. The further examination of this important question must be left for future work.

With regard even to those metals which are known to crystallize in the cubic system it is not safe to assume without some examination that the linear compressibility is the same in every direction, for few masses of metal are unicrystalline, but there are also regions of amorphous cementing material. In a number of cases I have tried, therefore, to justify this assumption. A number of years ago I measured the compressibility of commercial iron in the form of boiler plate along and across the direction of rolling,[4] and could find no difference in the linear compressibility. I have now repeated the same examination for rolled copper, and again could find no difference. Also, in several cases I have measured the compressibility, first in the shape of the unworked casting, and then after the casting had been extruded to a considerably smaller diameter, so that the crystalline grains were much distorted by working, and presumably considerable amorphous material was introduced. The general result of this work is that no perceptible change of linear compressibility is produced in this way, and it is highly probable that the assumption is correct that the ordinary cast or worked forms of those metals that crystallize in the cubic system have the same compressibility in every direction. The correctness of this assumption may further be checked by a comparison of the results obtained by my method with those obtained by the other methods, which are essentially methods for measuring the cubic compressibility.

The absolute linear compressibility of iron has been previously measured by me at two different temperatures.[5] I did not succeed in finding any departure of the compressibility from constancy over a pressure range of 10000 kg., but I did find an increase of compressibility at the higher temperature. This early work was unsatisfactory to me for two reasons. In the first place the theoretical considerations of Born would suggest that the departure of compressibility from constancy at high pressures should be large enough to measure even for a metal as little compressible as iron, and in the second place I could not help feeling that my temperature coefficient was too high. I therefore have made a fresh experimental attack on this question, greatly improving the experimental method, and have been able to definitely find and measure with some accuracy the change of compressibility of iron with pressure, and to show that the temperature coefficient is several times smaller than previously found.

This experimental work falls into two parts; first the redetermination of the absolute linear compressibility of iron, and second the determination of the relative linear compressibility of iron and 29 other solid metals.

THE ABSOLUTE COMPRESSIBILITY OF IRON.

The method adopted is in many respects like that previously used. An iron rod approximately 30 cm. long is placed inside a heavy cylinder and exposed to hydrostatic pressure exerted by a fluid filling the cylinder. The relative change of length of the rod and the cylinder is measured, and at the same time the change of length of the cylinder is measured at outside points. From these two the absolute change of length of the iron rod under pressure may be computed. The only assumption is that the change of length of the cylinder externally, where it can be measured, is the same as that internally, where it is in contact with the rod. This assumption is entirely justified if the cylinder extends some distance beyond the end of the rod, so that warping of the cross section due to end effects is negligible, and in any event the assumption is of little importance because the correction for the change of length of the cylinder is only a few per cent of the entire effect.

The essential difference between the old and the new method is in the means of measuring the relative change of length of the rod and the cylinder. Previously the rod carried a collar which bore against a fixed shoulder in the cylinder and whose relative displacement on the rod was measured after every application of pressure. This involved a complete setting up and disassembly of the apparatus for a single reading at any pressure. The present method is a continuous reading one. Attached to the end of the rod is a fine wire of high resistance alloy. This wire slides over a contact fixed to the cylinder. The sliding contact is made one potential terminal and another potential terminal is attached to a point of the wire. Current is passed through the wire independently of the potential terminals, entering the wire at one end and leaving at the other. The difference of potential between the fixed and the sliding point on the wire is measured for a given current, so that the resistance and hence the length of the wire between terminals may be calculated, and hence the relative motion of the rod and the cylinder. The reading is continuous, and after every change of pressure a new reading may be obtained in a few minutes, or as soon as the heat of compression is dissipated. Further, there is no error from the constant possibility of introducing minute particles of dirt every time the apparatus is opened, as there was previously.

The essentials of the apparatus are shown in Figure 1. The high resistance wire B (which is of nichrome) is mounted in a steel plunger A which is kept pressed by a spring against the upper end of the bar at S. The wire B slides over a contact at C which is attached to the plate D insulated by mica washers at E from the rest of the cylinder. At F the second potential terminal is soldered to the wire B, and at the end G is attached the current terminal. The other terminal is the cylinder itself, to which the lower end of B is grounded. The three wires from C, F, and G are attached at the upper end of the cylinder to a three terminal plug, of exactly the same design as used in previous work on electrical resistance under pressure. The plate D is kept rigidly in position by springs shown in section, which are compressed as the three terminal plug is screwed into place. There are various mechanical details not shown in the drawing to facilitate the attaching of the connections and the general assembly. The specimen S is fixed with respect to the cylinder at the lower end. S ends in a disc which is tightly pressed against a shoulder of the cylinder by a spring (which of course must be stiffer than the spring around A) which is compressed by the connections to the pressure producing apparatus at the lower end of the cylinder.

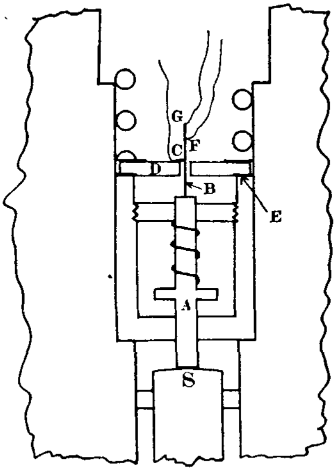

FIGURE 1. The sliding electrical contact device for measuring the absolute linear compressibility of iron.

The resistance between the points C and F was measured on the same potentiometer as was used in previous measurements of the effect of pressure on resistance, and needs no further description. The sensitiveness of the electrical arrangements was such that a motion of the wire of 1.5×10^{-6} cm. could be detected. Of course there were always extraneous disturbances, so that an accuracy corresponding to this sensitiveness could not be obtained, but it was nevertheless possible to measure very small motions with a highly gratifying accuracy; these measurements would have been entirely impossible without some such method. This method of measuring small displacements would seem to be applicable in a number of other places.

The external change of length of the cylinder was measured somewhat differently from formerly. This was done previously with a microscope, and there were difficulties at the higher temperature (50° in the former work). The arrangement of the pressure producing part of the apparatus was such that it was now much more convenient to use the rocking mirror device familiar to many engineers. A mirror was attached to a diamond sectioned staff, which could rock between two bars, one attached to the upper end and the other to the lower end of the cylinder. The motion of the mirror was read with a telescope and scale, and the elongation of the cylinder could at once be calculated in terms of the dimensions of the various parts. The conventional devices of three point contacts and springs to ensure freedom from back-lash, etc., were used, and the apparatus functioned satisfactorily in every particular. Two measuring devices were used, attached at opposite sides of the cylinder, so that any error due to any slight Bourdon spring action of the cylinder under pressure (there usually is such action in appreciable amount) was eliminated. The method of mounting the mirrors at a distance of 30 cm. or so from the cylinder and transmitting the motion to them by bars allowed the cylinder to be placed in a temperature bath, and the temperature to be controlled by a thermostat and stirrer, as in all these experiments. The stirring of the water produced only a slight mechanical shaking of the mirrors, too slight to introduce perceptible difficulty in making the readings.

It is obvious that there are various corrections to be applied to the readings as directly obtained. The most important of these is for the effect of pressure on the resistance of the nichrome wire B. This was determined by direct experiment, and is very low; in fact this, as well as the high specific resistance, was the reason for choosing this material for the wire. The resistance between terminals fixed to the wire was found to decrease linearly with pressure, the decrease for 12000 kg/cm^2

being 0.45%. This is much less than for any of the pure metals, and is seven times less than for manganin. It is possible to find other nickel-chromium alloys that have an even smaller coefficient, but this was so small that it was not necessary to go out of the way for other material.

Other corrections, not as important as the above, are for the temperature coefficient of resistance of nichrome wire, and for the difference of compressibility between the wire and iron. Both these corrections were determined by direct experiment; the determination of the compressibility will be described later.

Although there are several corrections to be applied, it is a unique advantage of this method that the corrections are very small, and that the greater part by far of the measured effect is the final effect that is wanted. The corrections just discussed for the change of resistance of the nichrome wire, etc., did not attain as much as 2% of the measured effect, and the correction for the external change of length of the cylinder was only 1.5%. On the other hand it frequently happens that the corrections in such methods as used by Richards or A. W. J. may be two or three times as large as the final effect wanted.

It is evident that the exact uniformity of cross section of the wire B is of vital importance if the changes of resistance are to give slight departures of compressibility from linearity with pressure. To test the uniformity of the wire a special arrangement was made consisting of two knife edges a fixed distance apart, mounted to slide along the wire, which was stretched along a meter stick. A constant current was passed along the wire, and the difference of potential between the two knife edges measured on the potentiometer. The wire was exceedingly uniform. In the first length tried, the extreme variation in a length of 20 cm. was less than 1/3000. The variation in a length of a millimeter or less was of course much smaller, and was obviously far too small to introduce any appreciable error.

The iron used for the measurement of linear compressibility was cut from a bar of American ingot iron. This iron is very pure, containing only 0.03% impurity, and was considerably purer than the piece of boiler plate whose compressibility was previously measured. The resistance under pressure of a piece cut from this same material has been previously measured.[6] The dimensions of the compressibility specimen were approximately 6 mm. in diameter and 30 cm. long. It was annealed at a red heat after machining.

The containing cylinder was especially made for this experiment and was of chrome vanadium steel from the Halcomb Steel Co., sold by them under the name of Type D. After machining it was heat treated by quenching from 930° for a few seconds in water and then finishing the quenching in oil. It was pressure seasoned before the measurements by an application of 12500 kg., and then machined to final size. No perceptible change of dimensions was produced by this preliminary treatment.

The measurements of the external change of length of the cylinder, which were made at the same time as the measurements of the change of relative length of cylinder and bar, were entirely satisfactory. The relation between pressure and elongation was linear within the limits of error, which were not more than a few per cent, and there was no perceptible hysteresis, showing that the elastic limit had not been exceeded. However, the absolute value of the elongation was abnormal in that it was only half that which would be computed from the ordinary elastic constants of steel, assuming perfect homogeneity. It is evident that the method of heat treatment must have introduced considerable internal stress.

Measurements were made at two temperatures, 30° and 75°, and at both temperatures the runs were repeated. The two runs at each temperature agreed within the limits of error. The magnitude of the effect was sufficient to give a displacement of 40 cm. on the wire of the potentiometer. The regular procedure was followed in making readings, which were at intervals of even thousands of kilograms with increasing pressure and at the odd intervals on decreasing pressure, the maximum being 12000. There was no perceptible hysteresis. At each temperature 27 readings were taken. Two of these had to be discarded, the observed points lying off a smooth curve by slightly over 1 cm. of the potentiometer wire; the average arithmetical deviation of the remaining 52 points from smooth curves was 0.6% of the maximum effect. The departure from linearity was perfectly well marked and unmistakable; at the middle of the range it amounted to 0.32%, and was sensibly the same at both temperatures. This deviation from linearity is thus twice as great as the average error of a single observation; hence it is evident that no high degree of accuracy can be claimed for the deviation from linearity (which determines at once the pressure coefficient of compressibility), and certainly one is not entitled to more than two figures in the pressure coefficient of compressibility.

With regard to the temperature coefficient of compressibility, the change with temperature was certainly established to be much smaller than had been previously found, but again in view of the smallness of the effect no great accuracy can be claimed for the temperature coefficient itself, and in fact not more than one significant figure could be obtained here.

After applying all corrections, including one due to the cubic compressibility not being exactly three times the linear, the following formulas were found to give the change of volume produced by any pressure p, expressed in kilograms per square centimeter.

$$\text{At } 30^\circ \qquad \frac{\Delta V}{V_0} = -10^{-7}(5.87 - 2.1 \times 10^{-5}p)\,p$$

$$\text{At } 75^\circ \qquad \frac{\Delta V}{V_0} = -10^{-7}(5.93 - 2.1 \times 10^{-5}p)\,p.$$

The V_0 of these formulas is in each case the volume under atmospheric pressure at 30°; the difference between the atmospheric volume at 30° and 75° would produce a change of somewhat less than one unit in the last place.

The fairest comparison of these results with those previously found is to be obtained by computing from the above formulas the average compressibility to 10000 kg., since the range of the previous work was 10000 kg., and no departure was found from linearity. The above formulas give as the average to 10000, 5.66×10^{-7} at 30° and 5.72×10^{-7} at 75°. These are materially smaller than the previous values, which were 5.83×10^{-7} at 0° and 6.01×10^{-7} at 50°. Probably part of the difference is to be explained by the greater purity of the present specimen, it being consistent with a number of my other observations that alloying iron increases its compressibility, but doubtless the important part of the difference is to be ascribed simply to the improvement in the method; certainly the present determination of the temperature effect is to be preferred to the previous one.

The new value for the compressibility of iron will not affect at all the compressibilities given by Richards, which involve my previous value for iron indirectly through the value for mercury, but will lower the absolute values of compressibility given by A. W. J. by 0.25×10^{-7}, since their values also depend on my value for iron. The fact that the compressibility of iron is now found to decrease somewhat at high pressures will also of course somewhat increase the estimate of A. W. J. of the decrease of compressibility with pressure.

The instantaneous compressibility, which I define as $\frac{1}{V_0}\frac{dV}{dp}$, may be found from the above formulas by differentiation. The change of compressibility with pressure is thus seen to be about 8.5%. One would not suspect so large a change from the slight apparent departure from linearity of the curve plotting change of length against pressure.

Compressibility of Metals Other than Iron.

METHOD.

The general idea of the method is the same as that used for the absolute linear compressibility of iron, namely a wire is attached to the specimen, which slides over a contact fixed to a comparison piece, and the relative motion of the wire is obtained from a measurement of the potential difference between the sliding contact and the contact fixed to the wire. In this case, however, the comparison piece is not the external cylinder itself, but is another piece of iron within the cylinder which is exposed to hydrostatic pressure all over, and which therefore experiences only a uniform compression. The amount of this compression is determined by the measurements on iron just described.

Several forms of apparatus were used, depending on the numerical value of compressibility and the shape in which the specimens could be obtained. For those substances whose compressibility is near to that of iron, an apparatus with a lever was used, by which the relative motion of the specimen and the surrounding iron was magnified about seven times. A sketch of the essential parts is shown in Figure 2. The specimen shown at S was, when convenient, made in the form of a turned bar 6 mm. in diameter. Over each end was slipped a well fitting steel cap, provided with steel points. At the lower end one of these points rested against the containing envelope of iron, and at the upper end the other point bore against the short arm of the lever. This lever had three point support, one at A, and two, one behind the other, at B. The lever was kept tightly pressed against the specimen by a spring of flat steel G,

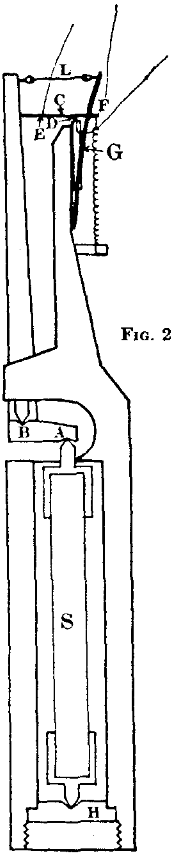

Fig. 2.

Figure 2. Device with lever magnification and sliding electrical contact for measuring the difference between the linear compressibility of iron and the specimen S.

pulling on the upper end of the lever through a link motion L. The nichrome measuring wire C is attached to the upper end of the lever, and slides over a contact at D which is attached to a plate insulated from the rest of the apparatus with a thin mica washer. At E is the potential terminal fixed to the wire. Flexible leads pass from E and D to the two potential terminals of the three terminal plug, and the current terminal is attached at F. The second current connection is simply the case, to which everything is grounded. The details of connecting the lever apparatus to the three terminal plug so that the specimen attached to the plug can be screwed into the pressure cylinder as one self-contained piece offer no difficulties of design, and need not be shown.

The apparatus as shown is flexible enough to be adapted to a wide variety of sorts of specimens. Variations in the diameter of the specimen may be allowed for either by making different steel caps, or preferably simply by making suitable steel collars to slip around the specimen by which it is held securely in the pointed end pieces. Variations in the length of the specimen are allowed for by changing the length of the dummy piece of iron at H. Specimens used with the apparatus as shown must be stiff enough to support the compressive stress due to the spring on the lever. But the compressibility of wires may also be measured by a simple change that will readily suggest itself, replacing the specimen S by a slotted frame of iron in which is laid the specimen in the form of wire, attached to the frame by a clamp at the lower end, and at the upper end attached to a clamp, which is in turn attached to the surrounding envelope by a pin. The wire is thus put under tension instead of compression. Measurements may be made on wires of any diameter large enough to stand the necessary tension. One must of course be sure that the elongation of the wire due to the tension of the spring is not large enough compared with the relative deformations produced by pressure to introduce any such complication as might arise from a change in Young's modulus under pressure. A simple calculation will enable one to stay on the safe side of this requirement.

The magnifying power of the lever had to be determined by a special arrangement, it not being possible to measure the dimensions directly accurately enough. An arrangement was made by which a micrometer screw could advance the short arm of the lever by a known small amount, the micrometer screw being provided with a vernier scale permitting readings to 1/20°. The accuracy obtainable with the special arrangement was considerably better than 1/10%.

For specimens whose compressibility was fairly high compared with that of iron the complications of the lever were dispensed with, and the relative change of length was measured directly by a wire attached directly to the end of the specimen. This is shown in Figure 3. The specimen is compressed against the lower end of the iron container by a spring M and also is pressed against the sliding contact D by a spring N.

The final measurements of compressibility were made with one or the other of three different pieces of apparatus; two of these magnified the motion with levers, one having a long envelope for long specimens (maximum capacity 16.5 cm.), and the other a short envelope for pieces of a maximum length of 2.5 cm. The third apparatus was for the direct measurement of the relative change of length, as described in the preceding paragraph.

Each form of apparatus was checked by making with it blank runs, using as the specimen a piece of the same iron as that whose absolute linear compressibility was measured with the other apparatus. If the compressibility of the envelope (which was made of commercial bessemer steel) was the same as that of pure iron, and if it were perfectly homogeneous, so that it experienced a volume compression without change of figure, then the changes of resistance of the wire under pressure should be such that when the various corrections were applied for the change in specific resistance of the wire under pressure, etc., there should be indicated no outstanding relative motion of specimen and surrounding envelope. Of course for one thing the compressibility of the pure iron and the bessemer steel was probably not exactly the same, so that perfect agreement was not to be expected, but nevertheless the correction so determined due to all these possible sources was very small indeed, and was about that due to the relative compression of the iron and the mica washers (which were 0.012 cm. thick) by which the fixed contact D was insulated. It is to be noticed that after this correction was applied the results gave accurately the difference

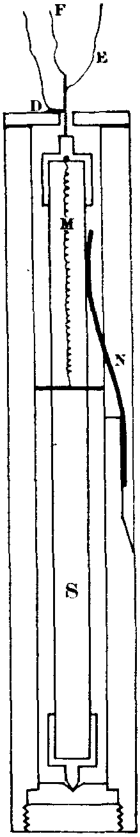

Fig. 3.

Figure 3. Device with sliding electrical contact for the direct measurement of the difference of linear compressibility between iron and the specimen S.

of compressibility between the specimen and pure iron, independent of any imperfections in the apparatus, which were thereby eliminated. Except for the convenience of using small corrections, it would have been quite possible to have made the envelope of some other metal, such as brass.

Besides the empirically determined correction just discussed a number of other corrections have to be applied for the change of resistance of the nichrome wire under pressure, for the difference of compressibility between nichrome and iron, and for the temperature effects. These corrections are essentially the same as those needed in getting the absolute linear compressibility of iron, and will not be separately discussed; they are small and involve no difficulty. The data requisite for making these corrections were determined by direct experiment. For those substances whose compressibility is near to that of iron the percentage magnitude of the correction on the difference of compressibility may of course be greater than in the experiment on the absolute linear compressibility of iron. The percentage magnitude of the corrections is approximately inversely proportional to the difference of compressibility.

The total motion of the wire was in all cases small, being seldom more than a small fraction of a millimeter. It will of course be understood that in dealing with such small quantities every feature of the design had to be carefully thought out, and that the apparatus as finally used embodied many details whose best form was found only after several trials, but which it will not be profitable to describe here. Many changes were made in several different pieces of apparatus, and readings were made continuously over a period of four months before the design used for the final measurements here recorded was attained. By far the most trouble was found with the lever apparatus. It is sufficient to mention here that the feature requiring the greatest care is the construction of the bearing points and seats of the lever. These must be of the greatest geometrical perfection, and highly polished, and will repay a considerable amount of labor. The radius of the bearing point must not be too small or the point will break, hence the necessity for a high polish. The result of failure to get proper bearing points is irregularity in the readings; the observations skip erratically about, or often show systematic departures from the correct course for considerable successions of readings.

The pressure was determined, as in all this work, from the changes in the resistance of a coil of manganin wire, and the corrections have been discussed in detail previously.[7] During the course of the work the coil was calibrated against the freezing point of mercury at 0° at sufficient intervals to insure the accuracy of the pressure measurements.

After the readings had been made of the potentiometer setting as a function of pressure over the entire pressure range, and the various corrections applied to these readings, the points were plotted, potentiometer setting against pressure. This plot is nearly a straight line. A straight line was now passed by computation through the zero point and the point at the highest pressure, and the intermediate points on the line computed corresponding to the intermediate observed pressures. It was necessary to use computation for the intermediate points, as graphical construction would not have been accurate enough. The difference between the linear values and those actually observed was now found by subtraction, and these differences were plotted on an enlarged scale. A smooth curve was now passed through these difference points. This curve could be drawn with sufficient accuracy in most cases free hand. In most cases this curve was symmetrical about the mean pressure (6000 kg.), and could be represented within the limits of error by a parabolic formula. This means that within the limits of error the change of volume is given by a two constant formula of the type used for iron above, namely $\frac{\Delta V}{V_0} = -(a + bp)p$. The change of volume under 12000 kg. was next computed, using as the most probable value of the change of resistance that obtained from the smoothed curve of departures from linearity. In computing this change of volume all the corrections were applied. The initial rate at which the volume changes with pressure was also computed. This initial rate was the average rate to 12000 corrected by a factor obtained from the smoothed difference curve. The correction factor was usually given with sufficient accuracy by doubling the ratio of the deviation at the mean pressure to the mean ordinate, and applying the result as a correction factor to the average rate to 12000. If the deviation from linearity was so great as to make this procedure inaccurate, the change of volume was calculated directly at several intermediate pressures. Having thus obtained the initial rate of change of volume, and the change under 12000, the constants of a formula like that above were at once determined. For several substances, particularly sodium and potassium, the deviation from linearity was so great that it was necessary to calculate from the smoothed curve the changes of volume at even thousand intervals of pressure, and tabulate these, instead of trying to reproduce the results by a single formula.

The procedure in making the calculations outlined in the paragraph above was applied independently at the two temperatures 30° and 75°. Departure from linearity was in most cases so small that it was useless to attempt to try to get the temperature coefficient of the departure. Accordingly in the calculations the mean of the departures found independently at the two temperatures was usually used as that most probably accurate.

The behavior of the metals under pressure gives in many cases a useful check on their probable homogeneity and therefore on the equality of their compressibility in all directions. If the metal is under no internal strains, the relation between deformation and pressure should be single valued, without hysteresis, and there should be no permanent change of dimensions, even after the first application of pressure. The absence of hysteresis could of course be checked by the readings themselves, but the freedom from set on the initial application of pressure was not so easy to determine, because there were sometimes slight initial irregularities in the apparatus itself which were smoothed out by the initial application of pressure. In all cases the metal was subjected to a preliminary application of pressure over the entire range before readings were begun, and the change of zero produced by this preliminary application was also recorded. This should give an upper limit to any actual change of dimensions produced in the specimen by pressure. If the substance is carefully prepared there should be no change, and this was in almost every case the fact. It was possible to make castings of lead, for example, that showed no perceptible permanent change of length after the initial application of 12000 kg. In one or two cases, however, there were comparatively large changes, and these were usually accompanied by hysteresis on subsequent applications of pressure, showing internal strains. These cases will be described in detail later.

There follows now the detailed presentation of data. First are given those metals crystallizing in the cubic system. There are included here a few metals whose structure has not yet been determined, since the chances are that any metal selected at random is cubic. The metals are arranged in order of compressibility, beginning with tungsten, the least compressible. After this are given data for several metals not crystallizing in the cubic system. The work on these latter metals must be extended. The present data are competent to give only an idea of the amount of variation to be expected in the compressibility in different directions.

METALS CRYSTALLIZING IN THE CUBIC SYSTEM.

Tungsten. It is well known that, because of its methods of preparation, this metal may show considerable differences of density, depending on the amount of mechanical working to which it has been subjected, the finer wires having the greater density. It therefore seemed to me desirable to measure the compressibility of two samples differing considerably in the amount of working to which they had been subjected. Through the kindness of the General Electric Company I obtained a number of samples of pure tungsten, and selected two from these best adapted to the measurements. The first was 0.48 cm. in diameter; it had been swaged to these dimensions from the original sintered bar, but had not been drawn through dies. It was mounted as a compression specimen in the lever apparatus, with a length of 8.68 cm. The density at room temperature was found by weighing in air and water to be 19.137. The second specimen was of drawn wire 0.051 cm. in diameter, and was mounted as a tension specimen in the lever apparatus. The length was about 10 cm. The density of this small sample could not be determined with sufficient accuracy by weighing, but tables compiled by the General Electric Company show that the density to be expected on the average for wire of this diameter is 19.48. The difference between the densities of the two samples is thus considerable; it is questionable how much is due to closing of the pores and how much is due to the breaking up of the crystalline structure by working, replacing the crystalline material by amorphous.

Regular readings were made on these two samples at 30° and 75°. The readings were very satisfactory, considering the small compressibility of this metal. Discarding the 8 most irregular points, the average numerical departure from a smooth curve of the remaining 48 points was 0.28% of the effect produced by the maximum pressure. The accuracy of the readings with the two different samples was essentially the same. The departure of the points from linearity, as compared with iron, was well marked, and at the mean pressure was about 2.5 times as great as the average error of a single observation. The deviations from linearity determined for each sample independently from the series of readings at the two temperatures differed by not more than 10% from the mean. The direction of this departure from linearity was such as to make the change of compressibility of tungsten with pressure less than that of iron, as would be expected from its smaller absolute compressibility. The final results are expressed in the following formulas, p being in kg/cm²:

$$\text{Swaged rod, At } 30^\circ \ \frac{\Delta V}{V_0} = -10^{-7}(2.93 - 1.5 \times 10^{-5}p)\,p$$

$$\text{At } 75^\circ \ \frac{\Delta V}{V_0} = -10^{-7}(2.95 - 1.5 \times 10^{-5}p)\,p$$

$$\text{Drawn wire, At } 30^\circ \ \frac{\Delta V}{V_0} = -10^{-7}(3.15 - 1.6 \times 10^{-5}p)\,p$$

$$\text{At } 75^\circ \ \frac{\Delta V}{V_0} = -10^{-7}(3.16 - 1.5 \times 10^{-5}p)\,p.$$

It is perhaps not to be expected that the drawn wire with the greater density should also have the greater compressibility. This is possibly due in part to the greater amount of the amorphous metal that the drawn wire doubtless contains, since it is usually true that an amorphous phase is more compressible than the corresponding crystalline one.

The only previous determination of the compressibility of tungsten seems to be by Richards.[1] His material was in the form of fused buttons, also obtained from the General Electric Company, and had a density at room temperature of 19.231. He finds for the initial compressibility 2.7×10^{-7}. The value is given to only two significant figures, and I gather from personal conversation with Professor Richards that he regards as possible an error of several units in the last place.

Platinum. Runs were made on two specimens, of quite different dimensions, in two different pieces of apparatus. I am indebted to the kindness of Baker and Company for the loan of the two specimens. The material was stated by them to be chemically pure platinum. The first specimen was 2.5 cm. long and 0.8 cm. in diameter. It was used as a compression specimen in the lever apparatus for short specimens. It was cut from a drawn rod of platinum, and before the measurements was annealed for several hours in an electric furnace at a temperature of 800°. The second specimen was of drawn wire, annealed, 0.065 cm. in diameter and about 10 cm. long, and was mounted as a tension specimen in the lever apparatus for long specimens.

The regular set of readings at two temperatures was made on each specimen. The readings with the shorter specimen were not nearly as regular as those with the longer wire, which indeed is not surprising in view of the smallness of the effect. The average arithmetical departure from a smooth curve of the 28 observed points with the short specimen (no discards) was 2% of the maximum pressure effect, and the departure from linearity, which was perfectly well marked, was 3% at the maximum. The series of readings at the two temperatures gave the same departure from linearity to one significant figure. It is obvious that a high degree of precision cannot be expected for the pressure coefficient of compressibility of this specimen. The average arithmetical departure from the smooth curve of all the readings on the wire (no discards) was 0.7% of the maximum pressure effect, but here the deviation from linearity, although perfectly evident, was only two thirds as great as the error of a single reading. The deviation from linearity at the higher temperature was about 50% greater than at the lower. The average value of the compressibility obtained from the wire sample may be expected to be relatively more accurate than that obtained from the massive specimen, but the differences are beyond the errors of the measurements, and point to a real difference between the two specimens. The pressure coefficient of the difference of compressibility between iron and platinum cannot obviously claim any great accuracy, but the actual pressure coefficient, obtained by combining these readings with the absolute values for iron, may be expected to be somewhat more accurate.

The results found are reproduced by the following formulas:

$$\text{Wire, At } 30^\circ \quad \frac{\Delta V}{V_0} = -10^{-7}(3.60 - 1.8 \times 10^{-5}p)\,p$$

$$\text{At } 75^\circ \quad \frac{\Delta V}{V_0} = -10^{-7}(3.64 - 1.8 \times 10^{-5}p)\,p$$

$$\text{Rod, At } 30^\circ \quad \frac{\Delta V}{V_0} = -10^{-7}(3.05 - 0.0 \times 10^{-5}p)\,p$$

$$\text{At } 75^\circ \quad \frac{\Delta V}{V_0} = -10^{-7}(3.09 - 0.0 \times 10^{-5}p)\,p.$$

It is of course not likely that the compressibility of the rod actually does not change with the pressure, but at any rate the change is small.

The initial compressibility of platinum has been found by Richards to be 3.7×10^{-7}. The agreement with the value found above for the wire is well within the limits of error. Richards also obtained his material from Baker and Company. He found for the density at 20°

21.31. The density of the larger of the two pieces measured above was 21.34 at 20°. Evidently the mechanical condition of these two specimens must have been nearly the same.

Molybdenum. The treatment of molybdenum was like that of tungsten. Through the kindness of the General Electric Company I obtained a number of pieces in different stages of mechanical working; the compressibility of two of these was measured. One specimen, which had been swaged only but not drawn, was used as a compression specimen in the lever apparatus; its diameter was 0.48 cm. and length about 10 cm. This specimen was in two pieces, held butting end to end in a steel sleeve. It is often possible in this way to fit together several shorter pieces and obtain effectively a longer piece, thus increasing the accuracy of the measurements. The density of this specimen at 20° was found by weighing to be 10.185. The second specimen was in the form of drawn wire 0.051 cm. in diameter and 10 cm. long, mounted as a tension specimen in the lever apparatus. Its density could not be obtained with sufficient accuracy by weighing, but the average density found by the General Electric Company for wire of this diameter is 10.20.

The regular series of runs, at 30° and 75°, were made for each specimen. The accuracy of the measurements was about the same for the two specimens. Discarding 5 points, the average arithmetical departure from a smooth curve of the remaining 50 points was 0.36% of the maximum pressure effect, and the departure from linearity (that is, the departure from linearity of the difference of compressibility of iron and molybdenum) was 1.40% of the maximum pressure effect. The departure from linearity of each specimen was sensibly the same at the two temperatures. The final results are:

$$\text{Swaged rod,}\quad \text{At } 30^\circ\ \frac{\Delta V}{V_0} = -10^{-7}(3.47 - 1.2\,p)\,p$$

$$\text{At } 75^\circ\ \frac{\Delta V}{V_0} = -10^{-7}(3.48 - 1.2\,p)\,p$$

$$\text{Drawn wire,}\quad \text{At } 30^\circ\ \frac{\Delta V}{V_0} = -10^{-7}(3.61 - 1.0\,p)\,p$$

$$\text{At } 75^\circ\ \frac{\Delta V}{V_0} = -10^{-7}(3.62 - 1.0\,p)\,p\,.$$

The compressibility of the drawn wire is seen to be higher than that of the swaged rod in spite of its higher density, as was also the case for tungsten, and probably for the same reason.

Richards has found for the initial compressibility of molybdenum in the form of fused buttons of density 10.21 the value 4.5×10^{-7}. This again is considerably higher than the value found above, but the discrepancy is not more than is to be expected when the small size of Richards' sample is considered, and the fact that one of his two results was 40% higher than the other. Richards' object in his measurement was only to obtain the order of magnitude.

Tantalum. This was in the form of a drawn wire 0.062 cm. in diameter and 10 cm. long, mounted as a tension specimen in the lever apparatus. I am indebted for this material to the kindness of the Fansteel Co., of North Chicago. It was stated by them to be of unusually high purity, but I have no analysis. As a partial means of estimating its purity I determined its temperature coefficient of resistance between 0° and 100°. The relation between temperature and resistance is sensibly linear over this range, and the average coefficient is 0.00335. This is materially higher than the coefficient of a piece which I had formerly obtained from the General Electric Company, namely 0.00293, and for which I have determined the effect of pressure on electrical resistance. The coefficient is less than that of a sample of Holborn,[8] 0.00347.

Two runs were made on this specimen, as usual, at 30° and 75°. The results at 30° were appreciably more regular than at 75°. At 30° the average arithmetical departure of the observed points from a smooth curve (no discards) was 1.4% of the maximum pressure effect, and at 75° it was 3.2%. The departure from linearity was sensibly the same at the two temperatures, and at its maximum was 6.8% of the pressure effect. It must be remembered that because of the small difference of compressibility between tantalum and iron a large percentage error in the difference of the two compressibilities need not mean a large error on the absolute compressibility.

The final results are given by the formulas:

$$\text{At } 30^\circ \quad \frac{\Delta V}{V_0} = -10^{-7}(4.79 - 0.25 \times 10^{-5}p)\,p$$

$$\text{At } 75^\circ \quad \frac{\Delta V}{V_0} = -10^{-7}(4.92 - 0.25 \times 10^{-5}p)\,p.$$

The close approach of the compressibility to constancy with pressure is to be noted.

The compressibility of tantalum at 20° over a small pressure range has been found by Richards to be 5.2×10^{-7}. His material was from the General Electric Company, and was in the form of fused buttons.

Palladium. Runs were made on two different samples, both of which I owe to the kindness of Baker and Company. They were said to be of the highest attainable purity, but I have no analysis. One of these was a massive specimen, 0.75 cm. in diameter and 2.5 cm. long. It was used as a compression specimen in the lever apparatus for short specimens. The second was in the form of wire, 0.062 cm. in diameter, 10 cm. long, and was mounted as a tension specimen in the lever apparatus for long specimens. Each sample was annealed after receiving from Baker, the massive specimen by heating to 800° for two hours in an electric furnace and slowly cooling, and the wire by heating in a Bunsen burner to a bright red. Two regular runs, at 30° and 75°, were made on each specimen.

The average arithmetical departure of the readings of the massive specimen from a smooth curve (no discards) was 2.2% of the maximum pressure effect. It was not possible to detect any departure from linearity. For the wire the average arithmetical departure was 3.0% of the maximum pressure effect, and again there was no detectible departure from linearity. The final results found for these two specimens are:

$$\text{Massive specimen,}\quad \text{At } 30^\circ\quad \frac{\Delta V}{V_0} = -10^{-7}(5.19 - 2.1 \times 10^{-5} p)\,p$$

$$\text{At } 75^\circ\quad \frac{\Delta V}{V_0} = -10^{-7}(5.11 - 2.0 \times 10^{-5} p)\,p$$

$$\text{Drawn wire,}\quad \text{At } 30^\circ\quad \frac{\Delta V}{V_0} = -10^{-7}(5.28 - 2.1 \times 10^{-5} p)\,p$$

$$\text{At } 75^\circ\quad \frac{\Delta V}{V_0} = -10^{-7}(5.31 - 2.1 \times 10^{-5} p)\,p.$$

It is to be noticed that the temperature coefficient of the massive palladium appears to be negative. This is a very unusual occurrence, and it is to be doubted whether the accuracy of the measurements is so high as to compel the conclusion that this is actually the case.

The initial compressibility of palladium at 20° has been found by Richards to be 5.3×10^{-7}, agreeing perfectly with the value found above for the wire. Richards' material was also obtained from Baker, and was a massive specimen weighing 94 gm. Its density was 12.14, against 11.97 for the massive specimen above.

Nickel. Measurements were made on samples from two sources.

One was a piece of commercial nickel obtained from the International Nickel Company in the form of a drawn bar 0.75 cm. in diameter and 16 cm. long. The purity was high commercial purity, a little better than 99%. It was annealed by heating for several hours to a bright red after its final machining, and was used as a compression specimen in the lever apparatus for long specimens. The second sample piece of nickel was obtained from the research laboratories of the Leeds and Northrup Company, and was of very unusually high purity. It is the same material as that for which I have already published data for the effect of tension on resistance[9] and pressure on thermal conductivity.[10] I have no chemical analysis, but have determined the temperature coefficient of resistance between 0° and 100° to have the mean value 0.00634, which is very high. This material was provided in the form of small cast ingots a couple of mm. thick and 5 or 6 cm. long. There were flaws in these castings large enough to be visible to the eye. I tried to get rid of these flaws by forging the castings to considerably smaller dimensions. The forged castings were mounted as compression specimens in the lever apparatus for long specimens, and their compressibility measured, but there was a rather large permanent set on the first application of pressure, and the relation between pressure and deformation showed a rather large amount of hysteresis. In order to get rid of the effect of the flaws, therefore, this forged piece was drawn down to wire of 0.079 cm. diameter, annealed at a bright red after the last drawing, and used as a tension specimen in the lever apparatus for long specimens.

The regular series of readings were made on each specimen, at 30° and 75°. The average arithmetical deviation from a smooth curve of the readings on the commercial rod (no discards) was 0.8% of the maximum pressure effect. The corresponding average deviation from a smooth curve of the readings on the wire, making three discards (it is usually true that the tension specimens do not give such regular results as the compression specimens) was 1.1% of the maximum effect. Neither specimen showed any perceptible deviation from linearity, which means of course that the variation of compressibility with pressure is the same for nickel as for iron.

The final results are as follows:

$$\text{Commercial rod, At } 30^\circ \quad \frac{\Delta V}{V_0} = -10^{-7}(5.25 - 2.1 \times 10^{-5}p)\,p$$

$$\text{At } 75^\circ \quad \frac{\Delta V}{V_0} = -10^{-7}(5.28 - 2.1 \times 10^{-5}p)\,p$$

Pure drawn wire, At 30° $\frac{\Delta V}{V_0} = -10^{-7}(5.29 - 2.1 \times 10^{-6}p)\,p$

At 75° $\frac{\Delta V}{V_0} = -10^{-7}(5.35 - 2.1 \times 10^{-6}p)\,p.$

A rough value for the average compressibility to 12000 kg. of the forging of pure nickel was 5.16×10^{-7}.

The values above for the two samples are seen to be nearly the same, and it is therefore probable that the usual impurities, such as cobalt and iron, have little effect on the compressibility.

The initial compressibility has been found by Richards to be 4.3×10^{-7}. His material was in the form of cubes, in which the pure metal is often supplied by chemical houses. Richards states that the nickel as originally provided was full of flaws into which the mercury was forced by pressure, and that the effect of these flaws was got rid of by heavy forging. My experience with the cast specimen above would strongly indicate (because of the hysteresis shown by this sample) that the forging must have introduced considerable internal strain, which is doubtless responsible for Richards' low value. The effects of hysteresis would be especially pronounced over a small pressure range, and would be in the direction to account for Richards' low value.

Cobalt. This I owe to the kindness of Professor C. C. Bidwell of Cornell University, by whom it was in turn obtained from Dr. Herbert T. Kalmus, who had prepared it for the Canadian Government. It was of high purity, and had the following analysis: Fe 0.14, Ni 0.00, S 0.019, Si 0.02, C 0.09, Co 99.73. It was in the form of wire about 0.075 cm. in diameter and 6 cm. long, and was mounted as a tension specimen in the lever apparatus for long specimens. The material had been formed into wire by swaging followed by drawing at a red heat. It was not perfectly straight as supplied. It is rather brittle to bending, but I was able to make it perfectly straight by rolling it between red hot iron plates, thus at the same time annealing it.

The two usual runs at 30° and 75° were made. The results were gratifyingly regular, considering the smallness of the effect. The average arithmetical departure from a straight line of the 28 readings (no discards) was 1.3% of the maximum pressure effect; 1.3% on the difference of compressibility means 0.1% on the actual compressibility. It was not possible to detect any departure from linearity.

The final results are expressed by the formulas:

At 30° $\frac{\Delta V}{V_0} = -10^{-7}(5.39 - 2.1 \times 10^{-6}p)\,p$

$$\text{At } 75^\circ \quad \frac{\Delta V}{V_0} = -10^{-7}\,(5.47 - 2.1 \times 10^{-5}p)\,p.$$

As far as I can find, the compressibility of cobalt has not been previously measured. The value found above, close to that for nickel and iron, is of the order of magnitude that one would expect.

Nichrome. The reason for determining the compressibility of this alloy is that the difference of compressibility between it and iron enters as a correction on the observed readings. The correction is very small, and at the maximum amounts to only 0.3 mm. on the slide wire of the potentiometer. The material was obtained from Driver Harris Company, and is of the grade known as Nichrome II. Its composition is Ni 80% and Cr 20%. The sample measured was from the same length as the wires attached to the lever or the specimen, by which the relative changes of length were measured. Its diameter was 0.030 cm. and length about 10 cm. It was mounted as a tension specimen in the lever apparatus for long specimens.

Because of the smallness of the correction it was necessary to measure the compressibility only at 30°. Here the mean departure from a smooth curve (no discards) was 1.5% of the maximum pressure effect, which corresponds to 0.09% on the actual compressibility. The deviation from linearity was sensibly not the same as for iron, and at the maximum was 5.5% of the maximum pressure effect. The results are contained in the formulas:

$$\text{At } 30^\circ \quad \frac{\Delta V}{V_0} = -10^{-7}\,(5.50 - 1.5 \times 10^{-5}p)\,p.$$

Gold. Measurements were made on two samples, both of which I owe to the kindness of Baker and Company. The specimens were stated to be of the highest purity, but I have no analysis. The density was found to be 19.272 at room temperature. Both specimens were in the form of drawn rod 0.75 cm. in diameter, and were annealed at a bright red before the measurements. The first was 2.5 cm. long, and was mounted as a compression specimen in the lever apparatus for short specimens; the second was 12 cm. long, and was mounted as a compression specimen in the lever apparatus for long specimens.

The compressibility of gold is exceedingly close to that of iron, and hence accurate results are not to be expected for the difference of compressibility. Apart, however, from the smallness of the effect, the apparatus did not function as well as usual during these measurements, there being minor electrical troubles, probably due to short circuits, so that it was necessary to repeat some of the readings. Four sets of readings were made on the shorter sample, two at 30° and two at 75°, and three sets on the longer sample, one at 30° and two at 75°. The readings on the short sample at 75° were never satisfactory, and those on the long sample at 75° left much to be desired. The readings at 30° were fairly satisfactory.

At 30° the average arithmetical deviation from a smooth line of the readings on the short sample was 5% of the maximum pressure effect. For the long sample the deviation at 30° averaged 5.9% and at 75° 4.2%. The results are given by the formulas:

$$\text{Short sample,}\quad \text{At } 30^\circ\ \frac{\Delta V}{V_0} = -\,10^{-7}\,(5.84 - 2.1 \times 10^{-5}p)\,p$$

$$\text{Long sample,}\quad \text{At } 30^\circ\ \frac{\Delta V}{V_0} = -\,10^{-7}\,(5.77 - 3.1 \times 10^{-5}p)\,p$$

$$\text{At } 75^\circ\ \frac{\Delta V}{V_0} = -\,10^{-7}\,(5.70 - 2.1 \times 10^{-5}p)\,p.$$

The details of the variation of the coefficients in these formulas probably are not accurate. Because of its greater length the results for the long sample are doubtless to be preferred. Probably the safest conclusion to draw from these measurements on gold is merely that at 30° the average compressibility to 12000 is approximately 5.40×10^{-7}, and at 75° it is 5.45×10^{-7}.

The compressibility of gold has been measured both by Richards and A. W. J. Richards finds the initial compressibility at 20° of gold of density 19.24 to be 6.3×10^{-7}, and A. W. J. find the compressibility to be constant with pressure, and its value over the range of 12000 kg. to be 5.6×10^{-7}. It is to be noticed that Richards makes gold more compressible than iron, and A. W. J. less. The measurements of this paper would seem to leave no room for doubt that it is actually less than iron. It is to be noticed that A. W. J. used a constant value for the compressibility of iron, namely 5.9×10^{-7}. If for this is substituted the new value found above for the average compressibility over the range of 12000, namely 5.62×10^{-7}, A. W. J.'s value becomes 5.3×10^{-7}, agreeing within one unit in their last significant figure with the value found above. It is possible that the high value of Richards is due to the fact that his gold was not annealed after the final drawing; the difference is in the same direction as produced by a similar effect in tungsten and molybdenum.

The low value of the compressibility of gold is somewhat surprising both when one considers the mechanical softness of the metal, and also that its position in the periodic table below copper and silver suggests a compressibility following the succession of values of these two other metals, which would lead to a figure more than twice as high as the actual one.

Copper. Copper from two sources was used. One was commercial drawn rod 0.6 cm. in diameter. Two sets of runs were made with this, the first with the rod in the commercial drawn condition, and the second after annealing. The second grade of material was pure copper from the Bureau of Standards, cut from one of their melting point samples, and had the following analysis: Sb 0.004, As 0.0026, S 0.0026, Cu 99.987. This second sample was measured only in the annealed condition. Both samples were mounted as compression specimens in the lever apparatus for long specimens; it was necessary to dowel together two pieces of the pure copper in order to get the requisite length, which was 12 cm. In addition to the runs on these long samples, at least four runs were made with preliminary forms of apparatus on shorter samples of the pure copper. The accuracy was not as high as in the final runs, and it is not necessary to describe the details, but within the somewhat wider limits of error, the results of these preliminary measurements agreed with the final results. It was during the preliminary measurements that an attempt was made to find a difference of compressibility of copper in different directions, with negative results.

The commercial copper was quite unusual in that it showed evidence of internal strains. The readings both before and after the annealing were affected by considerable hysteresis, and after the annealing the first application of pressure produced a rather large permanent distortion. The average arithmetical deviation from a smooth curve of the results for the unannealed copper (two discards) was 2.6 % of the maximum pressure effect. Almost the entire amount of this deviation is due, not to irregularity of the individual points, but to hysteresis, which at the maximum was 5.5% of the maximum pressure effect. For the commercial annealed copper the average arithmetical deviation from smoothness was 1.6% of the maximum effect, and the greatest width of the hysteresis loop was 5.5%. The loop was of a different shape from that of the unannealed copper. The pure copper from the Bureau of Standards showed no hysteresis; the average arithmetical departure (no discards) from a smooth curve was 0.47%, and the maximum departure from linearity was 1.44%. The results are contained in the following formulas:

Commercial drawn rod, unannealed,

$$\text{At } 30^\circ \quad \frac{\Delta V}{V_0} = -10^{-7}(7.32 - 2.7 \times 10^{-5}p)\,p$$

$$\text{At } 75^\circ \quad \frac{\Delta V}{V_0} = -10^{-7}(7.39 - 2.7 \times 10^{-5}p)\,p.$$

Same commercial rod, annealed,

$$\text{At } 30^\circ \quad \frac{\Delta V}{V_0} = -10^{-7}(7.29 - 2.7 \times 10^{-5}p)\,p$$

$$\text{At } 75^\circ \quad \frac{\Delta V}{V_0} = -10^{-7}(7.37 - 2.7 \times 10^{-5}p)\,p.$$

Pure copper,

$$\text{At } 30^\circ \quad \frac{\Delta V}{V_0} = -10^{-7}(7.19 - 2.6 \times 10^{-5}p)\,p$$

$$\text{At } 75^\circ \quad \frac{\Delta V}{V_0} = -10^{-7}(7.34 - 2.7 \times 10^{-5}p)\,p.$$

Richards has found the initial compressibility at 20° to be 7.4×10^{-7}. A. W. J. could find no departure from linearity. Correcting their results for the new value for iron, their average compressibility to 10000 was 7.1×10^{-7}, against 6.84×10^{-7} given by the formula above for pure copper.

Uranium. I am indebted for this to the kindness of Dr. A. W. Hull of the General Electric Company. It was furnished in the form of rolled strip 0.05 cm. thick, 0.6 cm. wide, and 5 cm. long. I have no chemical analysis. The temperature coefficient of the electrical resistance between 0° and 100° was 0.0022. Because there are no other measurements at present on the electrical properties of uranium, this does not mean much as to the purity, except that probably the purity was not high, but if in the future the properties of pure uranium are measured, this coefficient should give more definite information. This sample was mounted as a tension specimen in the lever apparatus for long specimens.

The usual two runs were made, at 30° and 75°. The average arithmetical departure from a smooth curve of all 28 points (no discards) was 1.0% of the maximum pressure effect. The average deviation at 75° was two or three times as great as at 30°. The maximum departure from linearity was 0.59% of the maximum pressure effect. It is therefore evident that the accuracy of the departure of the difference of compressibility between uranium and iron is not high. The results are embodied in the formulas:

$$\text{At } 30^\circ \quad \frac{\Delta V}{V_0} = -10^{-7}(9.66 - 2.5 \times 10^{-5}p)p$$

$$\text{At } 75^\circ \quad \frac{\Delta V}{V_0} = -10^{-7}(9.55 - 2.2 \times 10^{-5}p)p.$$

It is to be noticed that the temperature coefficient of compressibility seems to be negative. This is unusual, and perhaps is not genuine, but I give the results as they were found because the magnitude of this temperature effect is greater than the probable error.

The compressibility of uranium seems not to have been previously measured. The value found is not inconsistent with that which might be expected from its position in the periodic table, but this is always a rather inaccurate way of getting compressibility, and in the case of uranium, because of its position at the end of the list of the elements, it is a particularly poor way of guessing the compressibility.

Silver. This I owe to the courtesy of Baker and Co. I have no analysis, but it was said to be of the highest possible purity. It was provided in the form of drawn rod 0.6 cm. in diameter and 12 cm. long. It was annealed at a red heat before the measurements, and mounted as a compression specimen in the lever apparatus for long specimens. The density at 20° was 10.486. I also made measurements on another similar piece of material in the lever apparatus for short specimens. This was done before the final improvements had been made in the apparatus, and the results were rather irregular, but agree with those found for the other specimen within the limits of error.

The regular two runs at 30° and 75° were made. The average arithmetical departure from a smooth curve of the 28 readings (no discards) was 0.48% of the maximum pressure effect. The maximum departure from linearity, which was sensibly the same at the two temperatures, was 2.05%. The final results are given by the formulas;

$$\text{At } 30^\circ \quad \frac{\Delta V}{V_0} = -10^{-7}(9.87 - 4.4 \times 10^{-5}p)p$$

$$\text{At } 75^\circ \quad \frac{\Delta V}{V_0} = -10^{-7}(10.04 - 4.5 \times 10^{-5}p)p.$$

The initial compressibility found by Richards for silver of density 10.5 was 9.9 × 10^{-7}. A. W. J. found no departure from linearity with pressure; their mean value, corrected for the new value for iron, was 9.4×10^{-7}, both results agree very closely with the value above.

Aluminum. For these measurements I was fortunate to obtain two samples of exceptionally high purity. One was in the shape of a hard drawn rod and the other a casting, both 1.3 cm. in diameter. The density of both specimens at atmospheric temperature was the same, 2.700. The analysis of both specimens was also the same: Si 0.008, Fe, 0.013, Cu 0.014, Mn Nil, Al (by diff.) 99.965. For measurements these pieces were turned to a diameter of 0.6 cm., and were mounted as compression specimens in the lever apparatus for long specimens. The length of the casting was 7 cm. and that of the drawn piece 12 cm.

By way of curiosity I made measurements on the hard drawn piece first before annealing. I expected hysteresis and other evidence of internal stress, but to my surprise the results were perfectly smooth and gave evidence of nothing of the sort. These measurements were made only at one temperature and to a maximum pressure of 10000 instead of 12000, there being some temporary trouble with the pressure apparatus. This hard drawn piece was then annealed at 300° for several hours, and then the regular series of runs at 30° and 75° was made. The average arithmetical departure from a smooth curve of the 28 observed points (one discard) was 0.3% of the maximum pressure effect, and the maximum departure from linearity was 0.75%. The departure from linearity seemed to be somewhat greater at 30° than at 75°, but on the other hand the accuracy of the readings at 30° was materially greater than at 75°. The mean of the departures found at the two temperatures was used in the final computations.

The cast specimen was used without annealing; it showed no hysteresis or other evidence of internal strain. The average arithmetical departure from a smooth curve of the 28 observed points (one discard) was 0.3%, and the maximum departure from linearity 1.43%. The deviation from linearity was exactly the same at both temperatures, and this quantity seemed to be given much more satisfactorily by the measurements on the casting than by those on the drawn rod.

Besides the runs just described, at least six other complete sets of runs were made on aluminum, mostly with shorter specimens. A number of the preliminary forms of apparatus were tested by measuring with them the compressibility of aluminum. These early results were of course less accurate than the final ones, but agreed with them within the wider limits of error.

The final results are given by the formulas:

Drawn rod, Hard, At 30° $\frac{\Delta V}{V_0} = -10^{-7}(13.40 - 3.5 \times 10^{-5} p) p$

Drawn rod, annealed, At 30° $\frac{\Delta V}{V_0} = -10^{-7}(13.34 - 3.5 \times 10^{-5} p) p$

At 75° $\frac{\Delta V}{V_0} = -10^{-7}(13.91 - 3.5 \times 10^{-5} p) p$

Casting At 30° $\frac{\Delta V}{V_0} = -10^{-7}(13.43 - 5.0 \times 10^{-5} p) p$

At 75° $\frac{\Delta V}{V_0} = -10^{-7}(13.76 - 5.1 \times 10^{-5} p) p.$

Some time ago I found the average compressibility to 6000 kg. of a piece of commercial aluminum rod to be 11.7×10^{-7}, distinctly lower than the values found above. These early readings were not accurate enough to show a departure from linearity. The specimen showed set on the initial application of pressure, but otherwise there was no evidence, such as hysteresis, of internal strain. Richards has found the compressibility of aluminum of unstated analysis of density 2.60 to be 14.1×10^{-7}, somewhat higher than the above. A. W. J. for aluminum containing 0.235% Si and 0.016% Fe were not able to detect any departure from linearity over the range of 12000 kg., and give for the average compressibility (corrected by my new value for iron) 12.7×10^{-7}, which agrees to the third figure with the value given by the above formula for cast aluminum.

Germanium. The crystal structure of this substance has not yet been published; I owe to the kindness of Dr. A. W. Hull of the General Electric Company the information given in personal correspondence that X-ray analysis shows it to be cubic, of the same type of lattice as diamond and silicon.

For the sample of germanium I am indebted to the kindness of Professor C. C. Bidwell of Cornell University, who in turn obtained it from Professor L. M. Dennis of the Cornell Chemistry Department. It is the same specimen as that whose electrical properties he has reported in the Physical Review.[11] The material is evidently abnormal in some way, because the electrical resistance behaves normally below 100°, but above 100° decreases instead of increasing with rising temperature. The specimen was in the form of a casting about 2.5

cm. long and of square section, 0.5 cm. on the diagonal. Its density at 20° was 5.302. It was mounted as a compression specimen in the lever apparatus for short specimens. The compressibility measurements, as well as those on electrical resistance, showed that their is something abnormal about this substance. At 30° the points with increasing pressure lay regularly on a smooth curve, but at the maximum pressure there was an abrupt displacement corresponding to a shortening by an amount 7% of the maximum pressure shortening, and the subsequent points lay on another smooth curve displaced by this amount. The points at 75° showed no irregularity. On taking the apparatus apart, I found that one of the corners of the specimen was broken off. The abrupt displacement was ascribed to the breaking off of this chip, and no more was thought of it. The fractured material is more like a glass in appearance than a metal. Several days later, the specimen meanwhile having rested quietly wrapped in tissue paper, I found that one end of the specimen had spontaneously fractured into a great many small pieces, as badly annealed glass does sometimes under internal strain. It suggests itself that there may be another polymorphic modification at high pressures, the transition not being sharp, but viscous, and that it may have been the internal strains produced by the gradual transformation of this new modification that caused the fracture.

Making correction for the displacement from one smooth curve to another, the average arithmetical departure from a smooth curve of the 28 readings at both temperatures (no discards) was 0.33%, and the maximum deviation from linearity was 2.1% of the maximum pressure effect. There was no difference detectible between the deviations from linearity at the two temperatures.

The final results are given by the formulas:

$$\text{At } 30^\circ \quad \frac{\Delta V}{V_0} = -10^{-7}(13.78 - 6.8 \times 10^{-5}p)\,p$$

$$\text{At } 75^\circ \quad \frac{\Delta V}{V_0} = -10^{-7}(13.64 - 6.8 \times 10^{-5}p)\,p.$$

Again we find a compressibility less at the higher temperature. The variation with temperature seems to be well beyond the experimental error, and perhaps is not surprising in view of the abnormal behavior of the electrical resistance of this substance at higher temperatures.

There seem to be no previous measurements of compressibility.

Lead. This material was a melting point sample from the Bureau of Standards. The analysis of this specimen had not been completed. A presumably similar piece of purity 99.9948 is reported in detail in a previous paper.[12] Two samples were prepared in two different ways. The first was cast in a graphite mold, and chilled by lowering the mold slowly into water while the upper part of the casting was kept hot, in this way ensuring solidification from the bottom up without the formation of flaws. This was seasoned by heating in oil to 200° after casting. Its density at 20° was 11.347. It was machined to 0.75 cm. diameter and 16 cm. long, and was mounted in the apparatus for direct measurement without multiplication. The second piece was cast like the first, then extruded cold from a diameter of 1.5 to 0.75 cm., annealed at 230°, and measured in the same apparatus as the first piece. Its density at 20° was 11.337. Neither specimen showed any perceptible permanent set even after the first application of pressure.

Regular runs were made on each sample at 30° and 75°. The cast sample gave points whose arithmetical departure from a smooth curve (no discards) was 0.14%, and the maximum departure from linearity averaged 3.1%. The average arithmetical departure from a smooth curve of the points for the extruded sample (no discards) was 0.22%, and the maximum deviation from linearity was 2.2%. It is remarkable that both samples agreed in showing a departure from linearity nearly 50% greater at the lower temperature, but this seemed to me so unlikely, that I have preferred to regard it as due to a chance accumulation of error, and have used the mean of the deviations at the two temperatures in computing the final formulas, which are:

Casting, At 30° $$\frac{\Delta V}{V_0} = -10^{-7}(23.73 - 17.25 \times 10^{-5}p)\,p$$

At 75° $$\frac{\Delta V}{V_0} = -10^{-7}(24.33 - 17.7 \times 10^{-5}p)\,p$$

Extruded casting At 30° $$\frac{\Delta V}{V_0} = -10^{-7}(23.05 - 12.3 \times 10^{-5}p)\,p$$

At 75° $$\frac{\Delta V}{V_0} = -10^{-7}(23.63 - 12.3 \times 10^{-5}p)\,p.$$

It is to be noticed that these two samples differ chiefly in their pressure coefficient of compressibility; the average compressibility to 12000 kg. of the two samples is practically the same.

The initial compressibility of lead at 20° has been found by Richards to be 22.8×10^{-7}, appreciably smaller than the above. A. W. J. gives for the initial compressibility 21.7×10^{-7}. They also give a value for the decrease of compressibility with pressure, but they find a much smaller decrease than I do. Their decrease of instantaneous compressibility over a range of 10000 kg. (using my new values for the change of compressibility of iron with pressure) would be 1.2×10^{-7}, against a minimum value twice as great from the formulas above. A. W. J.'s average compressibility to 10000 kg. is 21.2×10^{-7} against 21.8 above for either sample.

Thallium. This material was prepared electrolytically by me several years ago, and I have already published data for some of its electrical properties. Judging by the temperature coefficient of resistance, its purity is high. Since the previous work the metal has been kept in a sealed glass tube. The metal was now fused under KCN, the fused button was hammered to fit the extrusion block, and it was extruded from a diameter of 1.2 to 0.6 cm. The final length was 13 cm. This was mounted in the apparatus for direct measurement without magnification. There was no perceptible set on the first application of pressure.

Discarding the three worst points, the average arithmetical deviation from a smooth curve of the remaining 27 points was 0.34%, and the maximum deviation from linearity was 4.0% of the maximum pressure effect. The deviation from linearity is unusually large for thallium, and furthermore it is not symmetrical about the mean pressure, so that the results cannot be represented by a too constant formula. I have therefore computed the change of volume at intervals of 3000 kg. (in terms of the volume at 30° and atmospheric pressure as unity) and give them in the following table.

TABLE I.

Change of Volume of Thallium under Pressure.

Pressure kg/cm²	$\Delta V/V_0$ 30°	$\Delta V/V_0$ 75°
3000	−.00978	−.01011
6000	.01872	.01908
9000	.02700	.02724
12000	.03505	.03554

Richards gives the initial compressibility of thallium which contained a slight amount of lead as 23×10^{-7}. The initial compressibility may be found graphically from the data of the table just given, or it may be computed. I have passed a three constant power series curve through the points at 3000, 6000, and 9000 kg., and from this find the initial compressibility at 30° to be 34.8, and at 75°, 36.7×10^{-7}. This is so much higher than the value of Richards that it seems difficult to ascribe to experimental error or slight impurity. The crystal structure of thallium does not seem to have been determined as yet; the difference in these values of compressibility strongly suggests that the crystal system cannot be cubic.

Cerium. This material I owe to the kindness of Dr. A. W. Hull of the General Electric Company. It was provided in the form of a square bar. I extruded it at about 400° to a round section 0.3 cm. in diameter, thereby reducing its section about four fold, and obtaining a piece 4.5 cm. long. This was mounted as a compression specimen in the lever apparatus for long specimens. The purity of this specimen was presumably not high, as its mean temperature coefficient of resistance between 0° and 100° was only 0.001, but since the compressibility seems never to have been measured, and impurity usually has only an additive effect on compressibility, I thought it of sufficient interest to run through the measurements.

The two regular runs were made, at 30° and 75°. Because of trouble with the pressure apparatus the maximum pressure of these runs was only 10000 instead of the usual 12000 kg., but it did not seem worth while to go to the labor of repeating the measurements over the greater range. The average departure from a smooth curve of the 23 observed points (no discards) was 0.47%, and the maximum departure from linearity was 2.2% of the maximum pressure effect. The final results are given by the formulas:

$$\text{At } 30^\circ \quad \frac{\Delta V}{V_0} = -10^{-7}(35.74 - 19.0 \times 10^{-5}p)\,p$$

$$\text{At } 75^\circ \quad \frac{\Delta V}{V_0} = -10^{-7}(35.80 - 19.7 \times 10^{-5}p)\,p.$$

This compressibility is not inconsistent with that to be expected for cerium from its position in the periodic table.

Calcium. Material from two sources was used. The first was obtained from the General Electric Company and was cut from the same block 1 inch in diameter as I used for previous determinations of the density and thermal expansion.[13] This block was perfectly free from any flaws on the exterior, but in cutting it in two preparatory to getting out a smaller specimen, an inclusion of slag was found in the very center. The previous measurement of density must therefore have a small error, and the value for the expansion also an error, but presumably not so large. On making a second cut no new inclusion was found. A clean piece was got from the original cylinder approximately 1.3 cm. in diameter, and this was extruded at a temperature of 450° to 0.6 cm. in diameter. A piece 10.5 cm. long was finally obtained and mounted in the apparatus for direct measurement. Its density at 20° was 1.555 against the previous value 1.556.

The second piece I owe to the kindness of Professor C. C. Bidwell. I do not know the origin, but it was stated to be of unusually high purity. This was originally of roughly square section 1.6 cm. in diameter. It was extruded in two steps at 450°, first to a round section 1.3 cm. in diameter, and then to 0.6 cm. diameter. The final length was the same as that of the other piece, and it was measured in the same apparatus. Its density at 20° was 1.532.

Professor F. A. Saunders was kind enough to make a spectroscopic analysis of these two samples for me, with the following results:

Calcium from General Electric Co.: Sr, considerable, perhaps 1 to 5% (2% would account for the difference of density), Cu trace, Mg trace, Al trace, Si trace.

Calcium from Professor Bidwell; remarkably pure, Mg trace, Cu trace, nothing else.

The regular runs at 30° and 75° were made on each specimen. The average arithmetical departure from a smooth curve of the points from the sample from the General Electric Co. (no discards) was 0.14%, and the maximum departure from linearity was 3.55% of the maximum pressure effect. The corresponding values for the sample of Professor Bidwell were (one discard) 0.11% and 3.7%. In spite of the high compressibility of calcium, the departure from linearity was symmetrical about the mean, so that it is possible to represent the results by a two constant formula. The results follow:

Calcium from General Electric Co.,

$$\text{At } 30^\circ \ \frac{\Delta V}{V_0} = -10^{-7}(59.46 - 48.8 \times 10^{-5}p)\,p$$

$$\text{At } 75^\circ \ \frac{\Delta V}{V_0} = -10^{-7}(59.67 - 48.8 \times 10^{-5}p)\,p.$$

Calcium from Professor Bidwell,

$$\text{At } 30^\circ \quad \frac{\Delta V}{V_0} = -10^{-7}(56.97 - 47.2 \times 10^{-5} p)\, p$$

$$\text{At } 75^\circ \quad \frac{\Delta V}{V_0} = -10^{-7}(58.50 - 52.7 \times 10^{-5} p)\, p.$$

The higher temperature coefficient of the second and purer sample is to be noticed; the pressure coefficient of the two samples is about the same.

The initial compressibility of calcium of density 1.54 has been found by Richards to be 56×10^{-7}. This agrees with the value above within the limits of error indicated by Richards' two significant figures.

Strontium. This material was from the same lot most kindly supplied me by Dr. B. L. Glascock as that for which I have previously determined the changes of resistance under pressure.[14] The purity is unusually high. The most homogeneous lump of this lot was chosen, and it was extruded in two stages to a final diameter of 0.3 cm. and machined to a length of 1.1 cm. The extrusion was performed at a temperature of 270° and under oil to avoid chemical action. The sample was mounted as a compression specimen in the lever apparatus for long specimens.

One preliminary run was made, in which it was not possible to reach more than 10,000 kg. because of trouble with the pressure apparatus, and then two regular runs to 12,000 at 30° and 75°. Within the limits of error the results of the first incomplete run agreed with the final results. The average departure of the 28 observed points of the regular runs from a smooth curve (no discards) was 0.35% and the maximum deviation from linearity was 3.9% of the maximum pressure effect. The deviation was the same at the two temperatures. In spite of the high compressibility of strontium the deviation from linearity was symmetrical about the mean pressure, so that it is possible to represent the changes of volume by a two constant formula, which follows:

$$\text{At } 30^\circ \quad \frac{\Delta V}{V_0} = -10^{-7}(81.87 - 72.5 \times 10^{-5} p)\, p$$

$$\text{At } 75^\circ \quad \frac{\Delta V}{V_0} = -10^{-7}(82.68 - 71.7 \times 10^{-5} p)\, p.$$

There seem to be no previous determinations of the compressibility of strontium.

Lithium. This was from Merck, obtained before the war, sealed in glass under oil, and was from the same lot as that used in determining the effect of pressure on resistance. A chemical analysis showed 0.7% Al and a trace of Fe. This was extruded cold to a diameter of 0.6 cm. The original piece was of square section; the extrusion produced little change of section or of length, but only a change in the shape of the section from square to round. The density at 20° was 0.546. The final length was 6.3 cm. and it was mounted as a compression specimen in the apparatus for direct measurement without magnification.

Two regular series of runs at 30° and 75° were made. The average departure of the 29 observed points from a straight line (no discards) was 0.15% and the mean departure from linearity was 4.9% of the maximum pressure effect. The departure from linearity was about 15% higher at the higher temperature than at the lower, a difference much beyond the limits of error. The final results are contained in the formulas:

$$\text{At } 30^\circ \quad \frac{\Delta V}{V_0} = -10^{-7}(86.92 - 97.5 \times 10^{-5}p)\,p$$

$$\text{At } 75^\circ \quad \frac{\Delta V}{V_0} = -10^{-7}(89.72 - 107.3 \times 10^{-5}p)\,p.$$

The initial compressibility at 20° of lithium of density 0.534 was found by Richards to be 88×10^{-7}. This agrees with the above within the limits of error indicated by Richards' two significant figures.

Sodium. This was from Eimer and Amend, obtained before the war. It was not the identical lot, but was obtained at about the same time and is presumably similar to the lot for which I have already determined the effect of pressure on the electrical conductivity and melting point. No chemical analysis has been made, but the sharpness and high value of the freezing point are evidence of its high purity. The specimen was prepared by cutting from the interior of a large coherent block of sodium a smaller piece, perfectly clean and with no inclusions of any sort, and forming this by cold extrusion and pressing in a special mold into a cylinder 0.9 cm. in diameter and 2.5 cm. long. This was mounted as a compression specimen in the apparatus for direct measurement without magnification.

The two regular runs were made at 30° and 75°. There was no incident at 30°; the points all lay smoothly and there was no perceptible permanent deformation after the run. At 75°, however, which is much nearer the melting point of this very soft substance, there was permanent set after the run, and it was obvious from the shape of the curve that the set had taken place during decreasing pressure, at the lower end of the pressure range. This set may easily have been due to viscosity in the transmitting medium. The last points at 75° were therefore discarded. Except for these points the mean departure of all the observed points from a smooth curve was 0.22%, and the maximum deviation from linearity 7.4% of the maximum pressure effect. The deviation from linearity was perceptibly greater at the higher temperature. Furthermore, the deviation is not symmetrical about the mean pressure, and it is not possible to represent the change of volume by a two constant formula in the pressure. In fact the deviation from linearity is so marked that I have thought it best to compute the changes of volume directly from the original data at every thousand kilogram interval at 30° and 75°. The values so computed are listed in the following table. The change of volume at each temperature is computed in terms of the volume at atmospheric pressure and 20° (room temperature) as unity.

TABLE II.

CHANGE OF VOLUME OF SODIUM UNDER PRESSURE.

Pressure kg/cm²	$\Delta V/V_0$ 30°	$\Delta V/V_0$ 75°
1000	−.0153	−.0166
2000	.0299	.0321
3000	.0437	.0466
4000	.0570	.0602
5000	.0697	.0732
6000	.0819	.0856
7000	.0937	.0976
8000	.1050	.1093
9000	.1159	.1206
10000	.1263	.1316
11000	.1365	.1423
12000	.1465	.1528

For comparison with the initial compressibility of Richards we may either plot the above results on a large scale and draw the tangent to the curve at the origin, or we may pass a power series curve through the low pressure points. I have passed a three constant curve through the points at 1000, 2000, and 3000 kg. at 30° and obtain the formula:

$$\frac{\Delta V}{V_0} = -1.562 \times 10^{-5} p + 3 \times 10^{-10} p^2 + 1.6 \times 10^{-13} p^3.$$

This formula is not adapted to represent the change of volume outside its own range; already at 4000 kg. the change of volume given by the formula is 0.0566 against 0.0570 experimental. The above formula would give for the initial compressibility 156.2×10^{-7}. Richards' value for the average compressibility over the range of 500 kg. is 153×10^{-7}. He states that over his range the compressibility did not depart sensibly from constancy, but the above formula would show that the average compressibility over the range of 500 kg. differs from its initial value by 1.5×10^{-7}. Making this correction, the agreement with the result of Richards (now 154.5×10^{-7}) becomes closer, and perhaps is as good as could be expected.[25]

Potassium. This was from the same lot as that for which the effect of pressure on freezing point [15] and electrical conductivity [16] has already been determined. The high value and sharpness of the freezing point are evidence of its high purity. The material was prepared for these measurements by first filtering by forcing in a melted condition through a fine gauze under oil (Nujol); it was then cast under oil into a coherent slug, cleaned by scraping the outside surface, and finally formed by squeezing in a mold into a cylinder 0.9 cm. in diameter and 1.7 cm. long. This was mounted as a compression specimen in the apparatus for direct measurement without magnification.

A considerably more elaborate series of measurements was made on this than on other materials. I had already found [12] that the behavior of electrical resistance under pressure was unusual in that above 6000 kg. the temperature coefficient of resistance, which is usually constant over the entire pressure range, begins to decrease rapidly. At the time of making this observation I ventured the guess on theoretical grounds that at high pressures the thermal expansion would show a large rate of decrease. In order to determine the behavior of the thermal expansion at high pressures the following measurements were made. It is to be noticed that theoretically a measurement of the compressibility at two different temperatures is capable of giving information as to the variation of thermal expansion with pressure, but the information so given is not very accurate. In the first place, the compressibility of potassium was measured in the ordinary way by two runs at 30° and 60°. The maximum pressure of these measurements was 10000 kg. instead of the usual 12000; the compressibility of potassium was so high that at 12000 kg. the relative deformation was beyond the range of the apparatus. At 60° the pressure was not allowed to fall below 500 kg., because of the proximity of the melting point. After these two series of runs over the pressure range, direct measurements of the thermal expansion were made by changing the temperature from 30° to 60° at constant position of the piston (constant mean pressure) at a number of different pressures distributed over the range. Two series of these temperature observations were made. The scheme of the method and the manner of computing the results is precisely the same as that which I have previously described in detail in connection with the measurement of the properties of liquids under pressure.[17] The only difference is that here the dimensions of the specimen are obtained by an electrical measurement instead of by measurement of the position of the piston. Is it not worth while again going into the details of the computations, which involve nothing not sufficiently obvious.

The average arithmetical departure from a smooth curve of the points of the two pressure runs (no discards) was 0.41% of the maximum pressure effect, and the average change corresponding to the thermal expansion under a 30° change of temperature was about ten times this. The measurements of thermal expansion, being much smaller, were not relatively so consistent among themselves. The first run for the determination of expansion gave scattering points at the high pressures, which, however, were not inconsistent with the later more accurate results. On the second run a better technique was used, and all the points except one lay on a smooth curve, this one lying off by some 25% of the observed effect. It is a question whether weight is to be attached to this discordant point. It has seemed to me that a certain amount of weight is to be attached to this point, as thus a curve is obtained without sudden changes of curvature, and the results tabulated have been thus computed. If no weight whatever is attached to this point, then the thermal expansion in the extreme case remains constant over the range from 6000 to 12000 kg., instead of dropping from 83 to 50, as shown.

The curvature of the graph of volume against pressure is so great that, as also in the case of sodium, it was necessary to compute the changes of volume directly from the original data at intervals of 1000 kg. In the following table is given the changes of volume at even 1000 intervals at the mean temperature of 45°, obtained by taking the mean of the two series of measurements at 30° and 60°, and there is also shown the mean thermal expansion over this range, obtained by dividing the smoothed observed expansion over the range from 30° to 60° by 30. The tabulated changes of volume and the expansion are those of a mass of potassium that occupies 1 c.c. at 20° at atmospheric

TABLE III.

Volume Changes of Potassium under Changes of Pressure and Temperature.

Pressure kg/cm²	$\Delta V/V_0$ at 45°
1000	−.0330
2000	.0615
3000	.0867
4000	.1095
5000	.1306
6000	.1504
7000	.1694
8000	.1877
9000	.2053
10000	.2222
11000	.2385
12000	.2544
	Mean Thermal Expansion between 30° and 60°
0	.000241
2000	148
4000	105
6000	083
8000	073
10000	060
12000	050

pressure. The initial point of the thermal expansion curve was taken from the work of others; these measurements cannot give it.

To obtain the initial compressibility I have proceeded as in the case of sodium, passing a three constant power series in the pressure through the three lowest points. The formula so found is:

$$\frac{\Delta V}{V_0} = -3.565 \times 10^{-5} p + 2.85 \times 10^{-9} p^2 - 2 \times 10^{-13} p^3.$$

The formula is not good outside its range; at 4000 kg. it gives for ΔV 0.1098 against 0.1091 experimental, and at 10000 kg. 0.27 against 0.22 experimental. The formula shows that the initial compressibility at 45° is 356.5×10^{-7}, and the average compressibility over the range of 500 kg. 342.3×10^{-7}. The above values for the dependence of thermal expansion on pressure show that the temperature correction on this compressibility for reducing from 45° to 20° is 12.5×10^{-7}, making the average compressibility over the first 500 kg. at 20° 329.8×10^{-7}. The value given by Richards is 311×10^{-7}. Perhaps this is as good as could be expected in view of the experimental error and the range of the extrapolation above.

Especial attention is to be paid to the behavior of the thermal expansion under pressure, which is that anticipated theoretically. There can be no question of the large decrease; at the minimum, which is improbable, it has decreased by a factor of 3, and more probably by a factor of 5. The compressibility in the same range has decreased by a factor of 2. This reverses the initial behavior, for at low pressures the relative decrease of compressibility with pressure is about 50% greater than that of thermal expansion. The difference of behavior of metallic potassium and an ordinary liquid is of interest. Some time ago I measured the compressibility and expansion under pressure of a number of organic liquids.[13] The behavior of all these liquids is on the average roughly the same. The total change of volume under 12000 kg. of these liquids is about the same as for solid potassium, being 0.292 for the liquids against 0.254 for potassium. But the compressibility of the liquids shows a change which is relatively high, and is also much higher than the change of expansion. In the range of 12000 kg. the average compressibility of the liquids has changed by a factor of 15.2, whereas the expansion has changed by a factor of 4.66; potassium on the other hand has a considerably larger relative change in the expansion than in the compressibility. The significance of the behavior of potassium I believe to be that the outer shells of the atoms are being pushed tightly into permanent contact, that the thermal agitation consists merely of a swinging back and forth within the atom of the heavy nucleus, and that the restoring force on the nucleus is proportional to a linear function of its displacement.

This completes the list of those substances which are known or presumed to crystallize in the cubic system, and for which therefore the assumption is justified that the linear compressibility is the same in all directions. Of the above list all except three have been definitely proved to be cubic. These three are Sr, Ur, and Tl. Sr resembles Ca chemically and is probably cubic. With regard to Ur there is not much basis for estimate. We have seen above that it is highly probable that Tl is not cubic because of the discrepancy between my value of the compressibility calculated above on the basis of uniform compressibility in all directions and that of Richards.

METALS CRYSTALLIZING NOT IN THE CUBIC SYSTEM.

No system except the cubic enjoys in general the property of having the same compressibility in all directions under hydrostatic pressure. It is, however, possible that there should be such relations between the elastic constants as to give this property to particular metals crystallizing in other systems, or it may be that the difference of compressibility in different directions is so small that experimentally it is not of importance. Now a study of crystal models makes it seem highly probable that in the hexagonal close packed arrangement of spheres the compressibility is the same in every direction. In fact the hexagonal and the cubic close packed arrangement of spheres are the same as far as the adjacent layers of atoms are concerned, so that any departure of the elastic properties of the hexagonal from that of the cubic close packed arrangements can be due only to forces between atoms separated from each other by at least one intervening atom, and these forces are presumably weak. Now of the metals studied here, magnesium crystallizes in the arrangement of hexagonally close packed spheres. (Of the metals listed above as cubic it will be found that Hull has evidence that two, cobalt and cerium, may also crystallize in the hexagonal system, but since the hexagonal arrangement in which these crystallize is the close packed arrangement of spheres, the compressibility would be expected to be the same in all directions, so that the possibility of the existence of two forms introduces no complications in the case of these two metals.)

It is probable therefore that the method of linear compressibility gives at once without question the cubic compressibility of magnesium, and the data are accordingly given here.

Magnesium. This material I owe to the kindness of Professor C. C. Bidwell, who stated that it was of unusually high purity, but the exact chemical analysis was not known. Professor F. A. Saunders was kind enough to make a spectroscopic analysis for me with the following results: Fe, none, Zn, perhaps none, Ca small, Cu small, Al faintest trace, Sr none, Cd faintest trace.

The material was originally provided in the form of a square bar, about 1.3 cm. on a side and 1 cm. long. I extruded this in two stages at 500° to a piece of round section 0.3 cm. in diameter. It was straightened after extruding by rolling between iron plates heated to a dull red, and then annealed by heating for several hours to 300°, sealed in a glass tube to protect from oxidation. The final specimen, which was 8 cm. long, was placed in a steel sleeve in order to adapt it to the apparatus, and mounted as a compression specimen in the apparatus for direct measurement without magnification.

The regular series of runs at 30° and 75° was made. The average arithmetical deviation from a smooth curve of the 29 observed points (one discard) was 0.49% and the maximum deviation from linearity was 3.8% of the maximum pressure effect. The maximum deviation from linearity was the same at the two temperatures. Within the limits of error the deviation from linearity is symmetrical about the mean pressure so that it was possible to represent the results by a two constant formula.

In addition to the measurements on this pure sample, in the early stages of the development work several series of readings were made on an extruded sample of commercial magnesium of unknown purity obtained from Eimer and Amend. These results agreed within the limits of error, two or three per cent, with those found for the pure sample. This makes it probable that small impurities do not exert an important effect on the compressibility, and also lends much plausibility to the contention that the compressibility of this material is the same in different directions.

The final results are given by the formulas:

$$\text{At } 30^\circ \quad \frac{\Delta V}{V_0} = -10^{-7}(29.60 - 20.3 \times 10^{-5}p)p$$

$$\text{At } 75^\circ \quad \frac{\Delta V}{V_0} = -10^{-7}(29.97 - 18.0 \times 10^{-5}p)p.$$

Richards gives for the average compressibility to 500 kg. (no distinction possible between the initial and the average compressibility) the value 29×10^{-7}, pressure being expressed in megabars. The correction for converting to kilograms would reduce this figure by about 2%. The formulas above give for the average compressibility to 500 kg. at 20° the value 29.4×10^{-7}. Considering the number of significant figures given by Richards, the agreement is within the possible experimental error, and thus we again have presumptive evidence of the equal compressibility of magnesium in all directions.

Magnesium is the only one of the non-cubic metals that may be expected to have equal compressibility in all directions. The other metals of this group measured were Bi, Sb, Cd, Sn, Zn, and Te. A complete description of their behavior would demand a determination of the compressibility in different directions specified with respect to the crystalline axes. This is much beyond the scope of the present work. For some of these metals the most that can be obtained from these measurements is some idea of the magnitude of the variation with direction that may be expected. This is given roughly by the variation in the numbers obtained for the compressibility with different methods of preparing the sample. The first strong evidence of this effect was found with bismuth.

Bismuth. The material used for these measurements was commercial electrolytic bismuth, obtained through the courtesy of the U. S. Lead Refinery, Inc. The properties of this bismuth have already been discussed in considerable detail [18]; the only measurable impurity is 0.03% of silver. Measurements were made on four samples. Two of these were castings, one cast in an iron mold, and the other in a thin walled graphite mold, chilled by slowly lowering into water. The other two specimens were extruded from the casting, the reduction in diameter being 2 to 1. Measurements on two of these samples, the casting in the iron mold, and one of the extruded pieces, were made with the preliminary apparatus, before the greatest accuracy was obtained. The initial compressibility at 30° of the extruded piece was 31.8×10^{-7}, and that of the casting 25.6×10^{-7}. The measurements of the casting were repeated to be sure that there was no error. Although these measurements were not as accurate as those finally made, still the accuracy was certainly one or two per cent, and the difference between these two samples was far beyond any possibility of experimental error.

Measurements on the two other samples, the casting made in graphite (density at 20° 9.803) and one of the extruded pieces (density at 20° 9.797) were made with the finally perfected apparatus, that for direct measurement without magnification. The pieces were mounted as compression specimens, and their length was 16.3 cm. The mean deviation from a smooth curve of the 28 readings on the extruded specimen (no discards) was 0.32% of the maximum pressure effect, and the maximum departure from linearity was 3.5%. The corresponding figures for the casting (no discards) were 0.24% for the deviation from a smooth curve and 3.07% for the maximum departure from linearity. The final results are given by the formulas:

$$\text{Extruded cylinder, At } 30^\circ \quad \frac{\Delta V}{V_0} = -10^{-7}(35.35 - 28.0 \times 10^{-5}p)\,p$$

$$\text{At } 75^\circ \quad \frac{\Delta V}{V_0} = -10^{-7}(35.94 - 28.7 \times 10^{-5}p)\,p$$

$$\text{Cast cylinder, At } 30^\circ \quad \frac{\Delta V}{V_0} = -10^{-7}(22.02 - 9.0 \times 10^{-5}p)\,p$$

$$\text{At } 75^\circ \quad \frac{\Delta V}{V_0} = -10^{-7}(22.11 - 9.0 \times 10^{-5}p)\,p.$$

These results were calculated on the assumption of equal compressibility in all directions. One third the difference between the results is a minimum measure of the difference of the linear compressibility in different directions in the crystal. It is to be noticed that both the pressure coefficient and the temperature coefficients are greater in that direction in which the linear compressibility is the greater.

Richards and A. W. J. have both measured the cubic compressibility, that is, the mean of the linear compressibility in all directions. Richards finds for the initial value of bismuth of density 9.80, 29×10^{-7}. A. W. J. find the same initial value, and the average compressibility to 10000, corrected by my new value for iron, has dropped by 2.3×10^{-7} below its initial value. It is seen that both the initial value and the pressure coefficient of A. W. J. are included in the range of values above.

It should be possible to calculate the difference of the linear compressibility in different directions from the elastic constants of bismuth, but these do not seem to have been determined with sufficient accuracy.

Tin. The material was a melting point sample from the Bureau of Standards, and had the following analysis: Pb 0.007, Cu 0.003, Fe 0.002, Sb not detected, As and S trace, Sn (by diff.) 99.988. It was investigated in the form of a simple casting, cast in graphite and annealed at a temperature of 150° for several hours after casting, of density 7.296 at 20°, or in the form of an extruded casting, the extrusion reducing the diameter from 1.2 cm. to 0.6 cm. The extrusion was performed at a temperature of about 125°, and after extrusion the piece was seasoned at 150° for several hours. The density of this piece at 20° was 7.302.

Two runs were made on the extruded pieces; one was with a preliminary form of apparatus and the results were not as accurate as those finally obtained. The preliminary run gave for the average compressibility to 12000 kg. at 30° 16.2×10^{-7}. The other run on the extruded piece was on a sample about 16 cm. long, mounted as a compression piece in the apparatus for direct measurement without magnification. The average arithmetical deviation from a smooth curve of the 40 readings (three discards) was 0.23% and the maximum deviation from linearity was 2.0% of the maximum pressure effect. The runs on the simple casting were also made in the final apparatus for direct measurement, and the specimen was also 16 cm. long. The mean arithmetical deviation from a smooth curve of the 28 readings (no discards) was 0.25%, and the maximum deviation from linearity was 1.09% of the maximum pressure effect. The values of cubic compressibility given by these two samples, assuming in the computation equal compressibility in all directions, was:

Extruded rod, At 30° $\frac{\Delta V}{V_0} = -10^{-7}(19.53 - 9.6 \times 10^{-5}p)\,p$

At 75° $\frac{\Delta V}{V_0} = -10^{-7}(20.11 - 9.9 \times 10^{-5}p)\,p$

Cast rod, At 30° $\frac{\Delta V}{V_0} = -10^{-7}(17.01 - 5.17 \times 10^{-5}p)\,p$

At 75° $\frac{\Delta V}{V_0} = -10^{-7}(17.37 - 5.75 \times 10^{-5}p)\,p.$

The difference between extreme values is not as large as in the case of bismuth. Again the direction in which the linear compressibility is the greatest is the direction in which the pressure and the temperature coefficients of compressibility are the greatest. The crystal structure of ordinary tin determined by X-ray methods is side-face-centered tetragonal.

The initial compressibility at 20° of tin of density 7.29 is given by Richards as 19×10^{-7}. A. W. J. give 18.5 for the initial and 17.0 ×

10^{-7} (corrected for the new value of iron) as the average to 10000. The initial values of these two observers fall within the extremes above, but the decrease of compressibility with pressure found by A. W. J. is somewhat greater than the larger of the two above values.

Antimony. Two different specimens from two different sources were used. One was so-called chemically pure antimony from the J. T. Baker Chemical Co. It was cast in a graphite mold, the mold being preheated to above the melting point of antimony, and chilled by slowly lowering into water. The second specimen was antimony from Kahlbaum, his "K" grade. I have no analysis, but antimony from this source is known to be of high purity. This also was cast, but by pouring into a groove machined in a massive iron bar, the bar being cold. The manner of chilling the two castings was therefore entirely different, and the crystalline orientation in the two castings should be different. The casting of Baker's antimony, which was 14.6 cm. long, was mounted as a compression specimen in the apparatus for direct measurement without magnification. Its density at 20° was 6.678. The other casting, which was given its final shape by grinding, was 2.3 cm. long, and was mounted as a compression specimen in the lever apparatus for short specimens.

The regular series of runs at 30° and 75° was made on each specimen. The average arithmetical departure from a smooth curve of the readings on Baker's antimony (one discard) was 0.48%, and the maximum departure from linearity was 1.67% of the maximum pressure effect. For the casting of Kahlbaum's antimony the average arithmetical deviation from a smooth curve (four discards) was 0.68%, and the maximum deviation from linearity was 2.7%. The results, computed on the assumption of equal compressibility in all directions, are:

Baker's antimony, slowly cooled in graphite,

$$\text{At } 30^\circ \quad \frac{V\Delta}{V_0} = -10^{-7}\,(14.69 - 6.2 \times 10^{-5} p)\,p$$

$$\text{At } 75^\circ \quad \frac{\Delta V}{V_0} = -10^{-7}\,(14.80 - 6.3 \times 10^{-5}\,p)\,p\,.$$

Kahlbaum's antimony, rapidly chilled in iron,

$$\text{At } 30^\circ \quad \frac{\Delta V}{V_0} = -10^{-7}\,(20.41 - 12.9 \times 10^{-5}\,p)\,p$$

$$\text{At } 75^\circ \quad \frac{\Delta V}{V_0} = -10^{-7}\,(20.50 - 12.9 \times 10^{-5}\,p)\,p\,.$$

The range of values is nearly as great as for bismuth. Here the temperature coefficient of compressibility is greater for that direction in which the linear compressibility is less, although the pressure coefficient of compressibility is less in this direction, as is to be expected.

For the initial value of the compressibility of antimony of density 6.71 Richards gives 24×10^{-7}. If there is no error, the significance of this result, when compared with that above, is merely that the two methods of casting adopted above did not bring out the extreme possible difference in the orientation of the crystalline grains.

Cadmium. Runs were made on three different samples. The first two were made from chemically pure cadmium from Eimer and Amend. The first was extruded at a temperature slightly below the melting point from a diameter of 1.2 to 0.6 cm., and was annealed at 230° for ten or fifteen minutes. Its density as 20° was 8.652. The second came from the same source and was cast in a graphite mold, preheated above the melting point, and the casting was cooled by lowering the mold slowly into water, and annealed after casting, as above. The density at 20° was 8.644. These two pieces were machined to a length of 16 cm. and mounted as compression specimens in the apparatus for direct measurement without magnification.

The source of the other specimen was Kahlbaum, his "K" grade. This is known to have only a few hundredths of per cent of impurity. This was cast in a special manner. The metal was melted in a tube of pyrex glass of diameter about 0.6 cm. in a cylindrical electric furnace mounted with its axis vertical. After temperature equilibrium was attained the glass tube was lowered slowly through the bottom of the furnace into the air of the room, thus allowing the metal in the lower part of the tube to solidify. The lowering was accomplished by clockwork, and was at the rate of about 12 cm. per hour. It will be seen that the conditions were highly favorable to a similar orientation of the crystals through the entire mass of the casting, except perhaps at the lower end where solidification began. It seems likely that in the long narrow tube, even if crystallization may originally begin about several nuclei, that eventually one surface of advance of the crystal, that on which growth is most rapid, will predominate over the others, and that after this, crystallization will consist merely in a single surface sweeping along the tube at a speed to keep pace with the lowering of the tube. This presumption as to the uniformity of crystal structure throughout the casting should be verifiable by X-ray analysis, or perhaps by a study of the etch figures; I have not yet made such an analysis. In the case of zinc and tellurium, which were cast in the same way, the appearance of the casting indicated almost unescapably the similarity of the crystalline orientation throughout the mass. At any rate, this method of casting seems to offer important possibilities, and has at least brought out great differences in compressibility. A casting made in this way I shall call a "unicrystalline" casting.

Regular runs were made on the first two samples of cadmium at 30° and 75°; on the third a run only at 30° was made. The results with these castings of cadmium had as a rule more error than the other compressibility measurements; the irregularity is probably a real phenomenon, and may be due to a slipping of the crystalline grains under pressure. In one case the permanent deformation after the initial application of pressure was considerably greater than for most other metals.

For the simple casting of Eimer and Amend cadmium the average arithmetical departure from a smooth curve of the 27 observed points (no discards) was 0.29% and the maximum deviation from linearity was 2.28% of the maximum pressure effect. For the extruded casting of Eimer and Amend the arithmetical departure from smoothness (five discards) was 0.44% and the maximum deviation from linearity 2.33%. The unicrystalline casting of Kahlbaum's cadmium at 30° (one discard) gave the figures respectively 1.8% and 2.03%. The results calculated from these different specimens, assuming equal compressibility in all directions, are:

Eimer and Amend, simple casting,

$$\text{At } 30^\circ \quad \frac{\Delta V}{V_0} = -10^{-7}\,(19.54 - 10.7 \times 10^{-5}\,p)\,p$$

$$\text{At } 75^\circ \quad \frac{\Delta V}{V_0} = -10^{-7}\,(20.19 - 11.0 \times 10^{-5}\,p)\,p\,.$$

Eimer and Amend, extruded casting,

$$\text{At } 30^\circ \quad \frac{V\Delta}{V_0} = -10^{-7}\,(14.17 - 7.5 \times 10^{-5}\,p)\,p$$

$$\text{At } 75^\circ \quad \frac{\Delta V}{V_0} = -10^{-7}\,(14.53 - 7.7 \times 10^{-5}\,p)\,p\,.$$

Kahlbaum's, unicrystalline casting,

$$\text{At } 30^\circ \quad \frac{\Delta V}{V_0} = -10^{-7}\,(8.57 - 3.8 \times 10^{-5}\,p)\,p\,.$$

The range in values is very striking, being almost two and a half fold. The difference between the figures for the ordinary and the unicrystalline casting tends to be the extreme. This is as one would expect, because the manner of chilling would tend to make that surface which grows along the axis of the unicrystalline cylinder grow at right angles to the exterior curved surface in the simple casting, advancing along the radius.

The initial compressibility at 20° of cadmium of density 8.60 was found by Richards to be 21×10^{-7}. A. W. J. found for the initial compressibility 22.2×10^{-7}, and for the average compressibility to 10000, 19.2×10^{-7}. This pressure coefficient of compressibility is at the extreme edge of the values found above, but the initial compressibility of these two observers is outside the range above, which would mean a greater variation of compressibility with direction than is suggested by my measurements.

Zinc. All the specimens were made from melting point samples of the Bureau of Standards of the following analysis: Pb 0.0026, Cd 0.0022, Fe 0.0034, Cu (not detected), Zn (by diff.) 99.992. More elaborate measurements were made on this than on any other metal, and the results show a greater effect of crystal structure. In general the results were much more irregular than for any other metal; there were more likely to be permanent distortions after the first application of pressure, there was quite often hysteresis, and the relation between pressure and deformation did not depart regularly from linearity, but on several occasions curves were found with distinct points of inflection. (The total departure from linearity of these abnormal cases was never large.) It will not pay to describe in detail all the anomalous effects found, but I will give merely the average compressibility between 0 and 12000 kg. at 30°. The effect of temperature was studied in only one or two cases.

With the preliminary apparatus measurements were made on three different samples. The first of these, a casting, had an average compressibility between 0 and 12000 kg. of 24×10^{-7}, the second, an extruded casting, had a mean compressibility of 12.8×10^{-7}, and the mean compressibility of the third, also an extruded casting, was 10.7×10^{-7}. Measurements with the final form of apparatus and another simple casting gave for the mean compressibility 9.04×10^{-7}, and on an extruded casting 8.58×10^{-7}. The density at 20° of the simple casting was 7.129, and of the extruded casting 7.134.

Finally a unicrystalline casting was made by the procedure described under cadmium, and measurements made of the compressibility of this in three mutual perpendicular directions. This was accomplished as follows. First the casting in its original form, 16 cm. long, was measured in the regular way as a compression specimen. From this casting were then cut four cubes 0.6 cm. on a side, marking the orientation with respect to the original casting, and these cubes were piled on top of each other, and the linear compressibility measured along the axis of the pile for two different ways of piling the cubes together, corresponding to two directions at right angles to the axis of the original casting. These measurements were made in the lever apparatus for long specimens. The mechanical properties of this casting were quite unusual in that it was very flexible, so that great difficulty was experienced during the machining of the cubes to prevent the casting from bending. The bending was not accompanied by the characteristic noise of the bending of an ordinary casting. It seems not unlikely that the stiffness of the ordinary zinc casting is due to a dovetailing together of crystal grains in different orientations, which cannot be deformed as a single homogeneous mass because of the great difference of the elastic constants of the grains in different directions.

Dr. G. L. Clark has been so kind as to make an X-ray analysis of this specimen of zinc for me, and finds, within the possible errors of the method, that the crystalline orientation is indeed the same throughout the specimen, and furthermore that the direction of the hexagonal axis is along the axis of the original casting within a few degrees.

The results found for the cubic compressibility of the unicrystalline casting in three mutually perpendicular directions, (that is, the linear compressibility in these directions multiplied by three) were: along the axis of the original casting 4.98×10^{-7}, one of the perpendicular directions, 15.9×10^{-7}, and the other perpendicular direction 21.4×10^{-7}. The average of these three is 14.1×10^{-7}, which is approximately the average cubical compressibility, and would be the result of a single measurement by the method of Richards or A. W. J. The initial value of Richards is 17×10^{-7}. The initial value of A. W. J. is 17.1×10^{-7}, and their average value to 12000, corrected according to the new values for iron, is 15.1×10^{-7}. The agreement of this last figure with the average above is as close as could be expected when one considers the possible error of my measurements on four small cubes piled together, and constitutes very strong evidence as to the reality of the large differences of linear compressibility found above in different directions.

Tellurium. This material I owe to the kindness of the Raritan Copper Works, from whom I obtained it a number of years ago. I have no chemical analysis. This was prepared as a unicrystalline casting in the manner already described for cadmium. The crystalline figures were very prominent in this casting, and show the orientation of the crystal to be the same throughout the entire casting. The appearance of the casting is somewhat tree-like, the axis of the tree is inclined at an angle of about 15° to the axis of the casting.

The specimen was mounted as a compression specimen (length 16 cm.) in the apparatus for direct measurement. Only a few measurements were made on it, because it was at once evident that the behavior is such as to be worthy of much more detailed study, which must be put off until a systematic investigation can be made of the variation of compressibility with direction of a large number of crystals. The measurements, as far as they went, were perfectly regular. There was very little permanent set on the first application of pressure, and the relation between distortion and pressure was sensibly linear. The startling result found for this substance was that its compressibility along the axis of the casting is negative; that is, that it lengthens along the axis when subjected to a uniform hydrostatic pressure all over. Its average *linear* compressibility in this direction was -2.36×10^{-7}.

Paradoxical as this behavior appears, there is nothing inherently impossible about it, and something analogous has been observed by Voigt in the negative Poisson's ratio in certain directions in pyrites. (This remark is merely by way of analogy; the linear compressibility of pyrites is not negative in any direction.)

Discussion of Results.

Although the experiments just described give the compressibility of some substances not measured before, and also I believe give the compressibilities with greater accuracy than hitherto attained, the principle interest in the measurements is in the variations of compressibility with temperature and pressure. The variation of thermal expansion with pressure is also given by the experiments, because of course it is a mathematical identity that the pressure coefficient of thermal expansion is the same as the temperature coefficient of compressibility.

The results of these experiments are collected in Table IV. In the first column is given the name and description of the substance, in the second column the initial compressibility at 30° multiplied by 10^7,

TABLE IV.

Collected Results.

Substance	χ_0 at 30° $\times 10^7$	$\left(\frac{1}{\chi}\frac{\partial \chi}{\partial p}\right)_0 \times 10^6$	$-\left(\frac{1}{\alpha}\frac{\partial \alpha}{\partial p}\right)_0 \times 10^6$	$\frac{-1}{\chi_0}\left(\frac{1}{\alpha}\frac{\partial \alpha}{\partial p}\right)_0$	$\left(\frac{1}{\chi^2}\frac{\partial \chi}{\partial p}\right)_0$
Al, hard drawn	13.40	.52	....	....	3.90
drawn and annealed	13.34	.50	1.76	13.2	3.96
cast	13.43	.74	1.02	7.6	5.70
Sb, Kahlbaum's cast	20.41	1.26	0.61	3.0	6.2
Baker's cast	14.69	.84	0.75	5.13	5.74
Bi, cast	22.02	.82	0.50	2.27	3.92
extruded	35.35	1.58	3.3	9.30	4.50
Cd, Eimer and Amend extruded	14.17	1.06	1.08	7.64	7.48
Eimer and Amend cast	19.54	1.10	1.89	9.67	5.60
Kahlbaum's unicrystalline	8.57	.88	....	....	10.32
Ca, G. E. Co.	59.46	1.64	.62	1.04	2.76
Prof. Bidwell's	56.97	1.66	4.5	7.90	2.90
Ce, extruded	35.74	1.06	....	....	2.98
Co, wire	5.39	.78	.48	8.93	14.28
Cu, Bureau of Standards	7.19	.72	.67	9.35	10.06
commercial, unannealed	7.32	.74	.31	4.23	10.06
commercial, annealed	7.29	.74	.36	4.94	10.06
Ge, casting	13.78	.98	....	....	7.16
Au, short specimen	5.78	.74	....	....	
long specimen	5.77	1.08	....	....	18.66
Fe, pure	5.87	.70	.37	6.30	12.16
Pb, cast	23.73	1.46	1.51	6.35	6.14
extruded	23.05	1.04	1.46	6.32	4.64
Li, extruded	86.92	2.24	3.3	3.8	2.58
Mg, extruded	29.60	1.36	1.05	3.54	4.64
Mo, drawn wire	3.61	.58	.2	5.54	16.12
swaged rod	3.47	.72	.2	5.76	20.8
Nichrome, wire	5.50	.54	....	....	9.94
Ni, commercial rod	5.25	.80	.16	3.04	15.2
pure wire	5.29	.80	.32	6.05	15.0
Pd, drawn rod	5.19	.80	....	....	15.54
drawn wire	5.28	.80	.18	3.41	15.02
Pt, drawn rod	3.05	.0	.33	10.8	.0
drawn wire	3.60	1.00	.33	9.2	28.0
K,	356.5	23.	20.	5.62	9.76
Ag, drawn rod	9.87	.90	.66	6.68	9.04
Na,	156.2	7.8	.37	4.43	4.92
Sr, extruded	81.87	1.78	....	....	2.16
Ta, wire	4.79	.10	1.26	2.63	2.18
Sn, extruded	19.53	.98	1.92	9.85	5.04
cast	17.01	.60	1.19	7.00	3.58
W, drawn wire	3.15	1.02	.2	1.24	32.2
swaged rod	2.93	1.02	.4	13.7	35.0
Te, unicrystalline casting	−7.1	....	....	....	....
Tl, extruded	34.2	3.36	1.53	4.48	1.94
Ur, rolled sheet	9.66	.52	....	....	5.36

(this is merely the coefficient "a" of the formula $\frac{\Delta V}{V_0} = -ap + bp^2$, where p is expressed in kg/cm²). In the third column is given the initial proportional change of compressibility with pressure, or $\left(\frac{1}{\chi}\frac{\partial \chi}{\partial p}\right)_0$. Here $\frac{\partial \chi}{\partial p}$ is $2b$ of the above formula for change of volume, and χ_0 is the "a" of the formula. In the fourth column is given the initial proportional change of thermal expansion with pressure $\left(\frac{1}{\alpha}\frac{\partial \alpha}{\partial p}\right)_0$. In calculating the figures of this column, the derivative $\left(\frac{\partial \alpha}{\partial p}\right)_0$, which is the same as $\left(\frac{\partial \chi}{\partial t}\right)_0$ or $\frac{\partial a}{\partial t}$, was computed from the detailed formulas already given for the compressibility at 30° and 75° merely by subtracting the value of "a" at 30° from the value at 75° and dividing by 45. The value of α_0 I took in most cases from the table in the paper of Richards [1] summarizing his compressibility data. In the fifth column is given the proportional change of expansion of the fourth column divided by the compressibility "a." Finally in the sixth column is given the proportional change of compressibility divided by the compressibility. An examination of the details of the measurements will show that less accuracy is to be attached to quantities involving changes with temperature than with pressure.

In every case the compressibility decreases with increasing pressure. This is entirely what we would expect, but I do not know that there is any necessity here, so that a case of increasing compressibility with pressure would violate any of the laws of physics. In fact it is not at all inconceivable that this may be found to be actually the case for some exceptional substances. For instance, there are a number of cases known in which the polymorphic form with the smaller volume has the larger compressibility, and it is conceivable that a change similar to one which in the case of polymorphism takes place abruptly might in some cases be brought about gradually by increasing pressure, so that the form stable at the higher pressures and therefore with the smaller volume should be the more compressible. In fact for potassium the proportional compressibility, $\frac{1}{v}\left(\frac{\partial v}{\partial p}\right)_T$, does increase slightly at the higher pressures.

The order of magnitude of the change of compressibility with pressure, when pressure is expressed as here in terms of kilograms per square centimeter, is seen to be 10^{-5}. The dimensions of this number are those of the reciprocal of a pressure.

The thermal expansion also decreases numerically with increasing pressure in nearly all the cases above, but there are at least two substances in the list above which give a negative change of thermal expansion with pressure that seems to be beyond the experimental error. These are germanium and uranium. Unfortunately the expansion at atmospheric pressure of these substances does not seem to have been determined, so that the proportional change could not be computed, and blanks were accordingly left in the third column of Table IV. The other blanks in this column are also due to missing data, either my own, or that of expansion at atmospheric pressure. Again the order of magnitude of the change of expansion with pressure is 10^{-5}.

The physical significance of the fifth and sixth columns is easy to see. We may write $\frac{1}{\chi^2}\frac{\partial\chi}{\partial p}$ as $\frac{1}{\chi}\left(\frac{1}{\chi}\frac{\partial\chi}{\partial p}\right)$. Now the dimensions of $1/\chi$ are those of a pressure, and numerically $1/\chi$ is the pressure required to halve the volume. Hence $\frac{1}{\chi^2}\frac{\partial\chi}{\partial p}$ is the proportional change of compressibility under a pressure that would halve the volume. Similarly $\frac{1}{\chi}\frac{1}{\alpha}\frac{\partial\alpha}{\partial p}$ is the proportional change of thermal expansion under a pressure which would halve the volume. Both of these quantities are dimensionless, and the most immediately interesting thing about them is their order of magnitude, which is that of a small number. That this would be found to be the case was anticipated some years ago by the theory of the solid state of Grüneisen.[19] Grüneisen's theory went farther, and gave an exact expression for this number in terms of the exponents in the assumed law of attraction and repulsion between the atoms. This exact expression does not seem, however, to fit at all well, and it is most doubtful whether the details of Grüneisen's theory can be maintained. The fact that these two ratios are of the order of magnitude of a small number would probably be given by a number of simple theories, simply because they are dimensionless.

The relative magnitude of the changes of compressibility and expansion with pressure is of some interest. The considerations which I have already mentioned in the case of potassium would lead one to expect that at very high pressures the thermal expansion becomes vanishingly small. But if one plots the above values, ruling out those metals which do not crystallize in the cubic system, it will be found that there are only three, Al, Pb, and Li, for which the thermal expansion initially decreases more rapidly than the compressibility. Even potassium is in the list of those whose compressibility at first decreases more rapidly than the expansion. But at high pressures we have seen that this state of affairs is reversed in the case of potassium. Computation will show that this is also the case for sodium, the expansion at 12000 kg. having dropped to about one third of its initial value, while the compressibility has dropped to two thirds. This may well be the behavior of all metals at sufficiently high pressures.

THEORETICAL CONSIDERATIONS.

By far the most successful theoretical attempt to account numerically for the compressibility of solid substances is that which Born[3] has developed and applied to crystals of the type of NaCl and also to CaF_2 and ZnS. The fundamental thesis of this theory is that the solid is maintained by electrostatic forces, there being in the position of the center of each chlorine atom, in NaCl for example, a single elementary negative charge, and at the center of the Na atom a single elementary positive charge. In addition to the forces between the charged ions, which act on each other according to the inverse square law, there are also mutual repulsions between adjacent atoms due to the electrons in the outer shells of the atoms. Equilibrium is due to a balance between the attractive and repulsive forces. It should be possible to calculate the repulsive forces as well as the attractive forces if the distribution of the electrons within the atom is known. From the assumption that the electrons are in cubical array in the interior of the atom, as they are supposed to be from other considerations, Born has deduced that the potential of the repulsive forces is as the inverse ninth power of the distance between atomic centers, and has furthermore shown that the inverse ninth power gives numerically very approximately the compressibility of crystals of the type of NaCl.

Naturally the first inquiry of an attempt to extend this theory to include metals is whether the fundamental thesis still holds, namely that the forces are essentially electrostatic in nature and are due to single elementary charges or small integral multiples of them situated at the centers of the atoms. A dimensional argument as to the order of magnitude of the quantities involved suggests that the same fundamental thesis does indeed hold. A quantity of the dimensions of compressibility ($M^{-1}LT^{-2}$) is to be built up from the electronic charge e (dimensions of e^2 are MLT^{-2}) and δ, the distance of separation of atomic centers (L). The required combination is at once found to be δ^4/e^2. The very fact that it is possible to build up a combination of these two quantities of the right dimensions is presumptive evidence of the correctness of our general considerations, because in general it would require three (instead of two) quantities to give in combination the dimensions of any one arbitrarily given quantity. This dimensional argument suggests, therefore, that compressibility should be of the order of magnitude of δ^4/e^2. Now assuming simple cubic structure, and expressing δ in terms of the quantities in terms of which it is determined in practise, that is in terms of atmoic weight, density, and the mass of the hydrogen atom, we find that the compressibility is of the order of magnitude of $\frac{(\text{Mass of H atom})^{\frac{4}{3}}}{e^2} \times \left(\frac{\text{At. Wt.}}{\text{Density}}\right)^{\frac{4}{3}}$, or substituting numerical values for e and the mass of the hydrogen atom,

$$\text{Compressibility is of order of } 8.6 \times 10^{-14} \left(\frac{\text{At. Wt.}}{\text{Density}}\right)^{\frac{4}{3}}.$$

Here of course, compressibility is in absolute units, that is, the unit of pressure is 1 dyne/cm^2.

In table V is given the compressibility in absolute units divided by $10^{-14}\left(\frac{\text{At. Wt.}}{\text{Density}}\right)^{\frac{4}{3}}$, for most of the metals measured above. It is seen that the numbers are of the order of magnitude of 8.6, as our argument suggests.

Although the argument is crude, it seems to me that the result is of great significance, because it suggests that in the metal, as well as in salts, the atoms become charged by losing valence electrons, and the metallic structure is held together by the forces between positively charged residues of the atoms and the lost electrons. It is furthermore to be presumed that the lost electrons take some definite position in the crystal lattice, as for instance, the measurements of Hull [20] make so probable in the case of calcium.

To make the next step in refinement in the computation, we must take account of the exact arrangement of the atoms in the lattice, and must also know the law of repulsion. As far as the arrangement of the atoms in the crystal structure goes, the results of X-ray analysis indicate very definitely in at least some cases what we must expect. Thus Hull [20] has found in the case of metallic calcium that the Ca atoms have exactly the same arrangement as the Ca atoms in CaF_2.

TABLE V.

COMPARISON OF COMPRESSIBILITY WITH THE ORDER OF MAGNITUDE SUGGESTED BY DIMENSIONAL ANALYSIS.

Metal	$\dfrac{\text{Compress}}{10^{-14}\left(\dfrac{\text{At Wt.}}{\text{Density}}\right)^{\frac{4}{3}}}$
Li	27.5
Na	23.0
K	22.1
Ca	7.43
Sr	7.30
Mg	8.8
Zn	7.8
Cd	6.7
Cu	5.2
Ag	4.4
Au	2.6
Ge	4.4
Sn	4.6
Pb	4.8
Sb	5.05
Bi	5.10
Mo	1.72
W	1.48
U	3.26
Fe	4.4
Ni	4.3
Co	4.1
Pd	2.7
Pt	1.9

The inference is strongly suggested that in metallic Ca the electrons take the place of the F atoms in CaF_2, and if this is the case, it should be possible to take over with slight modification the considerations which Born has applied to calculating the compressibility of CaF_2. There is also another group of metals about which we may make very plausible assumptions about the positions of the electrons in the lattice. The Na atoms by themselves or the Cl atoms by themselves form in NaCl a face centered cubic lattice. Hence it is natural to expect that the complete crystal structure (meaning by this the arrangement of ions and free electrons) of those metals which are univalent, and therefore readily lose a single electron, and whose atomic nuclei have been proved to crystallize face centered cubic, should be exactly like that of NaCl. Several of the metals of the above list are in this class: Cu, Ag, Au, and probably Pb.

In addition to the crystal structure, we must also know the law of repulsion. This is a much more uncertain matter. An uncritical extension of the argument given by Born would indicate a repulsion whose potential is as the inverse fifth power of the distance of separation of ions and electrons, since his first term in his fundamental development for the potential of a neutral cube and a single point charge is the inverse fifth. A more critical examination of this discussion will raise several doubts, however, and in fact it is very questionable to me how much significance is to be given to the inverse ninth power which Born deduces for NaCl. It is in the first place disconcerting to discover that the inverse fifth power term for a compound lattice of electrons and ions has the wrong sign, so that instead of a repulsion we have an attraction. However, in computing what the action is between a neutral cube and an electron there are several matters with regard to which we are in serious doubt and which may greatly affect the result. Thus the statement that the force between an ion and an external electron is of the wrong sign is based on the assumption that the electron is situated on one of the three axes of the cube perpendicular to the faces at their mid-points. This corresponds to the arrangement assumed by Born for the cubical atoms in NaCl. But it is evident that if the electron approaches the neutral cube along a diagonal, the force will be of repulsion instead of attraction. In crystals not of the simple type of NaCl it is not at all obvious what the arrangement of the cubical atoms in the crystal lattice is, for the symmetry of the crystal does not seem to be consistent with the cubical symmetry of the atom. Thus in the case of CaF_2, the Ca atom is surrounded by 12 other atoms symmetrically situated, so that the symmetry of the dodecahedron is indicated.

Even if the orientation of the atoms with respect to the electron is known and is that assumed by Born, still the inverse fifth power must be open to grave question. Born developed the potential of the cube at external points in a power series, of which the first term was the inverse fifth, the development being good at points whose distance from the center of the cube is large compared with the semi-diagonal, or in other words, the dimensions of the atom are assumed small in comparison with their distances apart. Now this is almost certainly not the case, but a number of lines of argument indicate that the atoms pretty completely fill the total space, and that their external shells are nearly in contact. Hence in most of the cases in practise we can certainly say that more terms than the first of the series are necessary to give an adequate representation, and that probably under many actual conditions the series is actually divergent.

There is another important consideration with regard to the series development. In computing the compressibility it is necessary to differentiate the expression for the potential. Now although the numerical magnitude of the potential itself may perhaps be given with sufficient approximation by the first term of the series, it is quite another matter to expect the derivative to be given by the derivative of the first term.

Apart from these considerations, which after all are concerned with the method of computing the results, there is still another consideration which reaches deeper, and involves the physical picture back of the computations. This has been suggested by Schottky [21] in a recent paper. He finds quite general theorems for electromagnetic systems like the system of ions and electrons in a crystal which connect the changes of internal kinetic and potential energy of the system with the external forces acting on the system. In particular, for a system under external hydrostatic pressure, Schottky obtains the result,

$$\begin{aligned} d\overline{L} &= -\, d\overline{U} + 3d\,(pV) \\ d\overline{E} &= 2d\overline{U} - 3d\,(pV), \end{aligned}$$

where L is the internal kinetic energy, E the internal electro-magnetic energy, and U the total energy of the system. The dashes indicate average values over a time sufficiently long for the average values to be constant. If in particular we now subject the system to an isothermal change of pressure, we have by thermodynamics

$$d\overline{U} = -\left\{\tau\left(\frac{\partial v}{\partial \tau}\right)_p + p\left(\frac{\partial v}{\partial p}\right)_\tau\right\} dp.$$

Substituting above,

$$d\bar{L} = \left\{ 3v + \tau \left(\frac{\partial v}{\partial \tau}\right)_p + 4p \left(\frac{\partial v}{\partial p}\right)_\tau \right\} dp.$$

Now in a solid under normal conditions we may suppose the kinetic energy of motion of centers of mass of the atoms to be a function of temperature only, so that at constant temperature this part of L is constant. But the equation above gives at moderate pressures an increase of L with pressure. As Schottky points out, this means an increase of internal kinetic energy of rotation of the electrons in the atoms about the nuclei, which again means a decrease of atomic radius. In other words, the atoms shrink in size as pressure increases. This reminds one strongly of the compressible atom of Richards.[22] What is more, the change of size is very considerable, so that when a substance is compressed the force between the atoms changes for two reasons, both of the same order of magnitude, one reason being the change in the distance between atomic centers, and the other the change in the dimensions of the atom. This will evidently introduce another term into Born's equation, and will essentially modify his results. The modified analysis is perhaps worth giving. While we are giving the analysis, we may as well carry the work one step further than Born has in his published papers, because we can thereby get the variation of compressibility with pressure.

If the atom is deformable, the coefficient of the repulsive force can no longer be regarded as a constant, but will depend on the distance of separation of the atoms. Born's analysis for the forces exerted by a cube suggests that this coefficient is proportional to the fourth power of the atomic radius.

We assume as the starting point, as does Born, that the potential energy per unit cell of the lattice is of the form

$$\Phi = -\frac{a}{\delta} + \frac{b}{\delta^n}.$$

δ is the lattice constant, and may be computed in terms of the lattice structure and its elementary charges. The coefficient b should be directly computable in terms of the details of the structure, but it must also satisfy a condition imposed by the stability of the lattice. n should also be computable in terms of the details of the structure, but again may also be found in another way, namely, from the compressibility. Thus we have here the possibility of two checks, one on n and one on b. Born has availed himself only of the check on n given by compressibility measurements.

All the following considerations take no account of temperature; they hold rigorously at absolute zero, but approximately at higher temperatures, because compressibility does not greatly depend on temperature.

Given a structure with a potential, like the above, and subjected to external pressure, then a necessary condition of equilibrium in general is that

$$\Phi + p\delta^3$$

shall be a minimum for changes of δ, or

$$\frac{d\Phi}{d\delta} + 3p\delta^2 = 0.$$

In particular, when there is no external pressure, and $\delta = \delta_0$,

$$\left.\frac{d\Phi}{d\delta}\right|_{\delta=\delta_0} = 0.$$

The compressibility we define in accordance with the experimental usage above as

$$\chi = \frac{1}{\delta_0{}^3}\frac{d\delta^3}{dp} = \frac{3\delta^2}{\delta_0{}^3}\frac{d\delta}{dp}.$$

We can get $\frac{d\delta}{dp}$ by differentiating the equation of stability, which is, solved for p,

$$p = -\frac{1}{3\delta^2}\frac{d\Phi}{d\delta}.$$

Hence
$$\frac{dp}{d\delta} = \frac{2}{3\delta^3}\frac{d\Phi}{d\delta} - \frac{1}{3\delta^2}\frac{d^2\Phi}{d\delta^2}.$$

We must next evaluate the derivatives of Φ.

$$\frac{d\Phi}{d\delta} = \frac{a}{\delta^2} - \frac{nb}{\delta^{n+1}} + \frac{1}{\delta^n}\frac{db}{d\delta},$$

$$\frac{d^2\Phi}{d\delta^2} = -\frac{2a}{\delta^3} + \frac{n(n+1)b}{\delta^{n+2}} - \frac{2n}{\delta^{n+1}}\frac{db}{d\delta} + \frac{1}{\delta^n}\frac{d^2b}{d\delta^2}.$$

With regard to b as a function of δ, we shall content ourselves with the next order of approximation beyond assuming that it is constant, and put

$$b = \alpha + \beta\,\delta .$$

Substituting this value back,

$$\frac{d\Phi}{d\delta} = \frac{a}{\delta^2} - \frac{n(\alpha+\beta\delta)}{\delta^{n+1}} + \frac{\beta}{\delta^n}$$

$$\frac{d^2\Phi}{d\delta^2} = -\frac{2a}{\delta^3} + \frac{n(n+1)}{\delta^{n+2}}(\alpha+\beta\delta) - \frac{2n}{\delta^{n+1}}\beta ,$$

and the special value of $\frac{d\Phi}{d\delta}$ at $\delta = \delta_0$ gives

$$\frac{a}{\delta_0{}^2} - \frac{n(\alpha+\beta\delta_0)}{\delta_0{}^{n+1}} + \frac{\beta}{\delta_0{}^n} = 0 .$$

Now substitute these values for the derivatives of Φ, and we get

$$\frac{dp}{d\delta} = \frac{4a}{3\delta^5} - \frac{n(n+3)}{3\delta^{n+4}}\alpha - \frac{(n+2)(n-1)}{3\delta^{n+3}}\beta,$$

$$\chi = \frac{3\delta^2}{\delta_0{}^3} - \frac{3\delta^5}{4a - \frac{n(n+3)}{\delta^{n-1}}\alpha - \frac{(n+2)(n-1)}{\delta^{n-2}}\beta} .$$

In this equation replace α in terms of δ_0 by the stability relation, put $\delta = \delta_0 + \Delta\delta_0$, where $\Delta\delta_0$ is small, so that

$$\left(\frac{\delta_0}{\delta}\right)^{n-1} = 1 - (n-1)\frac{\Delta\delta_0}{\delta_0},$$

$$\frac{1}{\delta^{n-1}} = \frac{1}{\delta_0{}^{n-1}}\left[1 - (n-1)\frac{\Delta\delta_0}{\delta_0}\right],$$

and

$$\frac{1}{\delta^{n-2}} = \frac{1}{\delta_0{}^{n-2}}\left[1 - (n-2)\frac{\Delta\delta_0}{\delta_0}\right],$$

replace $\frac{\Delta\delta_0}{\delta_0}$ by $\frac{1}{3}\,\frac{\Delta V}{V_0}$, and substitute back, obtaining

$$\chi = \frac{-9\delta_0^4}{(n-1)\left(a - \frac{\beta}{\delta_0^{n-2}}\right)}\left[1 + \tfrac{1}{3}\left\{7 + \frac{a(n+3) - \frac{2n+1}{\delta_0^{n-2}}\beta}{a - \frac{\beta}{\delta_0^{n-2}}}\right\}\frac{\Delta V}{V_0}\right].$$

The initial compressibility $\left(\frac{\Delta V}{V_0} = 0\right)$ is therefore,

$$\chi_0 = \frac{-9\,\delta_0^4}{(n-1)\left(a - \frac{\beta}{\delta_0^{n-2}}\right)}.$$

This agrees with Born's expression on putting $\beta = 0$.

Now apply this formula to numerical computation. What we are interested in is the order of magnitude of β, to see whether it checks with what might be expected by Schottky's theorem. We apply the formula to calcium. For this metal the evidence is particularly strong that we are dealing with a space lattice of doubly charged positive ions and electrons, the ions occupying the position of Ca in CaF_2, and the electrons the position of F. Born has worked out the analysis for this type of lattice, so that we may apply his results directly. We have numerically

$$\begin{aligned} a &= 38.7e^2 = 8.78 \times 10^{-18}, \\ \delta_0 &= 5.56 \times 10^{-8}, \\ \chi_0 &= -\,5.7 \times 10^{-12}. \end{aligned}$$

Let us assume at first that $\beta = 0$. Then solving for n, we find

$$n = 2.72,$$

which is much lower than the 5 suggested by Born's analysis for neutral cubes and electrons.

Let us now assume $n = 5$, and see what this gives for β.

$$\beta = \delta_0^3\left(a + \frac{9\delta_0^4}{4\chi_0}\right)$$

$$= 8.5 \times 10^{-40}$$

Now let us see what Schottky's theorem would lead us to expect for the order of magnitude of β. We have

$$\beta = \frac{db}{d\delta} = \frac{db}{dp}\frac{dp}{d\delta}$$

$$\frac{d\delta}{dp} = \frac{\delta\chi}{3}.$$

Hence $$\beta = \frac{3}{\chi\delta}\frac{db}{dp}$$

Now the analysis of Born (equation 4, page 235, Verh. D. Phys. Ges. 20, 1918) shows that b is of the form

$$b = \text{Const } e^2 r^4,$$

where r is the semidiagonal of the neutral cubic atom, and the constant, which varies with the type of lattice, will be of the order of magnitude of 10. (For instance, the coefficient of e^2 in the above expression for a is 38.7, for NaCl it is 13.94). Hence as a rough approximation put

$$b = 10\, e^2 r^4,$$

and $$\frac{db}{dp} = 40e^2r^3\frac{dr}{dp}.$$

Now Schottky's theorem gives $\frac{dr}{dp}$. At ordinary temperatures, at low pressures, his formula gives very closely

$$\frac{dL}{dp} = 3V,$$

and hence, applied to an elementary cell of the lattice,

$$\frac{dL}{dp} = 3\delta^3 = \frac{dL}{dr}\cdot\frac{dr}{dp}.$$

Now the variable part of L we take as the contribution of the 8 outer electrons rotating about a nuclear charge of 8 at distance r. (This assumes most of the deformation of the atom under pressure is confined to the outer shell). The kinetic energy of these electrons is one half their potential energy, or

$$L = \frac{32\, e^2}{r},$$

$$\frac{dL}{dr} = -\frac{64\, e^2}{r^2},$$

$$\frac{dr}{dp} = -\frac{3\delta^3\, r^2}{64\, e^2},$$

$$\frac{db}{dp} = -\frac{15}{8}\,\delta^3 r^5,$$

and finally,

$$\beta = -\frac{45}{8}\,\frac{\delta^2 r^5}{\chi}.$$

To find r, we assume that one side of the cubic atom is $\frac{2}{3}\,\delta$,

or
$$r = \frac{\delta_0}{\sqrt{3}} = 0.577\,\delta_0.$$

Hence, substituting numerical values

$$\beta = +\,1.04 \times 10^{-40}.$$

Now this is of the same sign, and of the same order of magnitude as the value required experimentally to make a plausible change in the value of n. When we are dealing with exponents as high as 40, it would seem that an agreement of order of magnitudes is of considerable significance. Hence we conclude that the Schottky effect must certainly be expected to play an important part (a conclusion also stated by Schottky, although without numerical computation). Consideration of the effect makes the value of n required to give the right compressibility larger than it otherwise would be.

The conclusion to be drawn from all these considerations I believe to be that we are not yet in a position to attach much significance to the precise value of n necessary to give the best value of compressibility.

An interesting suggestion as to the compressibility of solids at extremely high pressures is contained in Schottky's equation

$$\left(\frac{\partial \bar{L}}{\partial p}\right)_\tau = 3v + \tau\left(\frac{\partial v}{\partial \tau}\right)_p + 4p\left(\frac{\partial v}{\partial p}\right)_\tau.$$

The term $\tau \left(\frac{\partial v}{\partial \tau}\right)_p$ is small compared with $3v$, and since it becomes increasingly smaller at higher pressures, we will neglect it. We have seen that at low pressures $\left(\frac{\partial \bar{L}}{\partial p}\right)_\tau$ is positive. But the term $p \left(\frac{\partial v}{\partial p}\right)_\tau$ is negative, so that there is here the possibility of a reversal of sign of $\left(\frac{\partial \bar{L}}{\partial p}\right)_\tau$ at sufficiently high pressures. Now this is an exceedingly unlikely state of affairs, because it would mean that at low pressures the atoms shrink with increasing pressure, but that at sufficiently high pressures they begin to expand again. Hence if we assume that at the utmost $\left(\frac{\partial \bar{L}}{\partial p}\right)_\tau$ can only become zero, we find that the upper limit of $p \frac{1}{v} \left(\frac{\partial v}{\partial p}\right)_\tau$ is 3/4 (numerically). If this constitutes any real restriction, the place to look for it is in the compressible substances. Of the metals above, potassium is the most compressible. At 12000 kg., the maximum pressure of these experiments, $p \frac{1}{\tau} \left(\frac{\partial v}{\partial p}\right)_\tau$ has the value 0.19, and thus is still safely below the upper limit. However, if we plot $p \frac{1}{v} \left(\frac{\partial v}{\partial p}\right)_\tau$ against p, we obtain approximately a straight line, so that at some pressure below 50,000 kg. there must be an essential change in the character of the relation between pressure and volume. In this connection the possibility of a new polymorphic modification of potassium at high pressures must be kept in mind; it is natural to expect one in analogy with the new form of caesium which I have recently found. A new polymorphic form of potassium at very high pressures also seemed indicated to me a number of years ago [23] by the character of the curve giving change of volume on melting against pressure.

The extreme value of $p \frac{1}{v} \left(\frac{\partial v}{\partial p}\right)_\tau$ for potassium is somewhat higher than the value for the most compressible of the organic liquids investigated at high pressures, ether, which has the value 0.17 at 12000 kg.

Change of Compressibility with Pressure. The theory of Born as extended above gives the compressibility as a function of pressure (or change of volume). Neglecting at first the Schottky effect, or putting $\beta = 0$, we have

$$\chi = -\frac{9\delta_0^4}{a(n-1)}\left[1 + \frac{10+n}{3}\frac{\Delta V}{V_0}\right].$$

It is most interesting to notice that the lattice constants are not contained explicitly in the term giving the variation of compressibility with change of volume, but only the exponent n.

If we had defined the compressibility as $\frac{1}{\delta^3}\frac{d\delta^3}{dp}$ instead of as we did above, then the factor of $\frac{\Delta V}{V_0}$ would have become $\frac{7+n}{3}$. This agrees with a formula which Born has been so kind as to communicate to me in a personal letter.

At high pressures $\frac{\Delta V}{V_0}$ is negative, so that the formula predicts a numerical decrease of compressibility at high pressures, as is most natural. Furthermore, the compressibility decreases more rapidly than the volume itself, and by a factor of the order of a small integer. Now this factor has already been tabulated in Table IV in the column under $2b/a^2$. We see that this quantity is indeed of the predicted order of magnitude, so that Born's theory is suggestive also with regard to the pressure change of compressibility, as well as its absolute magnitude. But the exact value cannot be given by his expression, which demands values of n ranging from negative numbers for Ca and Sr to positive values as high as 95 for W. It is not surprising that the theory does not give precise results here. If we had to object to the single differentiation of the first term of a power series involved in computing the compressibility, still more must we object to the double differentiation involved in the computation of the change of compressibility with pressure.

It is difficult to know exactly how much significance should be attached to the prediction of a correct order of magnitude for the change of compressibility with pressure. Grüneisen's theory of the solid state, which rested on a physically very different picture from Born's, also gave the right order of magnitude.

It is perhaps interesting to consider whether the Schottky term gives plausible results here. If we assume as given by experiment both the compressibility and its variation with pressure, then we have two equations from which we may find both n and β.

We may write the factor of $\frac{\Delta V}{V_0}$ in the form

$$\tfrac{1}{3}\left[8 + 2n + \frac{a(n-1)(n-2)}{a\,\delta_0^4}\,\chi\right].$$

Now apply this again to calcium. We have the two equations:

$$\frac{9\delta_0^4}{(n-1)\left(a - \dfrac{\beta}{\delta_0^{n-2}}\right)} = 5.7 \times 10^{-12}$$

$$\tfrac{1}{3}\left[8 + 2n + \frac{a(n-1)(n-2)}{a\,\delta_0^4}\,\chi\right] = 2.9,$$

with the numerical values of a, δ_0, and χ already used. We notice that β does not enter the second of the two equations above, which is a quadratic in n. Solving this for n, we find

$$n = 7.34.$$

Now for β we have the equation

$$\beta = \delta_0^{n-2}\left[a + \frac{9\delta_0^4}{\chi\,(n-1)}\right].$$

Substituting numerical values, we find

$$\beta = 2.03 \times 10^{-56}.$$

The result found for n does not appear unreasonable, but the value for β appears to be of the wrong order of magnitude. It would seem, therefore, that the assumption of an inverse ninth power is so poor an approximation when computation involving two differentiations is involved that the inclusion of the Schottky term (which there is every reason to believe actually exists) cannot help matters.

It is to be noticed that if conversely we regard as sufficiently approximate the formula above for the initial compressibility with the Schottky term included, the value of n is very closely determined, if we know merely the order of magnitude of β, by the requirement that β/δ_0^{n-2} be of the same order as "a."

If instead now of assuming a specific form for the repulsive potential, we substitute an arbitrary function, a certain amount of information as to its behavior can be found in terms of the density, the compressibility, and the change of compressibility with pressure. Let us put in general:

$$\Phi(\delta) = -\frac{a}{\delta} + f(\delta).$$

As before, the general condition of equilibrium is $\Phi + p\,\delta^3$ is to be a minimum with respect to changes of δ. This condition gives:

$$p = -\frac{a}{3\delta^4} - \frac{f'}{3\delta^2}.$$

If δ_0 is the value of δ when $p = 0$,

$$a = -\delta_0^2 f'(\delta_0),$$

and if we furthermore write $(\delta/\delta_0)^3 = V/V_0$, we have the equation

$$f'(\delta) = f'(\delta_0)\left(\frac{V_0}{V}\right)^{\frac{2}{3}} - 3p\,\delta_0^2\left(\frac{V}{V_0}\right)^{\frac{2}{3}}.$$

Here V and p are measured quantities, so that if $f'(\delta_0)$ can be found, we have here the means of getting $f'(\delta)$ at any volume (or δ), and hence $f(\delta)$ itself, except for the constant of integration. The most information would be given by the most compressible metals. It is unfortunate that sodium and potassium, the most compressible of the metals above, do not crystallize face centered cubic, so that it is not easy to guess what the complete crystal structure of ions and electrons is, and hence not possible to compute the "a" or the $f'(\delta_0)$.

For less compressible metals, another way of treating the above general equation gives more information. For these metals, the compressibility is given with the accuracy of the experiment by a linear function of the pressure,

$$\chi = \chi_0(1 + \alpha p)$$

where α is an experimental constant, and is negative for all the metals studied above. Now the above equation for Φ can be treated in the general case exactly as we did in the special cases above. Writing again

$$\chi = \frac{1}{\delta_0^3}\frac{d\delta^3}{dp},$$

we may get χ by differentiating the equation of equilibrium, and keeping only small quantities of the first order, we may express χ as a function of p. It is not worth while reproducing here the algebraic details, but it will be readily found that

$$\chi = \frac{9\delta_0}{\dfrac{10a}{3\delta_0^3} - f''(\delta_0)} \left[1 + \left\{ \frac{14}{9} \chi_0 + \chi_0^2 \left(\frac{10a}{27\delta_0^4} + \frac{f'''(\delta_0)}{27\delta_0} \right) \right\} p \right].$$

This formula may be checked in special cases by substituting the particular values of f used above.

In this expression for χ, the term outside the square bracket is χ_0. The factor of p inside the bracket is the empirical constant α. A third relation is given by the stability relations under zero pressure. These three relations may be solved for the three derivatives of giving

$$\begin{cases} f'(\delta_0) = -\dfrac{a}{\delta_0^2} \\ f''(\delta_0) = -\dfrac{9\delta_0}{\chi_0} + \dfrac{10a}{3\delta_0^3} \\ f'''(\delta_0) = \left(-\dfrac{10a}{\delta_0^3} - 42\dfrac{\delta_0}{\chi_0} + 27\dfrac{\alpha\delta_0}{\chi_0^2} \right) \dfrac{1}{\delta_0}. \end{cases}$$

I am much indebted to Mr. J. C. Slater for pointing out to me an error in the original form of my formula for the third derivative.

These formulas may now be applied to numerical computation for those metals above which crystallize face-centered cubic, for in these cases we know the probable complete crystal structure, and can compute the "a." Given the structure, the only additional information needed to compute "a" is the magnitude of the charge on the ion. I have assumed that this charge is three electrons for Al, two for Ca, Fe, Co, Ni, Pd, and Pt, and one for Cu, Ag, Au, Ce, and Pb. With these assumptions the results tabulated in Table VI may be found.

In the first place we notice that the sign of any given derivative is the same for all metals, and that the signs are alternately negative and positive for the first, second, and third derivatives. This is the general character to be expected for a force increasing more and more rapidly as we approach the center of the atom (in particular, a potential as $1/\delta^n$ gives this relative arrangement of the derivatives).

In general the derivatives are numerically larger for the more in-

TABLE VI.

THE FIRST THREE DERIVATIVES OF THE REPULSIVE POTENTIAL.

Metal	$-f'(\delta_0)$	$f''(\delta_0)$	$-f'''(\delta_0)$
Al	17.3×10^{-3}	1.70×10^{6}	$.78 \times 10^{14}$
Ca	4.1	.33	.11
Fe	9.7	1.44	4.9
Co	10.0	1.52	6.6
Ni	10.1	1.54	7.1
Pd	8.1	1.34	7.0
Pt	8.2	1.65	19.2
Cu	2.4	.68	3.4
Ag	1.9	.52	2.1
Au	1.9	.78	7.8
Ce	1.2	.21	.12
Pb	1.3	.27	.55

compressible metals. This is what we would expect, the atoms of the less compressible metals being less deformable.

The order of magnitude of these derivatives now gives a little further information. We would expect that if f is one of the more ordinary mathematical functions the natural sphere of variability of f would have the radius δ, so that the proportional change of f and its various derivatives in a distance δ would be of the order of small numbers. This is true for the first derivative, as we see from the table, for $\delta_0 f''(\delta_0)/f'(\delta_0)$ is of the order of a small number. We can, if we like, reverse this principle, and get the order of magnitude of the unknown f by putting $\delta_0 f'(\delta_0)/f(\delta_0)$ equal to a small number. We shall find $f(\delta_0)$ of the order of 10^{-10}. This is what we would expect on other grounds, for it is also the order of a/δ_0. The same is true of the second derivative, for $\delta_0 f'''(\delta_0)/f''(\delta_0)$ is also of the order of a small number.

The fact that $f'(\delta_0)$, which measures roughly the force of repulsion between adjacent atoms, changes only a few fold in the range of δ_0, has an important bearing on our picture of the nature of the atomic boundaries. The meaning is that the atoms do not behave, when pushed into contact, as if they had rigid boundaries, like two bricks, for example. If this were the case, the relative increase of $f'(\delta_0)$ would be very much more rapid. In my previous work on polymorphic changes under pressure,[24] I found it useful to think of the atoms as more or less rigid bodies with definite shapes. We now see that this point of view is of restricted usefulness, and in particular is not suited to account for the behavior of the compressibility.

We have up to the present followed Born in entirely disregarding the effect of temperature. This may now be taken into consideration in the following way in the domain in which the temperature is high enough for the classical dynamics to apply, so that we may ascribe to each atom a kinetic energy equal to (3/2) kt. The condition of equilibrium may be conveniently expressed in terms of the "total heat" per unit cell of the lattice, and we will restrict the discussion to substances for which the complete lattice structure of both ions and electrons is like NaCl. For the total heat, H, we have in general the expression $H = E + pV$. Here V is to be taken as the volume of the cell of the lattice or δ^3. E, the energy of the cell of the lattice, is made up of two parts, a potential part, which we write as before as $-\frac{a}{\delta} + f(\delta)$, plus the kinetic part, due to temperature agitation, which is 6kt., since there are in the cell 4 atoms, each with energy (3/2)kt. (we suppose that the kinetic energy of the electrons may be disregarded). Hence

$$E = -\frac{a}{\delta} + f(\delta) + 6\,\text{kt}.$$

The equilibrium condition is that the total heat is constant for changes at constant entropy, and constant pressure. Hence we have

$$d\,(E + p\,\delta^3) = 0$$

where we are to change δ at constant pressure and entropy. This gives

$$\frac{a}{\delta^2} + f'(\delta) + 6\kappa\left(\frac{\partial\tau}{\partial\delta}\right)_s + 3p\delta^2 = 0.$$

To evaluate $\left(\frac{\partial\tau}{\partial\delta}\right)_s$ it is convenient to make connection with the ordinary formulas of thermodynamics for $\left(\frac{\partial\tau}{\partial v}\right)_s$, where v is the volume of material which under standard conditions occupies 1 c.c. We have

$$\left(\frac{\partial\tau}{\partial\delta}\right)_s = \left(\frac{\partial\tau}{\partial v}\right)_s \cdot \left(\frac{\partial v}{\partial\delta}\right)_s.$$

Now
$$v = N\delta^3$$
$$\left(\frac{\partial v}{\partial \delta}\right)_s = \frac{dv}{d\delta} = 3\,N\delta^2, \quad \text{and}$$
$$\left(\frac{d\tau}{\partial v}\right)_s = \frac{\tau\left(\frac{\partial v}{\partial \tau}\right)_p}{C_p\left(\frac{\partial v}{\partial p}\right)_\tau + \tau\left(\frac{\partial v}{\partial \tau}\right)_p^2}.$$

Hence substituting back, we have

$$f'(\delta) = -\frac{a}{\delta^2} - 3\delta^2\left[p + \frac{6\kappa\,N\tau\left(\frac{\partial v}{\partial \tau_p}\right)}{C_p\left(\frac{\partial v}{\partial p}\right)_\tau + \tau\left(\frac{\partial v}{\partial \tau}\right)_p^2}\right].$$

At zero pressure

$$f'(\delta_0) = -\frac{a}{\delta_0^2} - \frac{\frac{18\,\kappa\tau}{\delta_0}\left(\frac{\partial v}{\partial \tau}\right)_p}{C_p\left(\frac{\partial v}{\partial p}\right)_\tau + \tau\left(\frac{\partial v}{\partial \tau}\right)_p^2}$$

We have to compare this with the value $f'(\delta_0) = -\frac{a}{\delta_0^2}$, which we previously found, neglecting the temperature effect. For the case of copper, I have substituted numerical values into the second member of the right hand side of the last equation, obtaining 4×10^{-5}, compared with the value 2.4×10^{-3} for a/δ_0^2. Hence the effect of the temperature term is not important.

It is also probable that the new term will not greatly affect the values that we have found for $f''(\delta_0)$ and $f'''(\delta_0)$, or alter the conclusions that we have drawn from these numerical values, because in the case of copper, the numerical value of the new term which stands besides p in the next to the last equation above is 1×10^{10}, and is nearly constant, whereas p varies through a range of 1.2×10^{10}.

SUMMARY.

A new method has been developed by which the linear compressibility of metals or other solid substances with small compressibility may be measured to high pressures. The method is accurate enough to give change of compressibility with pressure and temperature. This paper contains the results of measurements by this method on 30 metals over a pressure range of 12000 kg/cm^2, and at 30° and 75°. The majority of these metals crystallize in the cubic system, or else in the hexagonal close packed arrangement of spheres, and for them the change of volume may be obtained immediately from the change of linear dimensions, because the compressibility is the same in all directions. Formulas are given for the change of volume as a function of pressure and temperature over the range of the measurements. In general the compressibility decreases with increasing pressure and increases with increasing temperature. The order of magnitude of the change of compressibility or thermal expansion with pressure is the same for all metals. The compressibility changes under pressure by a fraction which is a small number (varying from 2 to 30) times the proportional change of volume under the same pressure, and similarly the proportional change of thermal expansion under pressure is of the order of a small number times the corresponding proportional change of volume.

Six of the thirty substances do not crystallize cubic or in the hexagonal close packed arrangement of spheres, and for these it was established that there are large differences of compressibility in different directions; for one substance, tellurium, it was found that in one direction the linear compressibility is even negative.

In the theoretical discussion it was shown that it is very probable that the forces resisting compression in a metal are the same in nature as those in a salt of the type of NaCl, that is, the metal may be regarded as a lattice of ions and electrons acting on each other by electrostatic forces due to one or more single elementary charges. In addition there is a force of repulsion. Criticism is made of the details of Born's proof of the inverse ninth power law for the potential of the repulsion, and the conclusion drawn that we have not at present a detailed enough knowledge of the structure of the atom to determine the law of repulsion so accurately as to allow us to differentiate the formulas once and twice, as is necessary in computing the compressi-

bility and its change with pressure. It is shown by numerical calculation that the effect discussed by Schottky due to the deformation of the atoms under pressure is of such an order of magnitude that it must be taken into account.

Finally, if we do not assume the specific form of the repulsive potential, but replace it by an arbitrary function, it is shown that the first, second and third derivatives may be computed numerically from experimental data now available for a number of the metals of this investigation. From these numerical values the conclusion is drawn that the boundaries of the atoms cannot be as definite as appeared not unlikely from a study of polymorphic changes under pressure.

I am indebted to my assistant, Mr. I. M. Kerney, for help in making the greater number of the readings.

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A REVISION OF THE ATOMIC WEIGHT OF SILICON.

THE ANALYSIS OF SILICON TETRACHLORIDE AND TETRABROMIDE.

By Gregory P. Baxter, Philip F. Weatherill and Edward W. Scripture, Jr.

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Introduction.

At the present time the atomic weight of silicon depends largely upon the analysis of the chloride and bromide of this element. Pelouze [1] and Dumas [2] compared the tetrachloride with silver, while Schiel [3] determined the silver chloride obtained from the same substance. Thorpe and Young [4] converted the tetrabromide to silica, and Becker and Meyer [5] used a similar method with the tetrachloride. Two determinations of the density of silicon tetrafluoride have been made by Jaquerod and Tourpaian [6] and by Germann and Booth.[7] The results of all these determinations have been collected and recalculated by F. W. Clarke [8] and are summarized in the following table.

		Atomic Weight of Silicon
Pelouze	$SiCl_4$: 4 Ag	28.37
Dumas	$SiCl_4$: 4 Ag	28.08
Schiel	$SiCl_4$: 4 AgCl	27.95
Thorpe and Young	$SiBr_4$: SiO_2	28.38
Becker and Meyer	$SiCl_4$: SiO_2	28.23
Jaquerod and Tourpaian	Density SiF_4	28.50
Germann and Booth	Density SiF_4	28.31 [9]

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[9] Calculated by Clarke.

From these figures by the method of least squares Clarke computes the mean, 28.25, but calls attention to the uncertain nature of this value and to the desirability of new determinations of the constant in question.

In spite of the small percentage of silicon in the chloride and bromide, 16.5 and 8.6 respectively, so that the experimental error is magnified six and twelve times in calculating the atomic weight of silicon from the results of the analyses, on the whole these two substances seem to offer marked advantages for the purpose over any other compounds of silicon. Chief of these are the ease both of preparation and purification by distillation, and of analysis by comparison with silver.

A brief account of our preliminary experiments with silicon tetrachloride was published not long ago.[10] These experiments point to the value 28.11 for the atomic weight of silicon. Our subsequent experiments with both the chloride and bromide confirm the conclusion that silicon possesses an atomic weight lower than has been supposed, and indicate a value 28.06, which is even lower than the result of the preliminary experiments.

Purification of Materials.

Water. Ordinary distilled water was twice redistilled, once from alkaline permanganate and once from very dilute sulfuric acid. The condensers were block tin tubes, fitted to Pyrex flasks with constricted necks, which served as stills. The connection was made by a water seal, no cork or rubber being used. The water was collected in Jena flasks, generally just previous to use.

Nitric Acid. Concentrated C. P. nitric acid was distilled through a quartz condenser, the first two-thirds as well as the last tenth being rejected.

Sodium Hydroxide. The best commercial material was dissolved and the greater part of the carbonate was precipitated with barium hydroxide. The supernatant solution was centrifuged until clear and then evaporated to crystallization in a platinum dish. Centrifugal drainage of the crystals in platinum Gooch crucibles [11] was followed by rinsing with a small quantity of water and a second drainage. Recrystallization in the same way was continued until the mother liquor was free from chloride. This required from three to six crystallizations according to the quality of the original material.

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[10] Baxter, Weatherill and Holmes, Jour. Am. Chem. Soc., **42**, 1194 (1920).
[11] Baxter, Jour. Am. Chem. Soc., **30**, 286 (1908).

Two other specimens of hydroxide were made by combining barium hydroxide and sodium carbonate, after both substances had been crystallized until free from chloride. The resulting solution was then evaporated to crystallization, and the product was recrystallized.

Bromine. This material had been purified by Mr. A. F. Scott according to the method which has been frequently used at Harvard.[12] Crude bromine was first distilled from aqueous potassium bromide in order to remove chlorine. Next one-fourth of the product was converted to potassium bromide by means of recrystallized potassium oxalate, and the remaining three-fourths of the bromine was distilled from solution in this potassium bromide. All the product was then converted to potassium bromide by means of potassium oxalate, and the potassium bromide was fused in a platinum dish with enough recrystallized potassium permanganate to oxidize all organic matter. In order to obtain bromine the purified potassium bromide was dissolved in a solution of pure sulfuric acid. The excess of potassium permanganate contained by the salt caused a small quantity of bromine to be liberated. This bromine was removed by distillation since it might have contained a trace of iodine. Enough pure permanganate to liberate the greater portion of the bromine was then added, and the bromine was distilled into a receiver cooled with ice. In this last step the bromine received a third distillation from a bromide. The product was separated from the water, dried with resublimed phosphorus pentoxide and once redistilled.

Silver. This substance was prepared by standard methods. These consisted in brief of the following processes: double precipitation as chloride, followed by reduction with alkaline sugar solution, fusion of the metal on charcoal, solution and reprecipitation with ammonium formate, fusion on pure lime, electrolytic transport, fusion on pure lime in hydrogen, etching, drying in a vacuum at 500°. Recent work by Baxter and Parsons [13] and Baxter [14] as well as earlier work have shown that these processes yield a product of adequate purity.

Nitrogen. This gas was prepared by the Wanklyn process. Air was charged with ammonia and passed over hot copper catalyst. The excess of ammonia was removed by wash bottles containing dilute sulfuric acid. Hydrogen resulting from catalytic decomposition of the ammonia was next burned with hot copper oxide, and the gas was then purified and dried by silver nitrate solution, sodium hydroxide, concentrated sulfuric acid and phosphorus pentoxide. Finally last traces of oxygen were absorbed by hot copper oxide. The apparatus, which was constructed entirely of glass, is described in detail by Baxter and Grover.[15]

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[12] Baxter, Moore and Boylston, These Proceedings, **47**, 585 (1912); Jour. Amer. Chem. Soc., **34**, 1644 (1912); Baxter and Grover, Jour. Amer. Chem. Soc., **37**, 1029 (1915).

[13] Baxter and Parsons, Jour. Amer. Chem. Soc., **44**, 577 (1922).

[14] Baxter, Ibid., **44**, 591 (1922).

The Preparation of Silicon Tetrachloride.

The silicon tetrachloride was prepared by the action of chlorine on silicon, and was purified by distillation in exhausted glass vessels without exposure to air or moisture. Moisture was particularly to be avoided because of the decomposition products resulting from hydrolysis, i.e., silicic acid, silicon oxychlorides and hydrochloric acid. Because in preparing the chloride it was impossible entirely to avoid exposure to moist air, the process of purification was adapted for the removal of these impurities. In addition to the above impurities, the following were to be expected and provided for: silicon hexachloride and octachloride, titanium tetrachloride and carbon tetrachloride, the last two resulting from impurities of carbon and titanium in the silicon. Fortunately the boiling points of all the impurities are sufficiently far removed from that of silicon tetrachloride to lead to the expectation that they would be readily removed by fractional distillation. So far as can be told this proved to be the case. In the following table are given the boiling points of the impurities in question.

	bpt.		bpt.
$SiCl_4$	58°	HCl	−83°
Si_2Cl_6	146–148°	CCl_4	76.5°
Si_3Cl_8	200–215°	$TiCl_4$	136°
Si_2OCl_6	137–138°		

Several specimens of the tetrachloride were prepared, differing chiefly in the method and extent of the fractionation. One lot was purchased from the General Electric Company. This had been made from silicon and chlorine, and had been rectified. Our own preparations were made as follows: Dry chlorine was passed over silicon in a hard glass tube heated to redness, and the product was condensed. The chlorine was drawn from a large tank which served for all the preparations, and was purified and dried by passing through two towers filled with beads wet with water, then three similar towers containing concentrated sulfuric acid, and finally through a tube containing resublimed phosphorus pentoxide. The silicon, in the form of a coarse powder, was contained in a hard glass tube which could be electrically heated and which was inclined toward the large glass bulb cooled with ice which served as condenser. Connection with both the chlorine purifier and the condenser was made with dry ground joints, and all other connections in the apparatus were made with glass seals, except that between the chlorine tank and the purifying apparatus, which consisted of a very short rubber connector. Since the reaction is highly exothermic, comparatively little heat was required.

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[15] Baxter and Grover, Jour. Amer. Chem. Soc., **37,** 1037 (1915).

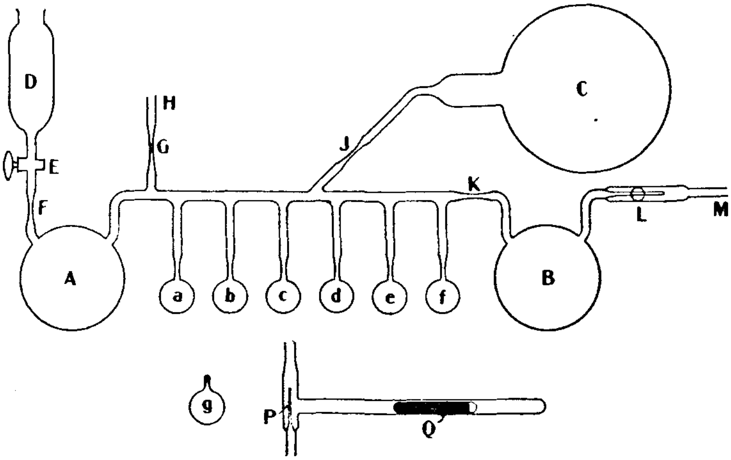

FIGURE 1.

The material obtained from the General Electric Company was a clear colorless liquid. The initial product of our own process was a dark colored liquid containing both dissolved and suspended ferric chloride, titanium chloride, chlorine, hydrochloric acid and probably some or all of the other impurities mentioned above.

Sample I was a mixture of General Electric Company product with some of our own. Sample II consisted entirely of General Electric Company material. Samples III and IV were two separate preparations of our own.

The fractional distillation of the first three samples was effected in apparatus of the general type shown in Figure 1. This apparatus resembles closely that used by Baxter, Moore and Boylston [16] for the preparation of phosphorus tribromide, by Baxter and Moore [17] for the preparation of phosphorus trichloride and by Baxter and Starkweather [18] for the preparation of tin tetrachloride. Two bulbs, *A* and *B*, of about 300 cc. capacity serve as stills or receivers. The separatory funnel *D* is employed during the introduction of the material. The flask *C* of one liter or more capacity is an expansion reservoir for removing permanent gases. The small bulbs *a*, *b*, *c*, etc. are fractionating bulbs. The various parts of the apparatus may be disconnected by sealing the capillaries *F*, *G*, *J*, *K*, etc. By means of the special joint *L*, the final receiver *B* may subsequently be connected with another similar system without exposure to air or moisture.

The special joint *L* is shown on a large scale in Figure 1. The sealed-in capillary *P* is closed at the end and is scratched with a file at several points to facilitate breaking. To break the joint the closed tube of glass weighted with mercury *Q* is allowed to strike the capillary with some force. Bruner and Bekier,[19] Briscoe and Little [20] and Baxter and Starkweather [18] used a similar device, while Richards and Craig [21] employed a magnetic hammer instead of one operated by gravity.

First the stopcock *E* is made tight by rings of grease on the ends of the cock and by pouring a small amount of mercury into the separatory funnel *D*. The whole apparatus is next exhausted through *H* and the capillary *G* is sealed. Next *D* is filled with silicon tetrachloride and the mercury and nearly all the tetrachloride are admitted to *A*. Again a small amount of mercury is poured into *D* to seal the stopcock and as soon as possible the capillary *F* is sealed. By sealing off the globe *C* at *J* a large part of the permanent gases originally dissolved in the liquid chloride can be removed. The more volatile fractions of material are condensed in some of the small bulbs, *a*, *b*, *c*, by means of liquid air or carbon dioxide-alcohol mixture, and are sealed off. Then the main portion of material is distilled into *B* which is sealed off at *K*, leaving the least volatile fraction in *A*. Subsequently, if desired, the residue in *A* can be distilled into *d*, *e*, *f*, for examination.

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[16] Baxter, Moore and Boylston, These Proceedings, **47** 590 (1912); Jour. Amer. Chem. Soc., **34**, 263 (1912); Z. anorg. Chem., **74**, 365 (1912).

[17] Baxter and Moore, Orig. Com. Eighth Internal. Cong. Appl. Chem., Vol. II, 21 (1912); Jour. Amer. Chem. Soc., **34**, 1644 (1912); Z. anorg. Chem., **80**, 189 (1913).

[18] Baxter and Starkweather, Jour. Amer. Chem. Soc., **42**, 907 (1920).

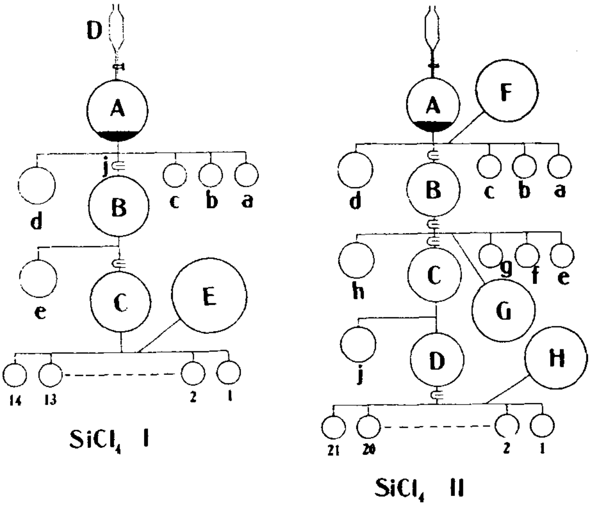
[19] Bruner and Bekier, Z. f. Elektrochem, **18**, 369 (1912).

[20] Briscoe and Little, Jour. Chem. Soc., **105**, 1321 (1914).

[21] Richards and Craig, Jour. Amer. Chem. Soc., **41**, 131 (1919)

A new distillation system can now be sealed on at *M* and the process repeated.

Sample I. All the details of the fractionation of Sample I are shown in Figure 2. The chloride was freed from suspended material by filtration through glass wool and was admitted to the exhausted bulb *A* together with several cubic centimeters of mercury. After standing for a week with occasional shaking, in order to remove the excess of chlorine as mercurous chloride, three small fractions, *a*, *b*, *c*, were removed and all but about one fifth the original chloride was distilled into *B*. From *B* about four fifths of the product was distilled into *C*, after sealing off the liter reservoir *E*, and finally all the material was collected in fourteen small bulbs, 1–14. The appearance of a bulb sealed off for analysis is shown at *g* (Fig. 1). In the figures, special joints are represented as at *j*, expansion reservoirs as at *E*, more volatile fractions to the right of the center, *a*, *b*, *c*, less volatile fractions to the left of the center, *d*, *e*. The volumes of the fractions are indicated roughly by the size of the circles. Fractions 3, 6, 9 and 12 were analyzed.



$SiCl_4$ I

$SiCl_4$ II

FIGURE 2.

Sample II. Figure 2 shows diagrammatically the fractionation of Sample II. After standing over mercury for some time, three small light fractions, *a*, *b*, *c*, were removed with liquid air and the greater part of the remainder was distilled into *B* leaving a residue *d* of about one-sixth. In spite of the use of the expansion reservoir *F*, distillation refused to begin when the bulbs, *a*, *b*, *c*, were cooled with carbon dioxide-alcohol mixture. Since, however, liquid air was effective in starting the distillation, and since thereafter carbon dioxide-alcohol refrigerant proved sufficient, it seems likely that hydrochloric acid gas was responsible for the delay. In the next two steps the expansion reservoir *G* and three light fractions, *e*, *f*, *g*, were removed, and two residues *h* and *j* were rejected when the bulk of the material was distilled successively into *C* and *D*. From *D* all the remainder was collected in twenty one small bulbs, except for a residue of obviously less volatile oily material, amounting to a few tenths of a cubic centimeter. Of the final fractions, Nos. 1, 7, 13 and 20 were analyzed.

In order to discover whether the non-volatile residue was diminishing in quantity with successive distillations, residue *j* was transferred to an exhausted bulb without exposure to air and fractionally distilled. Here the last cubic centimeter, as in the case of the residue in *D*, consisted of a colorless oily liquid, which could be distilled by warming but which was obviously less volatile than the tetrachloride. In the preparation of Sample III it was found that a part of this material is titanium tetrachloride. The remainder is apparently higher chlorides of silicon.

Sample III. In preparing Sample III, the product of the reaction of chlorine on silicon was collected in the bulb *A* (Fig. 3) which was then connected with an efficient water pump. The liquid boiled vigorously under these conditions, so that a large part of the excess of chlorine and probably much of the hydrochloric acid were removed at the outset. Furthermore the boiling of the tetrachloride must have flushed out most of the air originally in the bulb. After removal of the light fractions, *a*, *b*, *c*, with liquid air, the bulk of the material was distilled into bulb *B*, containing mercury, and allowed to stand for four months. In each of the fourteen distillations of the main portion, a residue of several cubic centimeters was left in the still, and usually was collected as two or more fractions. Furthermore nine more volatile fractions were rejected in the first four distillations. The final product was collected in twenty one small bulbs, of which Nos. 1, 13, 16 and 21 were analyzed. From the point of view of quantity of material bulb 14 represents about the middle of this series.

In the first seven distillations the oily, less volatile residue observed in the preparation of Sample II appeared, in gradually decreasing proportions. Some of this material was found by qualitative testing with hydrogen peroxide to contain titanium. The quantity of titanium contained in fraction *j* was very small, however, and none could be found in fraction *k*. Furthermore the amount found in fractions *d* and *e* seemed far too small to account for the quantity of less volatile residue, so that we are inclined to believe that a considerable portion of the residue consisted of higher chlorides of silicon.

Sample IV. Still a fourth sample of tetrachloride was purified as described under the preparation of Sample II. Owing to the discovery of the less volatile residue in Sample II, the fractions originally obtained in this way were not analyzed. After the preparation of Sample III, however, Mr. A. F. Scott, who was engaged in the Coolidge Memorial Laboratory in the purification of boron trichloride by fractional distillation, found that a Hempel column could be conveniently employed in a distillation at low temperatures, provided the

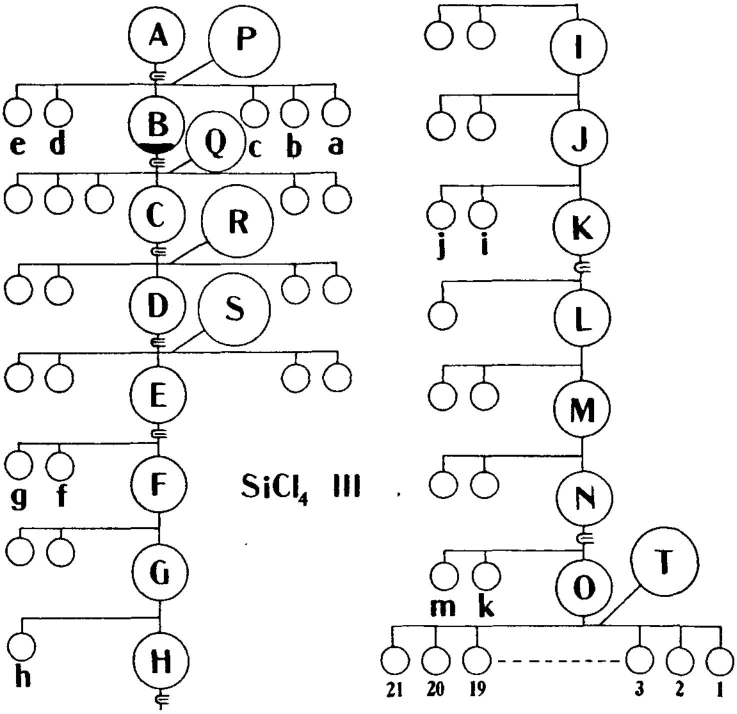


FIGURE 3.

column was chilled to a suitable temperature intermediate between that of the still and the condenser. Since it was a simple matter to apply this principle to the distillation of silicon tetrachloride, all of the fractions resulting from the first distillation, without regard to purity, together with a considerable number of rejected fractions obtained in the purification of Sample III, were combined in a large bottle by breaking the bulbs with a heavy glass rod, and the mixture was subjected to a new fractionation, as indicated in Figure 4.

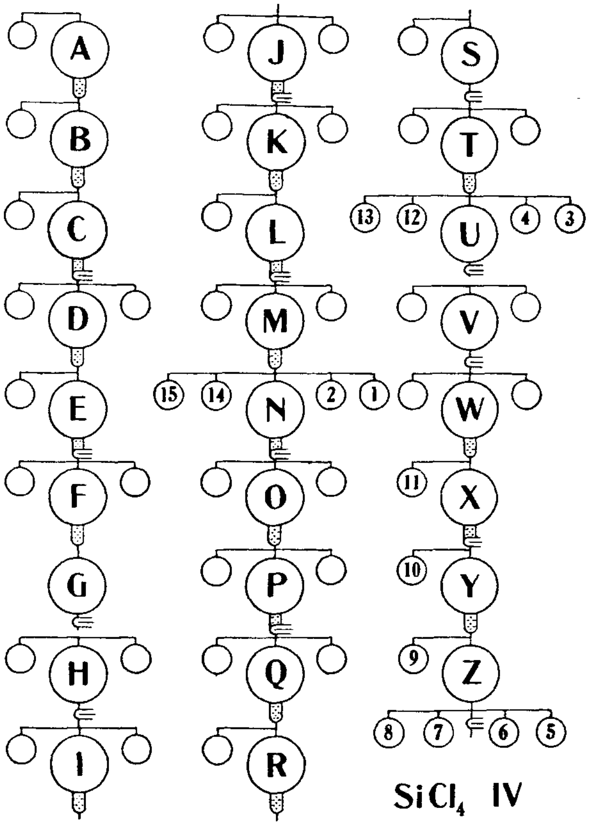


Figure 4.

The material was poured into bulb *A* through a glass funnel and the bulb, while immersed in hot water, was exhausted with an efficient water pump, through a drying tube containing sodium hydroxide. The tetrachloride was allowed to boil vigorously for a short time, until several cubic centimeters had evaporated, together, we hoped, with much of the hydrochloric acid formed by exposure to the moisture of the air. Repeated distillation was then carried out with apparatus similar to that indicated in Figure 5. The Hempel column was cooled with ice and water, while the receiver was chilled with carbon dioxide-alcohol mixture. A continuous reflux action was evident in the Hempel column throughout the distillation. In each distillation a few cubic centimeters of residue were rejected. Furthermore, frequently at the end of a distillation the temperatures of still and receiver were reversed so that the most volatile material was largely condensed in the still. Sometimes the same effect was produced by inserting an additional large bulb in the train and condensing the most volatile material in this bulb before the next distillation.

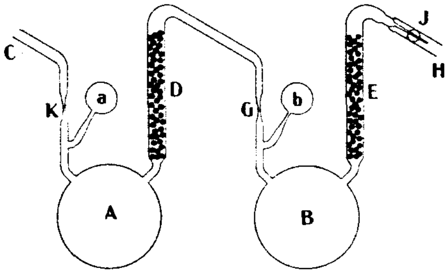


FIGURE 5.

After six distillations, one of the capillary connections in the apparatus was accidentally broken so that moist air was admitted. With as little exposure to the air as possible the chloride was introduced into bulb *H* and seven more distillations were carried out in the same way. In the last distillation the two most volatile fractions (1 and 2, Fig. 4) were collected for analysis, the main bulk of material was distilled into bulb *O* from bulb *N*, and the residue was collected in bulbs 14 and 15 for analysis.

Since the possibility of the existence of constant boiling distillates is always to be feared in fractional distillation, an effort was made to avoid or detect the effect of such a mixture by further distillation at a different pressure. To do this the still was immersed in warm water, the Hempel column was kept at about 5° and the receiver was surrounded with ice and water. In the seventh distillation at the higher pressure the two most volatile fractions, 3 and 4, were collected for analysis, as well as the two least volatile fractions, 12 and 13 (Fig. 4).

Unfortunately at this point the apparatus was again opened to the air by the breakage of a capillary. Although the quantity of material was by this time much diminished, about 30 cc., the remainder was introduced into bulb *V* and the bulb thoroughly flushed out with tetrachloride vapor. Further fractionation at the lower pressure, using carbon dioxide-alcohol as refrigerant, yielded three residual fractions, 9, 10 and 11, while the remainder was collected in the four fractions, 5, 6, 7 and 8.

Sample I was purified by Dr. E. O. Holmes, Jr., Samples II and III by Dr. Weatherill, and Sample IV by Mr. Scripture.

THE ANALYSIS OF SILICON TETRACHLORIDE.

After being weighed the bulb containing silicon tetrachloride was broken under an excess of sodium hydroxide and the glass was collected and weighed. The solution was diluted to considerable volume and made acid with nitric acid, and then was precipitated with a solution of a weighed very nearly equivalent amount of pure silver. The point of exact equivalence between chloride and silver was then found with the assistance of a nephelometer.

The bulb of material selected for analysis was soaked first in cleaning solution, then for several days in pure water. Sometimes the capillary was slightly scratched with a file. Next the bulb, suspended in a platinum wire basket, was weighed under water of known temperature. After being dried with a clean, nearly lintless cloth, the bulb was allowed to remain at least over night in a desiccator containing fused potassium hydroxide. Weighing by substitution followed, and the temperature, pressure, and humidity of the balance case were observed at the same time. From the loss in weight under water the volume of the bulb was computed, and then from the atmospheric conditions the buoyant effect of the air on the bulb. In weighing the fractions of Sample IV, the atmospheric density was found from the weight of a sealed, standardized globe.[22]

A solution of about fifty per cent excess of the quantity of pure sodium hydroxide (purified as described on page 247) necessary to react completely with the tetrachloride was filtered into a heavy walled 2 liter conical Jena flask, which was provided with a particularly well ground glass stopper. The volume of the sodium hydroxide solution varied from 200 to 400 c.c. according to the weight of the tetrachloride sample. The bulb was introduced and after thoroughly wetting the walls and stopper of the flask with the alkali, the bulb was broken by shaking the flask. The fog produced on breaking of the bulb apparently disappeared after fifteen or twenty minutes, but the flask was not opened for several hours, in order to make certain that no silicon tetrachloride or hydrochloric acid vapor was lost.

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[22] Baxter, Jour. Amer. Chem. Soc., **43**, 1317 (1921).

The solution was always perfectly clear, although in a few instances silicic acid separated in the end of the capillary of the bulb. This difficulty remedied itself in some instances, for on standing the silicic acid dissolved in the alkaline solution. In others, this process was assisted by breaking the capillary with a blunt rod. In no case was an analysis continued until all traces of such a deposit had disappeared, and two experiments were abandoned because of undissolved silicic acid in the capillary.

Next the contents of the flask were diluted to 700–800 cc. and were filtered through a quantitative filter into a glass-stoppered precipitating flask or bottle. The glass fragments were washed by decantation several times and were collected on the filter, and finally the filter paper was charred and the residue burned in a platinum crucible at as low a temperature as possible so as to avoid the danger of volatilizing alkali from the glass.

In order to find out whether it is possible to wash the filter free from sodium silicate and silicic acid, several blank experiments were carried out in which a similar alkaline solution of silicon tetrachloride was filtered in the same way through a quantitative filter, and after thorough washing with water the filter was burned. In still other experiments a weighed empty bulb similar to those used in collecting the silicon tetrachloride for analysis, but open to the air, was broken in the silicate solution and the glass fragments were collected and determined as above. In all these experiments the ash of the paper was found to be slightly in excess of the weight to be expected.

	Excess mg.
Without glass bulb	0.21
	0.15
	0.16
With glass bulb	0.14
	0.35
Average	0.20

In experiments where the foregoing method of determining the glass was employed (Series I and II) a negative correction of 0.20 mg. has therefore been applied.

Since we found that silicate solutions, of the concentration obtained in these analyses, when acidified with one per cent nitric acid remain clear indefinitely, the experiment was tried of washing the glass fragments and the filter paper with nitric acid of this concentration. Somewhat to our surprise not only did the excess in weight of the ash disappear, but a slight deficiency was found. Since this deficiency was practically the same whether or not a glass bulb was involved, it is apparent that some of the mineral constituents of the filter are extracted by either the alkaline silicate or the nitric acid.

The following table gives the results obtained in blank experiments in which one per cent nitric acid was used as washing liquid.

	Deficiency mg.
Without glass bulb	0.08
	0.08
	0.12
	0.01
	0.09
With glass bulb	0.13
	0.01
	0.02
Average	0.07

The method of filtering the sodium hydroxide solution of the silicon chloride through the filter and then washing the glass and filter with one per cent nitric acid was followed in all the analyses of Series III, and in Analyses 21 to 25 of Series V. In these experiments an average positive correction of 0.07 mg. was applied to the weight of the glass before subtracting the weight of the filter ash, 0.11 mg.

In all the analyses of Series IV and Analyses 26, 27 and 28 of Series V, the glass was washed with one per cent nitric acid, but a different variety of filter was employed. Blank experiments with these filters gave the following results:

	Excess mg.
Without glass bulb	0.05
	0.03

	Excess mg.
With glass bulb	0.04
	0.04
	−0.04
	−0.04
	0.09
	0.08
Average	0.03

In these experiments a negative correction of 0.03 mg. was applied to the weight of the glass.

The filtrate from the glass was then made acid by adding, in the form of 1 normal solution, a quantity of nitric acid equivalent to the sodium hydroxide used. Since a fifty per cent excess of sodium hydroxide was used, and since four ninths was changed to sodium chloride, after the addition of the nitric acid the total acidity was not far from 0.1 normal. This could do no harm, however, for Hönigschmid[23] has found that even three normal nitric acid liberates no hydrochloric acid or chlorine from a similar quantity of potassium chloride.

From the corrected weight of silicon tetrachloride the weight of silver necessary to precipitate the chloride was computed. This quantity was weighed out, chiefly in the form of a very few large buttons, the final adjustment being made with small electrolytic crystals. After careful solution of the silver in chloride-free nitric acid and elimination of nitrous acid, in a flask provided with a spray trap in the form of a column of bulbs ground into the neck, the solution was diluted to tenth normal concentration or less, and then was added slowly with constant agitation to the chloride solution contained in a glass stoppered flask or bottle. The mixture was allowed to stand for several days with occasional shaking, before testing for excess of chloride or silver in a nephelometer. If an excess of either was found, the deficiency of the other was made up by adding hundredth normal solution until the endpoint had been reached. Even then the solutions were allowed to stand for some weeks longer with occasional shaking in order to allow included or occluded material to be extracted from the precipitate. Only slight changes were ever produced by this standing.

The manipulations of precipitation and testing of the solutions were always carried out in ruby light. In using the nephelometer all the precautions noted by Richards and Wells[24] were observed, such as preparing the comparison tubes under as nearly as possible identical conditions of temperature, concentration, and time, allowing the tubes to come to constant ratio by standing for an hour or more, and taking the average of several readings of each ratio.

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[23] Hönigschmid, Z. Elektrochem., **26**, 403 (1920); Ber. d. d. chem. Gesell., **54**, B, 1873 (1921).

THE PREPARATION OF SILICON TETRABROMIDE.

To prepare silicon tetrabromide, dry nitrogen saturated with pure bromine at a temperature not far below the boiling point of bromine was passed over silicon at red heat in an all-glass apparatus very similar to that used for preparing the tetrachloride. Since the layer of silicon was long, very little bromine passed through the reaction tube into the receiver. About six hundred grams were secured in the single preparation which was made.

The bromide was then subjected to fractional distillation for removal of the impurities which experience with the chloride led us to expect. Since the boiling point of the tetrabromide, 153°, is far higher than that of the chloride, the distillations were carried out by heating the still instead of by cooling the receiver. Furthermore, Hempel columns were employed in most of the distillations.

The boiling points of the probable volatile impurities to be expected are as follows:

	bpt.		bpt.
$SiBr_4$	153°	HBr	−69°
Si_2Br_6	240°	CBr_4	190°
		$TiBr_4$	230°

Chlorobromides were not to be feared, since the bromine used was chlorine-free.

The general type of apparatus used in the distillations is shown in Figure 5. The material in the bulb *A*, which has been filled from a previous distillation and sealed off at *K*, is warmed in a water bath and is distilled through the Hempel column *D* into the bulb *B*, which is cooled with ice, leaving a residue of a few cubic centimeters in *A*. The bulb *A* is then separated by sealing the capillary *G*. *B* terminates in a special joint *J* through which it may be attached to another similar system. The residues left in *A* and *B* may then be distilled into the small bulbs *a* and *b* by inclining the bulbs to a suitable angle. The complete course of the tetrabromide distillation is shown in Figure 6.

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[24] Richards and Wells, Amer. Chem. Jour., **31**, 235 (1904); **35**, 510 (1906).

After the material had been collected in bulb *A*, a considerable amount of mercury was added and, with the bulb and contents nearly at 100°, the bulb was exhausted through a drying tube by a very efficient water pump. The liquid boiled vigorously so that a large part of the excess of bromine evaporated, together with tetrabromide vapor. The air must have been thoroughly flushed out by this boiling. Probably a large portion of hydrobromic acid formed by hydrolysis was removed at the same time. The bulb *A* was sealed off while the pump was operating. Upon shaking the bulb *A*, after cooling, the mercury

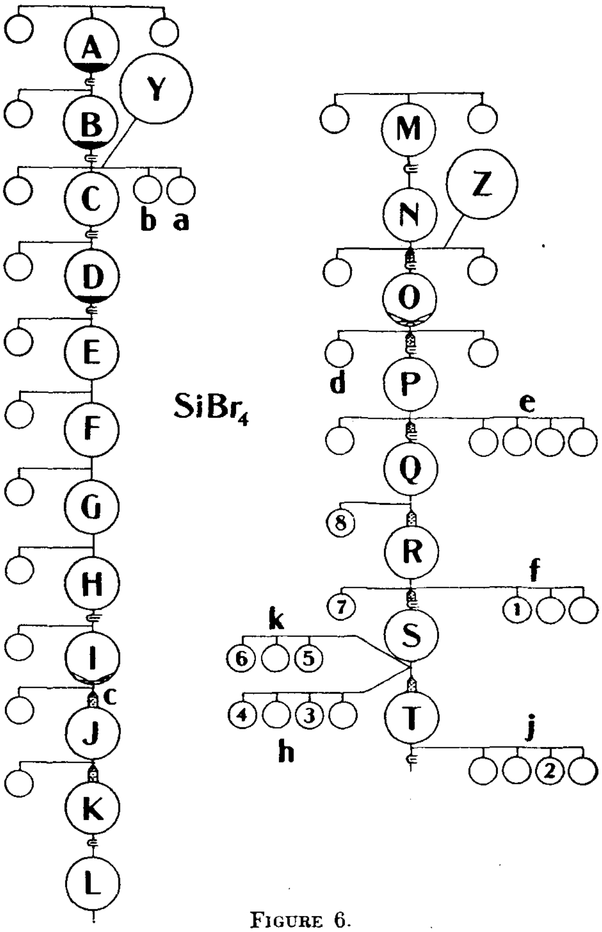

FIGURE 6.

combined with the greater part of the bromine. Bulb *B*, also containing mercury, was then attached and exhausted, and the material poured from *A* to *B*, where it was again shaken with fresh mercury. Connection was now made with the system *C*, *Y*, *a*, *b*, First the two liter expansion flask *Y* was sealed off. Two small fractions *a* and *b* were next removed by cooling with liquid air while *B* was warmed, and all but about 6 cc. was distilled into *C* and sealed off. The material in *a* and *b* was colored yellow, owing doubtless to free bromine. The main fraction in *C* also was slightly colored, as well as the residue in *B*. Another distillation into *D* which also contained mercury followed, and then a series of five distillations into *I*. In all these distillations a slight cloudiness persisted in the main body of material. Since we feared that this might be a bromide of mercury vaporizing with the silicon tetrabromide, several pieces of bright copper wire were placed in *I*. These pieces of wire were slightly discolored during the distillation from *I*, but since a fractionating column, as indicated at *c*, was used for the first time in the next distillation, it is uncertain whether the copper or the column was responsible for the great improvement in the appearance of the distillate; for from this point on it remained perfectly clear and colorless.

After distillation into the bulb *L*, the accidental breaking of a capillary admitted air and consequently a small amount of moisture to the tetrabromide. With as little exposure to air as possible the material was poured from *L* into a suitably prepared bulb *M* and again exhausted as before. After distillation into *N*, but before sealing off the connection to *M*, *N* was warmed and *M* was cooled so that several cubic centimeters were distilled back into *M*. This procedure is indicated in Figure 6 as if a light fraction had actually been collected from *N*. A similar procedure was followed in the next distillation.

The contents of the bulb *d*, which contained the heavy fraction remaining after the distillation from *O*, were tested for titanium, with a negative result.

Although a few pieces of copper wire were used in *O*, to remove a trace of bromine which was formed apparently by the exposure to the air in *L* and *M*, the first of the four light fractions removed from *P* had a slight yellow color; the remaining three were colorless.

After about half the material had been distilled from *S* to *T*, four small samples were collected as indicated at *h*, the remainder was distilled into *T*, leaving a residue which was collected in three portions, indicated at *k*.

The samples analyzed were, 1, the least volatile of three light fractions taken from *R*, 2, the second most volatile fraction from *T*, 3 and 4, middle fractions from *S*, 5 and 6, the least volatile material from *S*, 7, the residue left from the distillation from *R* and 8, a similar residue from *Q*.

THE ANALYSIS OF SILICON TETRABROMIDE.

The analysis of silicon tetrabromide was carried out almost exactly as described in the case of the tetrachloride. The glass of the bulb was washed with one per cent nitric acid and allowance made for reduction of the weight of the filter ash as described on page 258. When the end point of the comparison had been reached, the portions tested with silver and bromide solutions remained almost perfectly clear. This indicates the absence of appreciable amounts of chloride, and is to be expected in view of the pains taken to remove chlorine from the bromine.

Weighings were made on a No. 10 Troemner balance, sensitive to 0.02 mg. with a load of fifty grams. The beam was graduated to 0.05 milligram and the 5 mg. rider was used to determine all quantities larger than this, interpolation from the zero points being employed for amounts less than 0.05 milligram.

The weights were of gold plated brass, except the fractional weights, and were compared by the Richards [25] method.

All weighings were by substitution. In the case of the bulb and the silver, the weights were substituted for the object weighed. In the case of the glass, the crucible was substituted for a similar counterpoise.

Impure radium bromide was kept in the balance case to prevent electrostatic effects.

Vacuum corrections were applied as follows:

Weights	Density	Vacuum Correction per gram
Weights	8.3	
Ag	10.49	−0.000031
glass	2.5	+0.000335
air	0.001293 at 0° and 760 mm.	

Humidity was read on an accurate hair hygrometer.

The analyses of Series I, II, III, and analyses 21, 22, 23, 24 and 25 of Series V were performed by Dr. Weatherill. All analyses of Series IV and analyses 26, 27 and 28 of Series V were performed by Mr. Scripture.

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[25] T. W. Richards, Jour. Amer. Chem. Soc., **22**, 144 (1900).

TABLE I.

$SiCl_4 : 4 Ag$.

Ag = 107.880 Cl = 35.457

Number of Analysis	Fraction of $SiCl_4$	Weight of $SiCl_4$ in vacuum gm.	Weight of Ag in vacuum gm.	Weight of Ag added or subtracted in solution gm.	Corrected Weight of Ag in vacuum gm.	Ratio $SiCl_4$: 4 Ag	Atomic Weight of Silicon
				SERIES I. Sample I.			
1	12	10.43530	26.47403	+0.02120	26.49523	0.393856	28.129
2	3	5.97853	15.17909	+0.00395	15.18304	0.393764	28.089
3	9	8.79053	22.31981	+0.00150	22.32131	0.393818	28.112
4	6	6.83524	17.35557	+0.00060	17.35617	0.393822	28.114
					Average	0.393815	28.111
				SERIES II. Sample II.			
5	7	5.68295	14.43153	0.00000	14.43153	0.393787	28.099
6	13	5.01510	12.73571	0.00000	12.73571	0.393783	28.097
7	1	3.73771	9.49172	0.00000	9.49172	0.393786	28.099
8	20	7.58979	19.27465	−0.00590	19.26875	(0.393891)	(28.144)
				Average, omitting Analysis 8		0.393787	28.099
				SERIES III. Sample III.			
9	1	2.97268	7.54875	+0.00065	7.54940	0.393764	28.089
10	21	2.49266	6.33002	−0.00202	6.32800	(0.393910)	(28.152)
11	13	4.38690	11.14025	+0.00065	11.14090	0.393765	28.090
12	16	4.23780	11.76180	+0.00415	10.76595	0.393630	28.031
				Average, omitting Analysis 10		0.393711	28.070
				SERIES IV. Sample IV.			
13	1	6.32161	16.05564	+0.00040	16.05604	0.393722	28.071
14	10	2.50810	6.37019	+0.00040	6.37059	0.393700	28.061
15	4	6.30110	16.00486	−0.00030	16.00456	0.393707	28.064
16	8	7.02208	17.83944	−0.00410	17.83534	0.393717	28.069
17	11	4.14551	10.52927	+0.00015	10.52942	0.393707	28.065
18	12	6.43140	16.33562	−0.00010	16.33552	0.393707	28.064
19	14	6.17527	15.68345	+0.00080	15.68425	0.393724	28.072
20	6	6.56940	16.68581	−0.00010	16.68571	0.393714	28.068
					Average	0.393712	28.067

TABLE II.

$SiBr_4 : 4\ Ag$.

Series V.

Ag = 107.880 Br = 79.916

Number of Analysis	Fraction of $SiBr_4$	Weight of $SiBr_4$ in vacuum	Weight of Ag in vacuum	Weight of Ag added or subtracted in solution	Corrected Weight of Ag in vacuum	Ratio $SiBr_4$: 4 Ag	Atomic Weight of Silicon
		gm.	gm.	gm.	gm.		
21	1	6.29408	7.81228	−0.00153	7.81075	0.805823	28.064
22	8	7.08409	8.78878	+0.00164	8.79042	(0.805888)	(28.093)
23	2	8.75434	10.86317	+0.00070	10.86387	0.805821	28.064
24	4	6.09639	7.56496	+0.00070	7.56566	0.805797	28.054
25	5	5.89649	7.31704	+0.00050	7.31754	0.805802	28.056
26	3	5.32962	6.61348	+0.00050	6.61398	0.805811	28.060
27	7	5.87675	7.29263	+0.00050	7.29313	0.805792	28.052
28	6	4.16665	5.17077	−0.00010	5.17067	0.805824	28.065
			Average omitting Analysis 22			0.805810	28.059
			Average of Series IV and V				28.063

In the foregoing tables the analyses are arranged in the order in which they were carried out. A better comparison of the various fractions is obtained if the results are tabulated in the order of volatility. The following table gives the observed atomic weights in the order of decreasing volatility for the different chloride samples as well as for the bromide.

$SiCl_4$ I

Fraction	At.Wt.
3	28.089
6	28.114
9	28.112
12	28.129

$SiCl_4$ II

Fraction	At.Wt.
1	28.099
7	28.099
13	28.097
20	28.144

$SiCl_4$ III

Fraction	At.Wt.
1	28.089
13	28.090
16	28.031
21	28.152

$SiCl_4$ IV

Fraction	At.Wt.
1	28.071
4	28.064
6	28.068
8	28.067
10	28.063
11	28.064
12	28.065
14	28.071

$SiBr_4$

Fraction	At.Wt.
1	28.064
2	28.064
3	28.062
4	28.054
5	28.056
6	28.068
7	28.054
8	28.093

In the first three series, with the exception of Fraction 16, Series III, there is unmistakable evidence of slightly increasing atomic weight with decreasing volatility. It is especially noticeable that Fraction 12, Series I, Fraction 20, Series II, and Fraction 21, Series III, which were the third from the last, the next to the last and the last fractions respectively, all give values nearly alike and markedly higher than the others. As explained on page 252 this is probably due to less volatile chlorides of silicon which seem to be removed with considerable difficulty. The figures indicate, however, that the process of separation was still taking place even in the most carefully and elaborately distilled material, Sample III. In Sample IV the fractionation was many times as effective owing to the use of fractionating towers in nearly every distillation. The greater efficiency is obvious from the fact that even before the distillation was half completed, extreme fractions, Nos. 1 and 14, showed no indication of a difference in composition, and the uniformity of all the material from that point on is all that could be desired. There seems to be no question therefore that the results of the analyses of Sample IV should be given preference to those of the first three samples of chloride.

The silicon tetrabromide, although distilled at the beginning of the fractional distillation without the use of fractionating columns, ultimately was repeatedly distilled in much the same way as Sample IV of the chloride. Here also uniformity in composition over a wide range of fractions was secured. The fact that the least volatile fraction, No. 8, gave a distinctly higher result than the remainder, is counterbalanced by the fact that the three residual fractions from distillations immediately following, were apparently no different from the more volatile fractions.

Furthermore, the fourth chloride series and the bromide series yielded almost identical results. Therefore the average of the mean values from these two series, 28.063, is the most probable value which can be derived from these data. It seems unlikely that further fractional distillation would have affected this outcome.

It is interesting to compare the influence which various impurities would have on the results. In the following table are given the effects produced upon the apparent atomic weight of silicon by one-tenth of one per cent of the more likely impurities.

Si_2Cl_6	+0.009	Si_2Br_6	+0.009
Si_3Cl_8	+0.014	Si_3Br_8	+0.014
Si_2OCl_6	+0.024	HBr	−0.024
HCl	−0.024	CBr_4	−0.016
CCl_4	−0.016	$TiBr_4$	+0.020
$TiCl_4$	+0.020		

The bearing of the outcome of the foregoing work upon the isotopic character of silicon is an interesting one. Aston [26] finds ample evidence of the existence of two isotopes of this element, with masses 28 and 29. He then continues:

"The evidence of a silicon of atomic weight 30 is of a much more doubtful character. Its presence is suggested by the lines 30, 49, 68 and 87, but the possibility of hydrogen compounds makes the evidence somewhat untrustworthy, and no proof can be drawn from a second order line 15, as this is normally present and is due to CH_3. On the other hand, if we accept a mean atomic weight as high as 28.3, the relative intensity of the lines due to compounds of Si^{28} and Si^{29} indicates the probable presence of an isotope of higher mass."

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[26] Phil. Mag., **40**, 628 (1920).

It is obvious that the atomic weight derived in this paper is more nearly in accord with the evidence yielded by mass spectra as regards the proportion of the isotopic components of silicon, than the value 28.3 which has been in general use for some time. Assuming the existence of appreciable proportions of only the two isotopes 28 and 29, our result indicates that the ratio of lighter to heavier is about 14 to 1.

With at least two isotopes of silicon, chlorine, and bromine, there are evidently possible ten different tetrachlorides of silicon as well as ten different tetrabromides, with molecular weights ranging from 168, at intervals of one unit, to 177 in the case of the chloride, and from 344 to 353 in the case of the bromide. Of these, the two lightest, $Si^{28}(Cl^{35})_4$, 168, and $Si^{28}(Br^{79})_4$, 354, are presumably the most abundant and the most volatile.

The question therefore may be raised as to whether partial separation may have occurred during the fractionation of the two liquids.

If, however, separation actually occurs at all during the fractionation, it certainly is to be expected that it will continue to take place during the whole fractionation. The absence of any systematic difference in composition among the fractions analyzed in the case of Series IV and V, which covered a large proportion of the original material, is strong evidence that no important separation actually was taking place.

This is only to be expected from the fact that the conditions of distillation were far from ideal for the separation of isotopic substances, for as Mulliken and Harkins[27] and others have pointed out, such a separation is to be expected only when distillation takes place at extremely low pressures.

We are very greatly indebted to the Wolcott Gibbs and Bache Funds of the National Academy of Sciences for generous assistance in providing the necessary apparatus and materials.

SUMMARY.

1. Improved methods for fractional distillation out of contact with the air at various temperatures have been devised.

2. The analysis of pure silicon tetrachloride and tetrabromide has yielded the values 28.067 and 28.059 for the atomic weight of silicon. The average value is 28.063 (Ag = 107.880).

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[27] Jour. Amer. Chem. Soc., **44**, 143 (1922).

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CONTRIBUTION FROM THE OSBORN BOTANICAL LABORATORY.

THE CHILEAN SPECIES OF METZGERIA.

By Alexander W. Evans.

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THE CHILEAN SPECIES OF METZGERIA.

ALEXANDER W. EVANS.

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SPECIES of *Metzgeria* form a conspicuous part of the hepatic flora of Chile and the neighboring antarctic regions of South America. The first member of the genus to be reported definitely from this general area was the northern *M. furcata* (L.) Dumort., which was listed by Hooker and Taylor in 1844 (**8**, p. 480) under the name *Jungermannia furcata* L. Their report was based on specimens collected by Hooker during the voyage of the "Erebus" and "Terror," the locality being given as St. Martin's Cove, "Cape Horn." Three years later (**27**, p. 445) they gave somewhat fuller details, citing the species from Hermite Island, Cape Horn, and adding that it occurred there not only in the typical form but also as a var. *pubescens*, which they considered the same as the northern *Jungermannia pubescens* Schrank, usually known even at that early date as *M. pubescens* (Schrank) Raddi. In fact Montagne (**13**, p. 214) had already reported *M. pubescens* from the Straits of Magellan, on the basis of specimens collected by Jacquinot. The Synopsis Hepaticarum (**6**, pp. 504, 505), in 1846, accredited *M. furcata* to Cape Horn on the authority of Hooker and Taylor and *M. pubescens* to the Straits of Magellan on the authority of Montagne, but added nothing to the statements of these earlier writers. Four years later Montagne (**14**, p. 297) designated *M. furcata* as a widely distributed species in Chile, without giving definite localities, basing his statement on collections made by Gay.

In 1877 Lindberg published his Monographia Metzgeriae (**10**), in which he made a systematic use of anatomical features in characterizing the species and thus placed the study of the genus on a more scientific basis. Although the authors of the Synopsis recognized nine species in their treatment of *Metzgeria*, five of these really belong to the genus *Riccardia*, thus leaving a residue of only four species. Lindberg increased the number to eleven. Of the forms occurring in the Chilean region he was able to study some of the material collected by Hooker and by Jacquinot. He showed that Hooker's specimens of "*J. furcata* var. *pubescens*" and Jacquinot's specimens of "*Metzgeria pubescens*" were very distinct from the true *M. pubescens* of the Northern Hemisphere and that they represented an undescribed species. To this he gave the name *M. frontipilis.* He showed further that some of Hooker's specimens of "*J. furcata*" were distinct from the northern *M. furcata* and referred them to his new *M. hamata,* a species having a wide distribution in tropical and temperate regions. He made no definite allusion to Hooker's other specimens of "*J. furcata*" or to Gay's specimens of "*M. furcata*" but stated explicitly that he had seen no material of the true *M. furcata* from the American continent.

In 1885 Massalongo (**11**) issued his report on the Hepaticae collected by Spegazzini in the Fuegian region and recorded new localities for both *M. frontipilis* and *M. hamata.* He likewise restored *M. furcata* to a place in the flora, on the basis of sterile material from Basket Island, and described a new variety of this species from Staten Island under the name *β. decipiens.* This variety was said to be intermediate between the true *M. furcata* and *M. hamata,* and it was suggested that it might represent an undescribed species. In 1889 Bescherelle and Massalongo (**2**, pp. 246, 247) listed *M. frontipilis, M. furcata* var. *decipiens,* and *M. hamata* from numerous additional localities in Tierra del Fuego, the Straits of Magellan and southern Patagonia, basing their records on material collected by Savatier, Hariot, and other members of the French Scientific Expedition to Cape Horn. In the following year Schiffner (**16**), in his account of the Hepaticae collected by Naumann of the "Gazelle" Expedition, raised the var. *decipiens* to specific rank, under the name *M. decipiens* (Massal.) Schiffn. & Gottsche, and proposed *M. magellanica* Schiffn. & Gottsche as a new species. Both of these were found at Tuesday Bay in the Straits of Magellan. In addition he reported *M. frontipilis* and "*M. linearis* (Sw.) Lindb." from the same locality and also from Punta Arenas. The "*M. linearis,*" however, is not the same as the West Indian *M. linearis* (Sw.) Aust. but is merely a synonym of *M. hamata,* as Lindberg had already shown in his Monographia.

No additional species were reported from the region until 1899, when Stephani published a monograph of *Metzgeria* in the first volume of his Species Hepaticarum (**19**). In this important contribution he recognized *M. frontipilis, M. furcata,* and *M. hamata* as members of the flora but reduced *M. magellanica* to synonymy under *M. nitida* Mitt. (**12**, p. 243), a species based on Australian and New Zealand material, and expressed the opinion that *M. decipiens* probably represented another synonym of the same species. A third species that he included among the synonyms of *M. nitida* was his own *M. australis*

(**18**, p. 266), which had been described from specimens collected on Lord Howe's Island and in Australia. Two other well-known species that he accredited to the Chilean flora were the northern *M. conjugata* and *M. Liebmanniana* Lindenb. & Gottsche, a species originally known from Mexico. His record for *M. Liebmanniana* was based on a specimen collected at Valdivia by Hahn, and his record for *M. nitida* on Naumann's material from the Straits of Magellan. Most of his other data are more general, *M. conjugata* and *M. furcata* being cited merely from "Chile," *M. hamata* from "Patagonia" and *M. frontipilis* from the Straits of Magellan and "Chile." Under *M. furcata* and *M. hamata* he made no mention of Hooker as a collector and failed to cite either Hooker or Jacquinot in connection with *M. frontipilis*. He therefore gave no information regarding the earliest specimens of *Metzgeria* found in the region, including those upon which Lindberg's records were based.

In addition to the species already mentioned Stephani proposed as new no fewer than ten Chilean species, although he assigned to three of these a range extending far beyond the boundaries of the region. These species are the following, only the Chilean stations being indicated: *M. angusta*, "Chile" and Patagonia (*Dusén*); *M. chilensis*, "Chile" (*Dusén*); *M. corralensis*, Corral (*Krause*); *M. decrescens*, "Straits of Magellan" (*Dusén*); *M. Dusenii*, Desolation Island (*Dusén*); *M. glaberrima*, Straits of Magellan (*Spegazzini, Dusén*, "Gazelle" Expedition) and "Chile" (*Gay, Krause*); *M. Lechleri*, Arique (*Lechler*); *M. longiseta*, Straits of Magellan (Warnstorf Herbarium); *M. patagonica*, Newton Island (*Dusén*); and *M. terricola*, Straits of Magellan (*Savatier, Dusén*). It will be noted that Spegazzini and the "Gazelle" Expedition are mentioned in connection with *M. glaberrima* and that Savatier is named as one of the collectors of *M. terricola*. Since the collections of Spegazzini, Naumann and Savatier had already been reported upon (see **11**, **16**, and **2**), it is evident that the authors of these reports must have listed *M. glaberrima* and *M. terricola* under other names or else have made no allusion to the specimens here cited. Unfortunately Stephani throws no light upon these doubtful points. It will be noted further that more than half of the new species were based wholly or in part on material collected by Dusén. In regard to some of this material Stephani has given fuller details about localities in two subsequent papers (**21** and **22**), published respectively in 1900 and 1901.

In the first of these papers he listed *M. australis* as a valid species, apparently no longer regarding it as a synonym of *M. nitida*, and accredited it to Corral. He likewise cited *M. Dusenii* from the same locality and described *M. brevialata* from San Pedro Island as a new species. In the second paper he listed *M. pubescens* from Tierra del Fuego, although in his monograph he had restricted it to the Northern Hemisphere. Other data regarding stations will be noted later in connection with individual species. Strange to say Stephani made no mention in either paper of *M. chilensis*, *M. decrescens*, or *M. terricola*, in spite of the fact that he had used Dusén's specimens in drawing up the original descriptions of these species. It might at first appear that he had repudiated them, but this is clearly not the case, so far as *M. chilensis* and *M. terricola* are concerned, because he has referred to both of them in his later writings. In 1901, for example, he reported *M. chilensis* from Clarence Island in the Straits of Magellan (**22**, p. 4); in 1911 he reported the same species from Chiloé and Juan Fernandez (**23**, p. 10); and in 1916 he reported *M. terricola* from Bolivia (**25**, p. 180). These reports were based on specimens collected by Racovitza, Skottsberg and Herzog, respectively. In his report on Skottsberg's ample Chilean collections Stephani cited new stations for several other species of *Metzgeria* and also listed, as an addition to the flora, *M. albinea* Spruce, previously known from a single locality in Brazil. His record was based on a specimen from Huafo Island. In 1917 (**26**, p. 47) he made his last contribution to our knowledge of the Chilean *Metzgeriae* by proposing as a new species *M. antarctica* from Punta Arenas, basing his description on a specimen collected by Von Schrenk. According to his published writings Stephani recognizes twenty species of *Metzgeria* as members of the flora.

As in other large and natural genera of plants the species of *Metzgeria* are often difficult to distinguish. This is partly because most of the differential characters are drawn from variable structures and partly because the plants sometimes remain for a long time in an embryonic or juvenile stage of development, during which certain of the specific features fail to reveal themselves. Var. *ulvula* Nees of *M. furcata*, as shown by Goebel (**5**), is an excellent example of this condition, and equally good examples occur among the Chilean species. One unfortunate result of the difficulties involved in the determination of specimens has been an accumulation of incorrect records, not only in herbaria but also in the literature. This has been strikingly shown by Schiffner (**17**) in the case of *M. dichotoma* (Sw.) Nees, a species first described in 1788 and therefore one of the earliest to be recognized. In the Lindenberg Herbarium at Vienna he found eleven specimens bearing this name. Three of these were from Jamaica and represented original material of *Jungermannia dichotoma* from the Swartz collections; the others came from Guadeloupe, St. Vincent, Mexico, Brazil and Peru. The Jamaican specimens were the only ones that included plants of the true *M. dichotoma*, as this species is now understood, and two of these contained an admixture of *M. hamata;* the eight remaining specimens represented seven distinct species, three of which were at that time undescribed. In the Stephani Herbarium, now at Geneva, Schiffner found eight specimens bearing the name *M. dichotoma*, all of which had been collected in Brazil. Not one of these represented the true *M. dichotoma*. They represented instead three distinct species, not duplicated in the Lindenberg Herbarium. These two authoritative herbaria, therefore, which have served as the basis for many printed records, contained ten different species that had been incorrectly determined as *M. dichotoma*.

In view of these facts the writer has made an attempt to obtain for examination a full series of Chilean *Metzgeriae* and especially of specimens upon which printed reports have been based. This has been made possible through the kindness of correspondents and the curators of herbaria, and the writer would express his sincere thanks to all who have assisted him in his work. As a result of this study seven of the species based on Chilean material are reduced to synonymy, two species are proposed as new, and several incorrect determinations are rectified. These various changes reduce the number of known species to eleven, although certain fragmentary and undeterminable specimens indicate that this number is too low. In the citation of specimens the following abbreviations are used: B, Stephani collection in the Boissier Herbarium, University of Geneva; H, Cryptogamic Herbarium of Harvard University; M., Mitten Herbarium, at the New York Botanical Garden; Massal., collection of Professor Massalongo at Verona; Möll., collection of Dr. Möller at Stockholm; N. Y., herbarium of the New York Botanical Garden; S., collection of Professor Schiffner at Vienna; St., herbarium of the Swedish National Museum at Stockholm; U., herbarium of the University of Upsala; Y, herbarium of Yale University (including the private collection of the writer).

KEY TO THE SPECIES.

a. Upper surface of thallus densely covered with hairs
1. *M. frontipilis* (p. 276).
a. Upper surface of thallus naked . . . . *b.*
b. Costa bounded dorsally, on robust thalli, by more than two rows of cortical cells . . . . *c.*
b. Costa bounded dorsally by only two rows of cortical cells . . . . *d.*

c. Ventral surface of wings naked, the ventral hairs (when present) being restricted to the costa; gemmae lacking 2. *M. decrescens* (p. 279).
c. Ventral surface of thallus, including the wings, more or less densely covered with hairs; gemmae dorsal . . 3. *M. corralensis* (p. 285).
d. Costa bounded ventrally by four rows of cortical cells; gemmae marginal . . . . *e.*
d. Costa bounded ventrally by only two rows of cortical cells . . . . *f.*
e. Marginal hairs often in divaricate pairs; ventral surface of wings usually with scattered hairs . . . . 4. *M. divaricata* (p. 288).
e. Marginal hairs borne singly; ventral surface of wings naked. 5. *M. patagonica* (p. 291).
f. Gemmae lacking . . . . *g.*
f. Gemmae present, borne on more or less specialized branches becoming narrower toward the tips . . . . *i.*
g. Thallus plane or nearly so; marginal hairs usually borne singly and sometimes scantily developed or lacking . . . . *h.*
g. Thallus convex, the wings more or less revolute . . . . *j.*
h. Autoicous . . . . 6. *M. chilensis* (p. 294).
h. Dioicous . . . . 7. *M. decipiens* (p. 296).
i. Plants not turning bluish when dried; thallus and gemmae plane or nearly so . . . . 8. *M. epiphylla* (p. 303).
i. Plants turning bluish when dried; thallus and gemmae more or less convex . . . . 9. *M. violacea* (p. 306).
j. Marginal hairs borne singly and often scantily developed. 10. *M. magellanica* (p. 313).
j. Marginal hairs usually in pairs . . . . 11. *M. hamata* (p. 315).

1. METZGERIA FRONTIPILIS Lindb.

Metzgeria frontipilis Lindb. Acta Soc. F. et Fl. Fenn. $\mathbf{1}^2$: 14. *pl. 1, f. 2.* 1877.
Metzgeria brevialata Steph. Bihang K. Svenska Vet.-Akad. Handl. $\mathbf{26}^3$ (No. 6): 20. 1900.

SPECIMENS EXAMINED: Newton Island, May, 1896, *Dusén*, mixed with *113* (B., St., U., see **20**, p. 19); San Pedro Island, *Dusén 533* (Möll., type of *M. brevialata*); shores of Trinidad Canal, *Coppinger* (M.); York Bay, September, 1853, *Lechler 1354* (M., as *M. pubescens*); same locality, *Lechler* (N. Y., as *M. pubescens* var. *subglabra*); southern part of Smith Sound, near the entrance to the Straits of Magellan, 1883, *Görtner*, mixed with *Riccardia fuegiensis* and other bryophytes (Y., specimen received from the Warnstorf Herbarium); Puerto Angosto, Desolation Island, March, 1896, *Dusén*, mixed with *159* (B., M., St., U., see **21**, p. 10); Straits of Magellan, 1868, *Dow* (Y.); Rio Azopardo, Tierra del Fuego, March, 1896, *Dusén 72* (U., as *M. pubescens*, and listed under this name by Stephani, **21**, p. 10); without definite locality, Tierra del Fuego, 1896–97, *Hatcher* (Y., **3**, p. 426); Cape Horn, *Hooker* (M., **10**, p. 15; listed as *Jungermannia furcata β*.

pubescens by Taylor and Hooker, **27**, p. 445); Staten Island, 1882, *Spegazzini 65* in part (Massal., Y., **11**, p. 257).

The following additional stations may be cited from the literature: Hanover and Atalaya Islands, *Skottsberg* (**24**, p. 10); Clarence and Hoste Islands, *Hariot* (**2**, p. 247); Tuesday Bay and Punta Arenas, Straits of Magellan, *Naumann* (**16**, p. 43); Brecknock Pass, *Spegazzini* (**11**, p. 257); Tekenika Bay, Almirantazgo, and Lake Fagnano, Tierra del Fuego, *Skottsberg* (**23**, p. 9, and **24**, p. 10); Mount Sarmiento, Tierra del Fuego, *Spegazzini* (**11**, p. 257).

The presence of hairs on the dorsal surface of the thallus is the most remarkable feature of *M. frontipilis* and will at once distinguish it from all the other known species of the genus except *M. pubescens*. In the northern species, however, the thallus is scarcely convex and the hairs are equally abundant on the ventral surface, being situated on both costa and wings; whereas in *M. frontipilis* the thallus is usually distinctly convex and the ventral hairs are largely restricted to the costa, the wings being almost or entirely free from them. To a certain extent the female branches offer an exception to this description, so far as the arrangement of the hairs is concerned, since in these both surfaces are equally hairy throughout. The male branches are still unknown.

Other noteworthy characters are derived from the costa and especially from its cortical cells, which show marked variations in number. According to Lindberg's original account these cortical cells are in eight to twelve rows both dorsally and ventrally. Stephani (**19**, p. 932), with a larger series of specimens at his disposal, places the extremes at six and eighteen and associates the lower numbers with the branches and the higher with the main axis. As a matter of fact only four rows are present in some of the slender branches studied by the writer. Stephani brings out in addition the interesting fact that the wings are sometimes two or three cells thick at their junction with the costa, a feature overlooked by Lindberg.

Although the published descriptions of *M. frontipilis* state emphatically that the ventral surface of the wings is wholly naked, this condition is by no means always realized. The hairs show a tendency to encroach, as it were, upon the ventral surface, not only from the margins but also (less frequently) from the costa. This tendency sometimes expresses itself by a very slight displacement of the marginal hairs, so that they are not truly marginal (Fig. 1, A); but the displacement may be much more marked than this, so that the hairs appear on the ventral surface one, two, three or even four cells away from the margin. Occasional hairs of this type are to be expected on almost any thallus, but it is usually difficult to detect them, except in section, on account of the revolute wings. When such hairs are more abundant forms are produced like Dusén's specimens from the Rio Azopardo, which Stephani referred to *M. pubescens*. A comparison of these specimens with European material of *M. pubescens* shows at once that the two plants are not the same. In the Rio Azopardo specimens, the greater part of the ventral surface of the wings is still free from hairs, and the thallus shows the marked convexity characteristic of *M. frontipilis.*

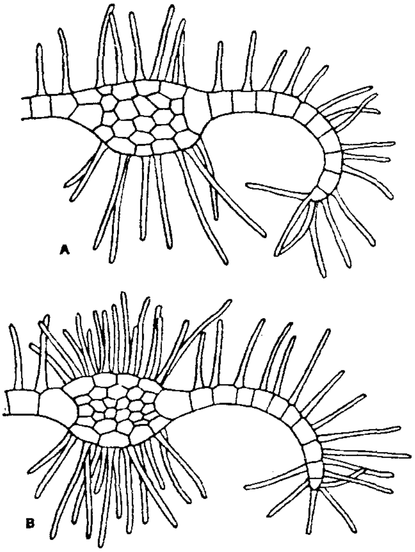

Fig. 1. METZGERIA FRONTIPILIS Lindb.

A, B. Transverse sections of rather slender thalli, × 100. A was drawn from a specimen collected on Desolation Island by Dusén, No. 159 in part; B, from the type material of *M. brevialata.*

The plants from San Pedro Island, upon which Stephani based his *M. brevialata*, represent a more aberrant type. According to his description the most important features of this species are the following: a large costa, bounded both dorsally and ventrally by ten rows of cortical cells on the main axes and by four rows on the ultimate branches; narrow wings, unistratose throughout and only slightly decurved, ten cells wide on the ultimate branches but only five cells wide on the main axes; and an abundance of hairs on both surfaces of costa and wings. The plants show that the description is essentially correct in most respects except that the numbers of cortical and alar cells are less definite than is implied. The thalli exhibit, however, two marked discrepancies, when compared with his description. In the first place the wings are not always unistratose throughout but may be bistratose at their junction with the costa; and in the second place the wings, except when very narrow, are distinctly decurved and show a broad band next to the costa entirely free from ventral hairs. The latter, to be sure, are abundant and show the extreme displacement noted above (FIG. 1, B), some of the hairs being three or four cells distant from the margin; but it is only on the narrowest wings that they are scattered over the entire ventral surface. The writer feels, therefore, that these specimens, although at first sight so striking, can not be separated specifically from *M. frontipilis*. They apparently represent a xerophytic modification of the species, both the narrow wings and the abundant hairs being in accordance with this view. It may be noted also in this connection that *M. pubescens* sometimes produces thalli with exceedingly narrow wings. Specimens showing this feature in a marked degree were discovered by Kaalaas near Christiania, Norway, and distributed by Schiffner in his Hepaticae Europaeae, No. 21, as forma *attenuata*. On some of the ultimate branches the wings are so strongly narrowed that they almost disappear, and the thalli thus acquire an appearance very different from that of typical *M. pubescens* with its broad wings.

2. METZGERIA DECRESCENS Steph.

Metzgeria decrescens Steph. Bull. Herb. Boissier **7**: 932. 1899.
Metzgeria terricola Steph. op. cit. 933. 1899.
Metzgeria longiseta Steph. op. cit. 934. 1899.
Metzgeria Dusenii Steph. op. cit. 942. 1899.

SPECIMENS EXAMINED: valley of the Aysen River, January, 1897, *Dusén 416* (M., TYPE; Möll., St., Y., as *M. glaberrima*, and listed under this name by Stephani, **20**, p. 20); Newton Island, May, 1896, *Dusén 113* in part (B., as *M. terricola*, and listed under this name by Stephani, **19**, p. 934; St., U., as *M. Dusenii*); southern part of Smith Sound, near the western entrance to the Straits of Magellan, 1883, *Görtner*, mixed with *Riccardia fuegiensis* and other bryophytes (Y., specimen received from the Warnstorf Herbarium); Alert Bay, western coast of Patagonia, 1882, *Coppinger* (N. Y.); Churucca, Desolation Island, *Savatier* (B., presumably the type of *M. terricola*, but listed as *M. furcata* var. *decipiens* by Bescherelle & Massalongo, **2**, p. 246); Puerto Angosto, Desolation Island, March, 1896, *Dusén 159* in part (M., St., U., type of *M. Dusenii*); Straits of Magellan, July, 1885, collector unnamed (B., type of *M. longiseta*, specimen received from the Warnstorf Herbarium); Cape Horn, *Hooker*, mixed with *M. decipiens* (M.).

The following additional station may be cited from the literature: Corral, *Dusén* (**20**, p. 19, as *M. Dusenii*).

In his monograph of the genus *Metzgeria* (**19**) Stephani divides the species into the two groups *Pinnatae* and *Furcatae*. Unfortunately, as Schiffner has since emphasized (**17**, p. 184), the distinction between these groups is based on variable features, since a thallus may be more or less pinnate in one part and distinctly furcate in another. Even in the more typically pinnate species, such as *M. filicina* Mitt. of the Andes, *M. pubescens* and *M. frontipilis*, the differentiation between an axis and its branches is relatively slight and expresses itself quantitatively rather than qualitatively. In certain species which Stephani includes among the *Pinnatae*, such as *M. decrescens*, he utilizes these quantitative differences in determining the relative rank of the branches in a branch-system. In the main axis of *M. decrescens*, according to his description, the costa is bounded dorsally by five rows of cortical cells and below by seven; in the pinnae (or branches of the first rank) the numbers are four and five, respectively; while in the pinnules (or branches of the second rank) the costa is bounded both dorsally and ventrally by two rows of cortical cells. He describes further a decrease in the width of the wings, corresponding with this decrease in the number of cortical cells. Apparently on the basis of these differences he states that the thallus is irregularly pinnately branched and adds that long pinnae and pinnules are mixed with much shorter ones.

A careful study of the type specimen of *M. decrescens* in the Mitten Herbarium shows that a pinnate habit is not apparent and that the branches arise in the usual dichotomous manner. These branches, to be sure, show the differences described by Stephani, and there are also intermediate conditions connecting his three types; but these differences do not by any means determine the relative rank of the branches. When an axis forks, for example, either or both of the branches may have a more complex costa and broader wings than the original axis, and the costa of a branch may increase in complexity with its growth in length. The differences are apparently due to nutritive causes and merely indicate that the costa and wings are variable with respect to the number of their component cells.

The plants of *M. decrescens* are whitish or yellowish green and apparently grow in dense mats. The living portion of the thallus may reach a length of 4–5 cm., while the width is mostly 0.9–1.3 mm. The wings are so strongly revolute that their margins sometimes meet, the thallus thus acquiring a cylindrical or subcylindrical form. The successive dichotomies may be as much as 1 cm. apart or as little as 1 mm., an individual thallus rarely showing both extremes. According to Stephani the ventral surface is naked, the hairs being restricted to the margin. In most cases this description applies, but occasionally a few scattered hairs arise from the ventral surface of the costa. The crowded marginal hairs, as he notes, occur singly and are usually straight. In well-developed plants a hair (FIG. 2, A) is found between every two marginal cells, and when the revolute wings approach each other closely the hairs form a delicate weft between them, making it difficult to study the features of the costa without spreading the wings apart. The hairs are mostly 0.2–0.4 mm. long and 10–20 μ in width, tending to taper from the base.

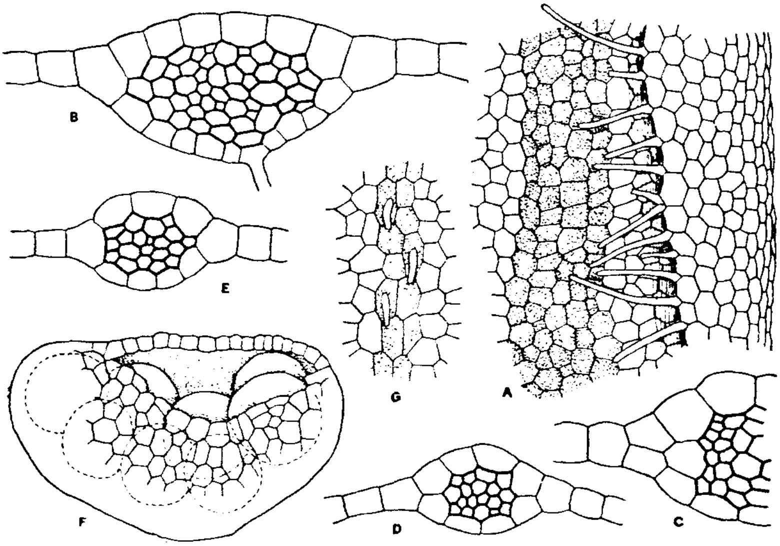

FIG. 2. METZGERIA DECRESCENS Steph.

A. Part of a slender thallus, ventral view, × 50; there are only two or three rows of cortical cells, the external stippled row on each side representing a bistratose transition-region between costa and wings. B–E. Transverse sections of costae, × 100. F. Male branch, × 100. G. Costa and adjacent alar cells of a male branch, showing slime-papillae, × 100. The figures were all drawn from the type material.

The alar cells have thin or slightly thickened walls, and trigones are either absent or minute and inconspicuous. Stephani gives the size of the cells as 54 × 45 μ. According to the writer's measurements the average size of the cells in the type specimen is about 45 × 34 μ, while the general average derived from the eight specimens listed above is 43 × 33 μ. At the same time one specimen gave an average as high as 55 × 35 μ, so that Stephani's figures are not excessive. The lowest average obtained was 36 × 32 μ. It should be remembered in this connection that, in species of *Metzgeria*, considerable variation in the size of the cells is to be expected, not only when different thalli, but also when different parts of the same thallus are compared. Data derived from cell-measurements must therefore be used with caution in distinguishing species.

The costa of *M. decrescens* (Fig. 2, B–E) yields some of the most distinctive characters of the species. It not only shows the marked variation in the number of cortical cells, to which attention has already been called, but it often shows in addition another feature unusual in the genus. In most of the *Metzgeriae* the transition between the unistratose wings and the multistratose costa is very abrupt, a condition clearly shown in many of the published figures. When, however, the costa is large and complex, as it is for example in *M. frontipilis*, the transition between the wings and the costa may be more gradual, a narrow band of large cells two or three cells thick being interpolated between the unistratose portion of a wing and the costa. In *M. decrescens*, although the costa is not particularly complex, both types of structure occur, just as they do in *M. frontipilis*. In some of the branches the abrupt transition is present (Fig. 2, E), in others the gradual transition (Fig. 2, B–D), and the latter is not necessarily associated with the more complex costae. When there is a gradual transition this is usually clearly apparent even when an intact thallus is examined. Under these circumstances the costa seems to be poorly defined, and the bistratose or tristratose band becomes evident by careful focusing, the outlines of the cells in the superimposed layers not corresponding.

The type material of *M. decrescens* shows male branches in abundance but no female branches. The male branches, which seem to be the only ventral branches present, have involute margins and are strongly incurved, although the apex does not approach the base very closely (Fig. 2, F). They measure about 0.45 × 0.3 mm. in well-developed examples and are wholly destitute of hairs, the only appendicular organs developed being the slime-papillae (Fig. 2, G). The alar cells are more delicate than those of a vegetative thallus and average only 25 μ in diameter. In some of the other specimens studied a few female branches with calyptras were found. These organs bore scattered hairs and attained a length in some cases of 3–3.5 mm., the diameter being 0.6–0.8 mm. Unfortunately the female branches themselves were so old and battered that their distinctive features could not be determined.

The three synonyms included under *M. decrescens* may now be considered. The first, *M. terricola,* was based on two specimens, one collected by Savatier on Desolation Island and the other by Dusén on Newton Island. According to Stephani *M. terricola* shows a variability in the number of cortical costal cells, comparable with what is found in *M. decrescens.* The wings of the thallus, moreover, are revolute in much the same way, while the cells are said to average about 54 × 36 μ, measurements which diverge but slightly from those given for *M. decrescens.* The following represent the most important differential characters indicated: the presence of a few setulae on the ventral surface of the thallus and the occurrence of the marginal hairs in pairs.

In Savatier's specimens, which may perhaps be regarded as the type, the costa is essentially like that of the type specimen of *M. decrescens;* and, although ventral hairs are sometimes present on the costa, this is equally true of *M. decrescens.* The marginal hairs, moreover, so far as the writer can determine, are invariably borne singly. In Dusén's material some of the thalli are like Savatier's, but others show crowded marginal cilia in pairs. The latter, however, are associated with costae that are bounded constantly, both dorsally and ventrally, by only two rows of cortical cells, while the cells of the revolute wings are considerably larger, averaging about 70 × 40 μ. In the writer's opinion the thalli with the geminate marginal hairs should be referred to *M. hamata,* although they evidently formed a part of the material from which the description of *M. terricola* was drawn. If these thalli are eliminated there is apparently nothing whatever to distinguish *M. terricola* from *M. decrescens.* The writer regrets that he has not seen Herzog's specimens of "*M. terricola,*" to which allusion has already been made (see page 274).

The second synonym, *M. longiseta,* was based on a specimen from the Straits of Magellan, the collector's name not being given. In his account of this species Stephani calls attention to the strongly convex thallus, the variable number of cortical costal cells, the long marginal hairs borne singly, and the alar cells averaging about 54 × 36 μ, four characters which *M. longiseta* clearly shares with *M. decrescens.* He mentions also the fact that the wings are two or three cells thick near the costa. This, as has been shown, is another characteristic feature of *M. decrescens,* although the original description does not allude to it. The differential characters of *M. longiseta* are apparently drawn from the marginal hairs, which are described as "hamate," and from the costa. The latter is said to be strongly convex dorsally and nearly plane ventrally and to have a thickness of five cells. The dorsal cortical cells, furthermore, are said to be convex and much larger than the internal and ventral cortical cells, which are said to be subequal in size. Unfortunately the type specimen does not support this description very convincingly. The marginal hairs are very rarely hamate, most of them being straight or irregularly curved and contorted; while the costa, as shown by cross sections, may project ventrally slightly more than dorsally. The dorsal cortical cells, moreover, measure about 50 μ in width, the ventral about 40 μ, and the internal cells, which may be in more than three layers, measure about 30 μ. These observations show that Stephani's differential characters are far from constant, and yet with the withdrawal of these his descriptions of *M. longiseta* and *M. decrescens* are almost identical.

Although the first two synonyms of *M. decrescens* are placed by Stephani among the *Pinnatae*, the third, *M. Dusenii*, is placed among the *Furcatae*. It is based upon material collected by Dusén on Desolation Island and the three original specimens examined have all been badly mixed with *M. frontipilis*. Stephani's description of *M. Dusenii* would seem to indicate that the species was much less variable in its costal features than *M. decrescens*, since the cortical cells are said to be in four rows both dorsally and ventrally. It is added that the dorsal surface is convex and the ventral smooth, that costal hairs are lacking, that the dorsal cortical cells are large and projecting, and that the ventral cells are much smaller. The original material shows at once the inconstancy of these features. Although some of the thalli show four rows of cortical cells on both surfaces, deviations from this number are frequent; some of the branches, for example, show only two or three such rows, while five rows of ventral cells were observed in at least one instance. Costal hairs, moreover, can be demonstrated by careful search in spite of their great infrequency, and they are really not much rarer than in the type of *M. decrescens*. The costal cells, finally, show deviations from the description. In a series of sections examined by the writer, the costa was found to be distinctly convex ventrally, while the ventral cortical cells measured 38 μ in width and were thus only slightly narrower than the dorsal cells, which measured 42 μ. Aside from the characters which have just been discussed the description of *M. Dusenii* agrees in all essential respects with that of *M. decrescens*, since the thallus is said to be strongly convex with naked wings and long marginal hairs borne singly, while the alar cells are said to average about 45 × 36 μ. The Desolation Island specimens are perhaps a trifle less robust than the type of *M. decrescens* from the Aysen Valley, and the branches tend to be shorter, but these differences are too slight and too inconstant to be of much significance.

It is interesting to note that Dusén's material from Newton Island, No. 113, has been differently determined by Stephani at different times. The specimen in the Boissier Herbarium bears the name *M. terricola,* while those at Stockholm and Upsala bear the name *M. Dusenii.* In the writer's opinion, as indicated above, these specimens are clearly the same and represent *M. decrescens.* It might appear from his determinations that Stephani recognized the identity of his *M. terricola* and *M. Dusenii* and wished to supplant one name by the other. Unfortunately this assumption is contradicted by his published writings.

3. METZGERIA CORRALENSIS Steph.

Metzgeria corralensis Steph. Bull. Herb. Boissier **7**: 933. 1899.
Metzgeria Lechleri Steph. op. cit. 942. 1899.

SPECIMENS EXAMINED: without definite locality or date, *Gay* (Mont., as *M. furcata,* and listed under this name by Montagne, **14**, p. 297); Corral, no date, *Krause* (B., TYPE); Arique, no date, *Lechler 652* (M., unnamed but probably representing the type of *M. Lechleri*); Valdivia, 1887–88, *Hahn* (B., as *M. Liebmanniana* and listed under this name by Stephani, **19**, p. 935); same locality, date and collector (B., apparently a part of the same collection as the preceding but bearing a manuscript name); Osarno Volcano, date and collector's name not given (M.).

Stephani places *M. corralensis* among the *Pinnatae* and describes the thallus as remotely pinnate. At the same time he makes no allowance for variability in the number of costal cells, as he did in *M. decrescens,* stating definitely that the dorsal cortical cells are in four rows and the ventral in eight. The type specimen shows that these numbers are too rigid. Although the dorsal cortical cells are usually in four rows (FIG. 3, A, B), the number really varies from two to five, and the ventral rows are frequently fewer than eight (FIG. 3, B). In spite of this variability a pinnate habit is no more apparent in *M. corralensis* than in *M. decrescens.*

The species varies in color from a pale yellowish green to a dull green and is fairly robust. According to Stephani the thallus sometimes attains a length of 4 cm. The width is mostly 1–1.5 mm. but may be as much as 2.5 mm. in well-developed plants. The wings are plane or somewhat convex and are mostly fifteen to twenty-five cells across in the type material, although Stephani gives the width as only twelve cells. According to his statements the alar cells measure 36 × 27 μ, and these figures agree pretty closely with the general average of 33 × 26 μ, obtained from the five specimens listed above. The cells, as he notes, are essentially thin-walled throughout, although vague indications of trigones are sometimes present.

In well-developed thalli the whole ventral surface, including both the costa and the wings, is covered over with crowded hairs, giving it a pubescent appearance. These hairs are mostly 0.1–0.3 mm. in length and 10–12 μ in diameter. Those along the margin, which are essentially like the others, sometimes spread widely and sometimes grow downward; they usually arise singly, as Stephani notes, but twinned hairs may occasionally be demonstrated. Between the densely pubescent ventral surface, which is doubtless typical for the species, and a smooth or nearly smooth condition, all gradations occur, although it is doubtful if a thallus is ever smooth throughout.

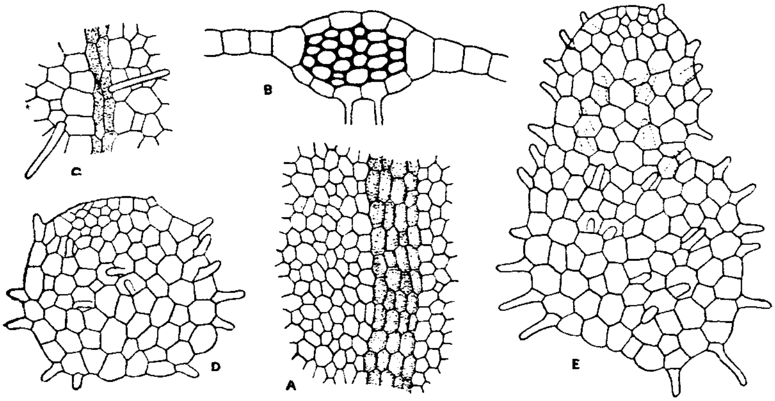

FIG. 3. METZGERIA CORRALENSIS Steph.

A. Part of a thallus showing costa and adjoining cells of wings, dorsal view, × 50. B. Transverse section of costa, × 100. C. Costa and adjoining alar cells of a male branch, showing ventral hairs, × 100. D. Gemma at time of separation, × 100. E. Germinating gemma, × 100. A, D and E were drawn from a specimen collected on the Osarno Volcano; B and C, from a specimen collected at Arique by Lechler.

In the specimens studied by the writer a few male branches are present and are mostly 0.2–0.35 mm. in length by 0.25–0.4 in width. The wings are involute and the costa so strongly incurved that the apex almost reaches the base, the branch thus acquiring a spherical or subspherical form. According to Stephani the surface is smooth, but this is rarely the case, from one to a dozen hairs being usually present (FIG. 3, C).

If Stephani's descriptions of *M. Lechleri* and *M. corralensis* are compared it will be seen that they agree in most important respects, even though he places *M. Lechleri* among the *Furcatae*. The most important differences that he brings out are derived from the costae and alar cells, the features of which in typical *M. corralensis* have already been discussed. According to his description the costa of *M. Lechleri* is bounded both dorsally and ventrally by four rows of cortical cells, while the alar cells have firm walls distinctly thickened at the angles. The study of Lechler's Arique specimens in the Mitten Herbarium, which agree in most respects with Stephani's description, brings out the fact that the cortical costal cells are inconstant in number, just as in *M. corralensis*. The alar cells, moreover, although slightly thickened, do not show conspicuous trigones; in fact it is usually difficult to make them out at all. Since the differences between the species thus break down, and since the Arique specimens are essentially like Krause's type, the writer feels convinced that the two species are synonymous.

The importance of gemmae in distinguishing species of *Metzgeria* has already been emphasized by the writer in another connection (**4**). In *M. corralensis* the gemmae are dorsal and are borne on ordinary vegetative branches, the growth of which is apparently unlimited. As in *M. crassipilis* (Lindb.) Evans of the eastern United States (see **4**, p. 282) and other species having dorsal gemmae, many thalli are not gemmiparous at all, while others produce the gemmae in great profusion. In the case of *M. corralensis* the early stages of development have not been studied, but their adult features will be described.

At the time of separation (FIG. 3, D) the gemmae vary somewhat in size but most of them are 0.18–0.27 mm. long and 0.16–0.24 mm. wide; they may be orbicular, but the width is usually a little less or a little more than the length. A gemma is six to eight cells across and has a single apical cell. What may be described as the dorsal surface is convex and usually shows from two to six short and rudimentary hairs. The gemma bears in addition from three to eight marginal hairs on each side, and these may be truly marginal or slightly displaced to the ventral surface, which seems otherwise to be perfectly smooth.

When a gemma germinates its apical cell continues (or resumes) its activities and gives rise to a flat, strap-shaped thallus which tends to be narrower than the gemma itself (Fig. 3, E) being often only four cells wide. While this is going on the hairs on the gemma increase somewhat in length, and similar superficial and marginal hairs appear on the flat extension. The superficial hairs are always more numerous on one surface than on the other and may be confined to one surface. Sometimes the more hairy surface of the extension is continuous with the hairy convex surface of the gemma and sometimes with the smooth concave surface, these observations apparently showing that the dorsiventrality of the gemma is not firmly fixed but that a reversal of the dorsiventrality may take place at germination.

The presence of superficial hairs on the gemmae of *M. corralensis* and on the young thalli to which they give rise are perhaps the most distinctive features of these structures. Except for these peculiarities the gemmae and young plants are much like those of *M. crassipilis* and *M. Liebmanniana.* The latter species, in fact, is closely related to *M. corralensis*, differing from it mainly in its greater size; and it is therefore not surprising that specimens of the Chilean species have been referred to *M. Liebmanniana.*

4. **Metzgeria divaricata** sp. nov.

Grayish or yellowish green, scattered or growing in depressed mats, more or less firmly attached to the substratum: thallus prostrate, repeatedly dichotomous but rarely branching ventrally, plane or slightly convex, well-developed thalli mostly 0.6–1.2 mm. wide, the forks mostly 2–8 mm. apart; costa bounded dorsally by two rows of cortical cells and ventrally by four; wings mostly eight to fifteen cells broad, the cells mostly 38 × 31 μ, the walls thin or slightly thickened and sometimes with more or less distinct trigones and nodular intermediate thickenings; hairs varying greatly in abundance; marginal hairs in the hairiest and most characteristic plants occurring in divaricate pairs, ventral hairs under these circumstances numerous on the wings and especially on the costa; hairs averaging about 0.15 mm. in length and 10–12 μ in width, often branched at the apex and acting as rhizoids: inflorescence dioicous: ♂ branches sometimes borne in considerable abundance, subspherical, usually bearing on the ventral surface from one to five scattered hairs, 0.33–0.36 mm. long and 0.33–0.45 mm. in width: ♀ branch broadly obcordate, 0.25–04. mm. long and 0.45–06. mm. wide, hairs abundant along the margin and also scattered over the ventral surface, especially in the median part; calyptra about 2 mm. long and 0.9 mm. wide, more or less hairy throughout but especially in the upper half: gemmae rarely abundant, marginal, borne on undifferentiated branches, oblong, flat or nearly so, usually with crowded rudiments of marginal and sometimes paired hairs slightly displaced to one surface.

SPECIMENS EXAMINED: Chile, without definite locality or date, *Neger 68* (B., as *M. conjugata,* and listed under this name by Stephani, **19**, p. 951); near Santiago, 1882, *Philippi 24* (B., as *M. furcata,* and listed under this name by Stephani, **9**, p. 941); Concepcion 1905–06, *Thaxter 90, G* (H., Y.); San Antonio, Pudeto River, Chiloé, July, 1908, *Halle & Skottsberg 257* (U., as *M. Lechleri,* and listed under this name by Stephani, **24**, p. 10). No. 90, collected by Professor Roland Thaxter, may be designated the type; No. 257, from Chiloé, is poorly developed and somewhat doubtful.

In *M. divaricata* and the species that follow the structure of the costa is far more constant than in *M. frontipilis, M. decrescens* and *M. corralensis.* This does not mean that an absolute constancy is to be expected. In *M. divaricata,* for example, the ventral cortical cells may be in five rows instead of four, even at some little distance from a fork; it simply means that deviations from the typical numbers are infrequent enough to be ignored.

As noted in the description the ventral hairs vary greatly in abundance. In the more extreme development of these hairs the entire ventral surface appears loosely pubescent, and the marginal hairs occur between every two marginal cells. In typical cases these marginal hairs are paired and spread so widely apart that they form a straight line perpendicular to the margin. As a rule the outer hair of each pair is truly marginal and the inner ventrally displaced. Sometimes, however, the outer hair is slightly displaced too, and a semblance of displacement is often brought about by the slight convexity of the wing-margins. When a long series of these paired and divaricate marginal hairs is present the thallus acquires a very striking and distinctive appearance (FIG. 4, A). Unfortunately the condition just described is not always realized. Sometimes, for example, one part of a thallus may be pubescent, while other parts produce hairs sparingly or not at all. An entire thallus, in fact, may be sparingly hairy throughout, and most of the marginal hairs present may be borne singly. Even under such circumstances, however, a prolonged search will usually bring to view an occasional pair of the characteristic marginal hairs.

Marginal gemmae occur abundantly on some of the plants collected by Neger but are apparently absent from all the other specimens. The gemmiparous branches are essentially like the others and present no evidence of limitation in growth. The gemmae are usually scattered, although a crowded series is sometimes to be observed, and the mother-cells of the gemmae arise directly from the marginal cells, just as in *M. furcata* (**4**, p. 277). At the time of separation the gemmae vary considerably in size, average examples measuring perhaps 0.3–0.4 mm. in length and 0.15–0.2 mm. in width. Most of them are oblong in form, six to eight cells across, and show an indistinct stalk and a single apical cell. Crowded rudiments of marginal hairs, slightly displaced to one surface, are usually present and not infrequently show a paired arrangement. Otherwise the gemmae are scarcely differentiated. In germination (Fig. 4, B–D) the young plant is at first nothing more than a slightly narrower extension of the gemma, although in one somewhat older example a rudimentary costa was present with a wing three cells wide on each side. No late stages of germination have been observed.

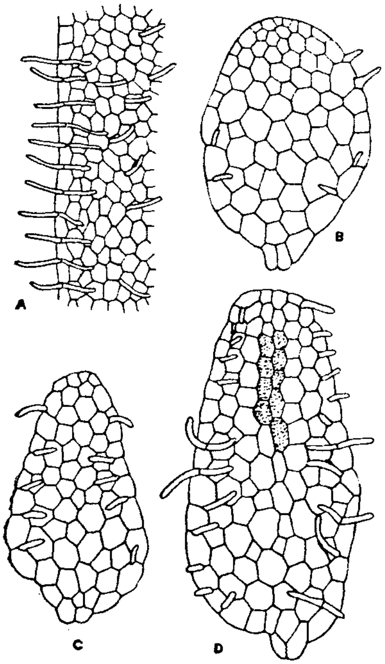

Fig. 4. Metzgeria divaricata Evans.

A. Marginal portion of a thallus-wing, ventral view, × 50. B–D. Germinating gemmae, × 100. A was drawn from the type material; B–D, from a specimen collected in Chile by Neger, No. 68.

The list of specimens cited brings out the fact that *M. divaricata*, apparently on account of its variability, has been confused with three other species of *Metzgeria*. In the structure of the costa with its two rows of dorsal and four rows of ventral cortical cells it agrees with *M. conjugata* and *M. furcata;* in having ventral hairs, sometimes produced in considerable abundance, it agrees with *M. corralensis*. It is, however, amply distinct from all three species. It differs from *M. conjugata* in being dioicous and in having gemmae and ventral alar hairs, while it differs from *M. furcata* in having paired marginal hairs. When strongly pubescent it resembles *M. corralensis* rather markedly but is distinguished by the more definite structure of the costa, by the occurrence of the marginal hairs in divaricate pairs, and by the marginal gemmae.

5. METZGERIA PATAGONICA Steph.

Metzgeria patagonica Steph. Bull. Herb. Boissier **7**: 940. 1899.

SPECIMENS EXAMINED: Newton Island, May, 1896, *Dusén 24* (M., U., TYPE).

The following additional station may be cited from the literature: Escapada Island, Skyring, *Skottsberg* (**24**, p. 11).

According to the original description of this well-marked species the wings of the thallus are strongly decurved and often revolute, but a supplementary note adds that the specimens are "etiolated" and that the normal structure is to be found only on the younger "innovations." In the material studied by the writer most of the thalli are perfectly plane and only a few of the branches show revolute margins. At the same time the plane thalli can hardly be regarded as abnormal; they do not present the appearance of being etiolated, and the presence of female branches in some abundance shows that the plants are by no means in a juvenile stage of development. The soluble yellow substance, to which Stephani calls attention in a later paper (**20**, p. 20), is very much in evidence when the specimens are soaked in water.

The thalli of *M. patagonica* are pale green, often deeply tinged with yellow, and grow in depressed mats. The width is mostly 1–1.5 mm. and the length may be as much as 3 cm. Measured in cells the wings are usually fifteen to twenty-five cells across. Although ventral branching sometimes occurs, dichotomous branching is far more common, the successive forks being mostly 1–5 mm. part.

Hairs are rarely abundant and many regions are nearly or quite free from them. The marginal hairs are straight and seem to be invariably borne singly. They are usually slightly displaced to the ventral surface, tending to extend at right angles to the wings, but they may be truly marginal and lie in the same plane as the wings. The hairs are about 10 μ in diameter and rather short, the length being usually only 0.1–0.12 mm. Although the wings are naked the costa sometimes bears loose and scattered clusters of hairs, essentially like the marginal hairs but sometimes a trifle longer. Apparently in either position a hair has the power of branching at the tip and acting as an organ of attachment.

The costa shows the same structure as that of *M. divaricata,* being bounded dorsally by two rows of cortical cells and below by four. The alar cells, according to Stephani, measure 54 × 40 μ, those near the costa being 72 × 40 μ. The writer's measurements give an average of 41 × 33 μ and do not indicate that the cells near the costa are appreciably longer than the others. The cells throughout have rather firm walls, which often show nodular intermediate thickenings as Stephani notes, but the thickened angles that he likewise emphasizes are difficult to demonstrate.

No male branches have been seen by the writer and the original description does not mention them. Female branches are often abundantly produced, and it is a noteworthy fact that a female thallus sometimes becomes gemmiparous shortly after it has borne the sexual branches. Some of the latter are small and undeveloped, but most of them are of fair size (Fig. 5, A), measuring perhaps 0.5–0.7 mm. in length and 0.9–1 mm. in width. The outline, which is broadly orbicular with a deep apical indentation, does not show clearly without spreading the branches out flat, owing to their strong concavity. The margin shows crowded hairs borne singly, each representing the outgrowth of a small cell situated between two larger cells, just as in the normal vegetative thalli of most *Metzgeriae.* On the ventral surface the thickened median portion bears a dense cluster of hairs, and a few other hairs are scattered over the unistratose portion.

Mention has just been made of gemmiparous plants, although Stephani does not allude to them. As a matter of fact the gemmae of *M. patagonica,* which are marginal in position, yield some of the most distinctive characters of the species. The gemmiparous branches are at first scarcely modified but rapidly decrease in width after the formation of the gemmae has been initiated. When the wings have been reduced to a width of four or five cells the growth of the branch comes to an end. The reduction in the width of the wings is often accompanied by a simplification in the structure of the costa, the rows of ventral cortical cells being only two or three. In the formation of the gemmae their mother-cells are derived directly from the marginal cells of the branch, without a preliminary cell-division. The gemmae may be scattered or crowded, a long series of adjoining marginal cells sometimes giving rise to a continuous row of gemmae. The latter tend to appear in acropetal succession and yet show many exceptions to this arrangement.

At the time of separation the gemmae are flat and unistratose structures, orbicular to oblong in outline, broadening out abruptly from a two-celled and often indistinct stalk, and showing a broad and rounded apex with a single apical cell. They are mostly 0.25–0.3 mm. long and 0.18–0.25 mm. wide, being composed of six to eight indefinite rows of cells. On each side six to eight hairs are usually present, and these are commonly (but not invariably) arranged in pairs. The hairs extend almost at right angles to the surface of the gemma and, when paired, spread in opposite directions. The majority are strongly curved and might often be described as hamate. Only the earliest stages of germination have been observed and in these the young plants have simply repeated the features of the gemmae, except that they have sometimes been a little narrower (FIG. 5, B). In other words they have remained flat and unistratose thalli, showing no signs of dorsiventrality and tending to produce a succession of paired and divergent marginal hairs.

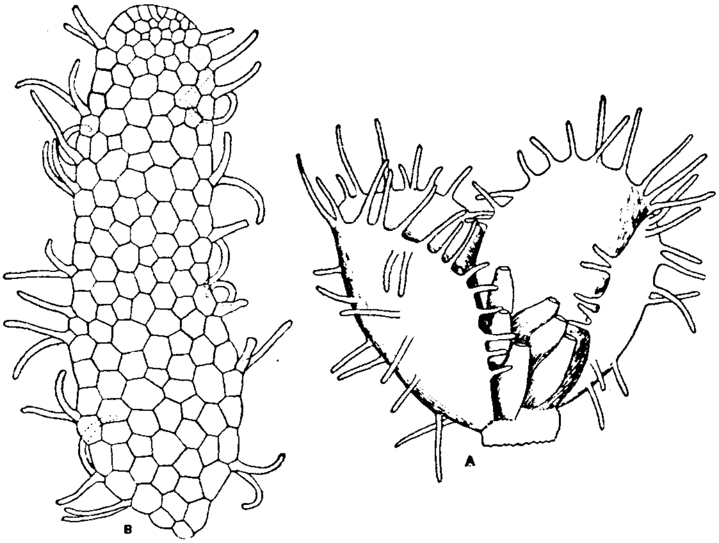

FIG. 5. METZGERIA PATAGONICA Steph.

A. Female branch, × 50. B. Germinating gemma, × 100. The figures were both drawn from the type material.

Marginal gemmae with hooked hairs have been described in *M.*

uncigera Evans of the West Indies and Florida (**4**, p. 273), a species in which the vegetative thallus bears straight hairs arising singly. Dorsal gemmae with hooked hairs have been described in two West Indian species, *M. dichotoma* and *M. vivipara* Evans (**4**, pp. 285, 288), in both of which the vegetative thallus bears straight marginal hairs, again arising singly. In *M. vivipara* twinned hairs occur as a rare exception, the hairs being usually borne singly; in the other two species twinned hairs are apparently never found. *M. patagonica* shows a new combination of characters — marginal gemmae with curved or hooked hairs arising in pairs and a vegetative thallus with straight marginal hairs arising singly. It is this unusual association that separates the species most sharply from its allies.

Of course the structure of the costa allies *M. patagonica* with *M. conjugata* and *M. furcata*, as well as with the preceding species. In *M. conjugata*, however, no gemmae are produced and the marginal hairs of the thallus are normally borne in pairs; in *M. furcata* the hairs of the gemmae, if present at all, are straight and arise singly; while in *M. divaricata* the marginal hairs of both thallus and gemmae often arise in pairs but are straight. Aside from these differences *M. patagonica* can be distinguished from *M. conjugata* by its dioicous inflorescence and from the other two species by its lack of ventral hairs on the wings.

6. METZGERIA CHILENSIS Steph.

Metzgeria chilensis Steph. Bull. Herb. Boissier **7**: 937. 1899.

SPECIMENS EXAMINED: Quiriquina Island, near Concepcion, no date, *Dusén 179* (M., TYPE).

The following additional stations may be cited from the literature: Clarence Island, *Racovitza* (**22**, p. 4); Quicavi, Chiloé, *Skottsberg* (**24**, p. 10); Juan Fernandez, *Skottsberg* (**24**, p. 10); New Zealand, *Colenso* (**19**, p. 937). The Juan Fernandez specimen is clearly distinct from the true *M. chilensis;* the other specimens have not been seen by the writer.

The species was based on two specimens, one from Chile and the other from New Zealand. The Chilean specimen is naturally to be regarded as the type, but the original description was probably partly drawn from the New Zealand specimen, since it does not agree in all respects with Dusén's material.

The plants in the Mitten Herbarium are very fragmentary and grew in a loose depressed mat in admixture with other bryophytes. The thallus is mostly 0.5–0.9 mm. wide and attains a length of 1–1.5 cm. The wings, although described as almost revolute by Stephani, are flat or even slightly concave and are mostly six to twelve cells wide. The normal branching is dichotomous with the forks 2–10 mm. apart, but ventral branching is not exceptional.

The marginal hairs vary greatly in abundance. In some places they may be absent altogether; in other places, even on the same thallus, they may be as numerous as the marginal cells, a single hair arising between every two cells. In most cases the hairs are slightly displaced to the ventral surface, but they may be truly marginal, and it is not unusual for the apex to be branched and to act as an organ of attachment. The longest hair seen was 0.3 mm. long but most of them were 0.1 mm. or less in length, the average diameter being about 10 μ. The ventral surface of the wings is apparently wholly free from hairs, but the costa bears them in loose clusters or scattered and is rarely free from hairs for any great distance. These costal hairs are essentially like the marginal hairs but tend to be a little longer.

The costa is bounded both dorsally and ventrally by two rows of cortical cells, a type of structure found also in all the following species. The alar cells average about 35 × 27 μ, although Stephani's measurements gave 54 × 36 μ. The walls are slightly thickened and sometimes show minute trigones and occasional nodular intermediate thickenings.

According to Stephani the inflorescence is dioicous. The type specimen, however, is clearly autoicous, the male and female branches often occurring in close proximity. The male branches are mostly 0.3–0.4 mm. long and 0.25–0.3 mm. wide and are ellipsoidal in form, the margins being involute and the costa so strongly incurved that it approaches the base without reaching it. Except for the slime-papillae the surface is smooth. The female branches, which are more or less concave and obcordate in outline, are mostly 0.4–0.45 mm. long and 0.45–0.6 mm. wide. The marginal hairs grow out from small cells but are not numerous; the ventral hairs may be restricted to a cluster of six to twelve on the thickened median portion, but one to three scattered hairs may be present also on the wings. No gemmae have been observed.

The autoicous inflorescence will at once distinguish *M. chilensis* from all the other Chilean species. It agrees in this unusual feature with *M. conjugata*, but in that species the ventral cortical cells of the costa are in four rows and the marginal hairs often in pairs. The only other South American species to which an autoicous inflorescence has been assigned is *M. albinea* Spruce, which further agrees with *M. chilensis* in the structure of the costa. In *M. albinea*, however, the marginal hairs are in pairs. Aside from the inflorescence *M. chilensis* approaches the following species very closely.

7. METZGERIA DECIPIENS (Massal.) Schiffn. & Gottsche.

Metzgeria furcata β. decipiens Massal. Nuovo Gior. Bot. Ital. **17**: 256. *pl. 28, f. 36.* 1885.
Metzgeria decipiens Schiffn. & Gottsche in Schiffner, Forschungsreise "Gazelle" **4**^{4}: 43. 1890.
Metzgeria glaberrima Steph. Bull. Herb. Boissier **7**: 939. 1899.
Metzgeria nuda Steph. Kungl. Svenska Vet.-Akad. Handl. **46**^{9}: 10. *f. 3a.* 1911.

SPECIMENS EXAMINED: Valdivia, 1887, *Hahn* (S.); Corral, 1905–06, *Thaxter 1f, 2c, 78, 110, 122, 124, 138, 141* (H., Y.); valley of the Aysen River, 1897, *Dusén 283* (Möll., as *M. glaberrima*, and listed under this name by Stephani, **20**, p. 20); Puerto Chacabuco, 1908, *Halle 256* (St., as *M. glaberrima*, and listed under this name by Stephani, **24**, p. 10); near the mouth of the Rio Pudeto, Chiloé, 1908, *Halle 256* (St., as *M. glaberrima*, and listed under this name by Stephani, **24** p. 10); Guaitecas Islands, 1897, *Dusén 394* (M., Möll., St., as *M. glaberrima*, and listed under this name by Stephani, **20**, p. 20); Port Gallant, Straits of Magellan, 1896, *Dusén* (N. Y., St., as *M. glaberrima*); Tuesday Bay, Straits of Magellan, 1876, *Naumann* (S., Y., listed by Schiffner, **16**, p. 43); Grappler Bay, Straits of Magellan, 1893, *Douglas* (H., Y.); Rio Azopardo, Tierra del Fuego, 1896, *Dusén 71* (U., as *M. glaberrima*, and listed under this name by Stephani, **21**, p. 10); Rio Olivia, Tierra del Fuego, 1902, *Skottsberg* (St., as *M. glaberrima*, and listed under this name by Stephani, **23**, p. 9); Cape Horn and Hermite Island, *Hooker* (M., as *M. furcata*, and listed under this name by Hooker and Taylor, **8**, p. 480); near Basil Hall, Staten Island, 1882, *Spegazzini* (Massal., Y., TYPE of *M. furcata β. decipiens*). The following three specimens from the Falkland Islands have likewise been examined: Port Stanley, 1902, *Skottsberg* (St., as *M. glaberrima*, and listed under this name by Stephani, **23**, p. 9); same locality, 1905, *Thaxter* (H., Y.); near Port Stanley, 1907, *Skottsberg 356* (U., type of *M. nuda*).

The following additional stations from the literature may be cited: Wellington and Desolation Islands, *Savatier;* and Hoste Island, *Hyades* (**2**, p. 246, as *M. furcata* var. *β. decipiens*).

The following stations for *M. glaberrima* may likewise be cited:

near Puerto Varas, *Dusén* (**20**, p. 20); Skyring and Dawson Island, *Skottsberg* (**24**, p. 10); Desolation Island, *Dusén* (**21**, p. 10); Ushuaia, Tierra del Fuego, *Skottsberg* (**23**, p. 9). Also the following stations beyond the boundaries of Chile: New Zealand and Australia, several collectors (**19**, p. 939); Antipodes Islands (**24**, p. 10).

As here understood *M. decipiens* is probably the commonest and most widely distributed *Metzgeria* in Chile. It exhibits a great deal of variation in size and particularly in width, in the number and distribution of its hairs and in the measurements of its alar cells. It shows, however, the following apparently constant features; a flat or slightly convex thallus; a costa bounded both dorsally and ventrally by two rows of cortical cells; a lack of ventral alar hairs; and a lack of gemmae. Another feature almost as constant is the presence of ventral vegetative branches. It is of course difficult to establish the absolute constancy of any characters in so variable a genus as *Metzgeria*, especially characters of a negative nature, but the writer has found no exceptions to the four first enumerated after a detailed study of the numerous specimens cited.

The plants are pale yellowish green and are sometimes scattered but usually form depressed and layered mats of considerable extent. They are frequently found on trees but are by no means restricted to such localities; in rare instances, in fact, they are epiphyllous in habit. The living portion of a thallus is usually 1–2 cm. long, while the width is mostly 0.8–1.2 mm. These figures represent the mean averages obtained from six specimens. The narrowest thallus seen, however, was only 0.2 mm. wide, while the widest was 1.8 mm. Measured in cells an average wing is usually thirteen to seventeen cells across; a very narrow wing, however, may be as little as two cells and a very wide one as much as twenty-seven cells. The ventral branches are sometimes so abundant that they largely replace the normal branches. When the latter occur to the usual extent the successive dichotomies are mostly 1–3 mm. apart. A ventral branch broadens out abruptly from a narrow stalk-like base and quickly acquires a normal width, often in fact just beyond the margin of the higher axis. Sometimes the branch spreads widely or obliquely; sometimes it grows in the same direction as the higher axis. Under the latter circumstances the axis is usually soon limited in growth; and, if the process is repeated, a more or less definite sympodium may be the result.

Hairs occur in two positions — along the margin and on the ventral surface of the costa. The marginal hairs (Fig. 6, A) are by far the more numerous and are sometimes very abundantly produced. In other cases, however, a prolonged search is necessary before any hairs at all can be demonstrated, and there are many intermediate conditions between these extremes. A thallus, in fact, may produce hairs abundantly in one part and be hairless or nearly so in another. When the marginal hairs are crowded a single hair usually arises between every two marginal cells, but sometimes the hairs arise in pairs more or less frequently. When borne singly they are either truly marginal or slightly displaced to the ventral surface. The hairs are usually straight and measure 0.15–0.3 mm. in length by 10–12 μ in width. In rare instances they are branched at the apex and act as organs of attachment. Costal hairs are usually exceedingly rare, and in many individual thalli none at all can be demonstrated, as Stephani notes under *M. glaberrima*. When they occur they are either scattered or in small irregular clusters and are essentially like the marginal hairs.

The alar cells vary considerably in size (Fig. 6, A–D), not only in different thalli but also (in some cases at least) in different parts of the same thallus. In Spegazzini's material from Staten Island, for example, the cells in most places averaged about 48 × 36 μ, while a branch of a thallus yielding these higher measurements in its other parts had cells averaging only 35 × 29 μ. Taking the mean averages of fourteen specimens the cells measure about 38 × 29 μ, the highest average being 48 × 26 μ and the lowest 31 × 22 μ. Stephani's measurements of *M. glaberrima*, 36 × 36 μ, agree closely with the general average. It must of course be remembered that individual alar cells may deviate rather widely from these average measurements. The cells have thin or slightly thickened walls, and trigones are either minute or absent altogether.

The male branches of *M. decipiens* present few distinctive features. They are almost globular in form, the wings being involute and the costa so strongly incurved that the apex almost reaches the base (Fig. 6, E). The largest example measured was about 0.35 mm. in diameter. Hairs are entirely absent, but the usual slime-papillae are of course present.

The female branches (Fig. 6, F–I) are broadly obovate and vary from plane to convex when viewed from the ventral surface. Exclusive of the hairs they are usually 0.3–0.4 mm. in length and 0.45–0.75 mm. in width. Along the margin the hairs are crowded but apparently never in pairs, each hair representing the outgrowth of an ordinary marginal cell. On the ventral surface the hairs, if developed at all, are restricted to the thickened median portion, where from one to perhaps twenty may be present, the number being usually larger if

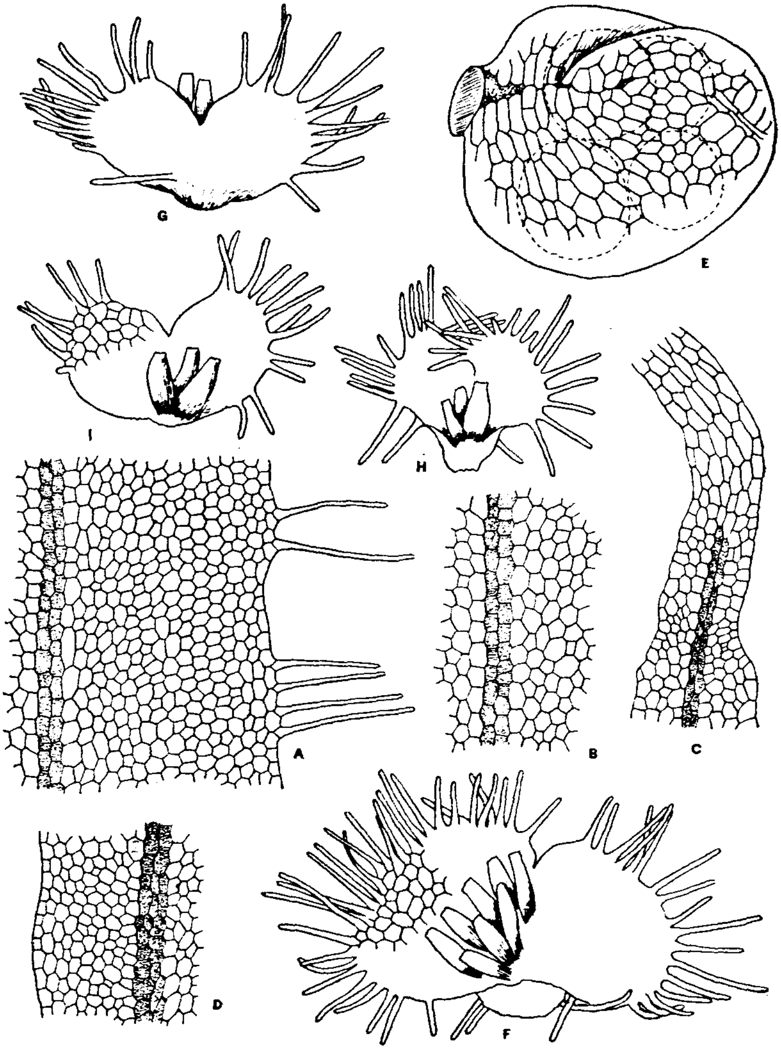

Fig. 6. Metzgeria decipiens (Massal.) Schiffn. & Gottsche.

A–D. Portions of thalli, ventral view, × 50. E. Male branch, × 100. F–I. Female branches, × 100. A, F, and G were drawn from a specimen collected by Naumann at Tuesday Bay; B and H, from a specimen of *M. glaberrima* collected on the Guaitecas Islands by Dusén, No. 394; C, from a specimen of *M. glaberrima* collected on Tierra del Fuego by Dusén, No. 71; D, E and I, from a specimen collected at Corral by Thaxter, No. 78.

fertilization has taken place. The calyptra at maturity is clavate, measuring about 1.2 mm. in length and 0.6 mm. in diameter in the upper part, exclusive of the hairs. The latter are densely crowded above the middle and more sparingly developed toward the base; some of them attain a length 0.4 mm.

On one of the specimens from Corral, No. 138, several capsules are present and give an opportunity for describing the valves, our knowledge of which in *Metzgeria* is still very incomplete. The mature capsule is dark brown and oval, measuring about 0.6 mm. in length and 0.4 mm. in diameter. The valves, when spread out flat, measure 0.6×0.3 mm. and are composed, as is uniformly the case in *Metzgeria*, of two layers of cells. Those of the outer layer (FIG. 7, A) are more or less subject to variation but in the more typical cases extend lengthwise and are two or three times as long as broad, a valve being mostly fourteen to sixteen cells across. The local thickenings of the cell-walls are conspicuous and are largely (but not wholly) confined to the inner longitudinal walls, that is, to the walls turned toward the middle of the valve. The median wall thus shows two rows of thickenings, which alternate with one another, whereas each other longitudinal wall shows only one such row. As a rule three or four thickenings are present in each cell; they extend from the surface through the thickness of the layer but are not prolonged on either tangential wall.

Although the cells of the inner layer (FIG. 7, B) do not exactly correspond with those of the outer layer, they are of about the same size and shape. With rare exceptions each cell shows from three to six transverse bands of thickening on the inner tangential walls, these bands being prolonged down the radial walls but not forming complete rings. Sometimes, however, the bands are less developed and do not extend wholly across the tangential walls, gradually fading out toward the outer boundary of the cell.

The spores are pale yellow, minutely punctulate, and 14–16 μ in diameter. The elaters are mostly 0.3–0.4 mm. in length and 6 μ in diameter in the middle, tapering gradually to the extremities. Each one bears the usual broad band of thickening extending its entire length.

If the account of the capsule-valves as given above is compared with the description of Andreas (**1**, p. 195) certain interesting differences become apparent. According to his statements the three rows of cells of the outer layer that come next to the edge of the valve have their thickenings definitely restricted to the inner longitudinal walls.

In the remaining cells, however, the thickenings are arranged more irregularly and the median wall of each valve is entirely free from them. In all probability these differences are specific in character, and it is natural to assume that the capsule-valves in *Metzgeria* may be as useful in distinguishing closely related species as in the related genus *Riccardia*. It is to be regretted that Andreas does not indicate the species from which his description was drawn.

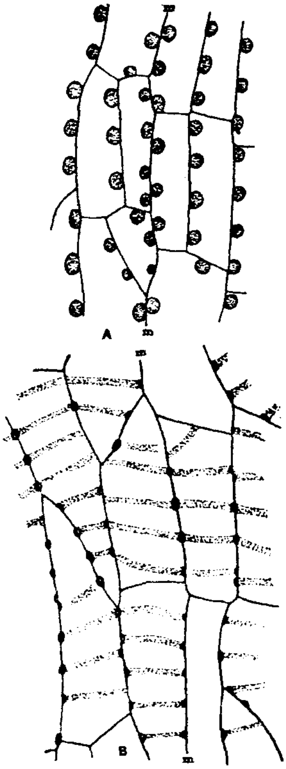

Fig. 7. Metzgeria decipiens (Massal.) Schiffn. & Gottsche.

A. Cells from outer layer of a capsule-valve, × 300; *m*, median wall of valve. B. Cells from the inner layer of a capsule-valve, × 300; *m*, median wall of valve. The figures were both drawn from a specimen collected at Corral by Thaxter, No. 138.

Although no gemmae have been observed in *M. decipiens*, a single case of regeneration from a marginal cell has been demonstrated. The product of regeneration in this instance bore a strong resemblance to a gemma, but its true nature was made evident by the zone of dead cells separating it from the rest of the thallus. Attention may be called also to the ease with which the species reverts to a more juvenile condition. The narrow and relatively hairless thalli, which have been described, represent cases of such reversion, and these often reach a more extreme state by losing their costae altogether, thus becoming reduced to uniform, unistratose bands of cells (Fig. 6, C). The prevalence and long duration of these reversionary forms have added to the difficulties of recognizing and defining the species.

In reducing *M. decipiens* to doubtful synonymy under *M. nitida* (see p. 272) Stephani criticised Schiffner for basing a new species on material so poorly developed that it could not be definitely determined. In the writer's opinion this criticism is unjustified. In the first place *M. decipiens* was really based on *M. furcata* var. *decipiens* of Massalongo, and the figures drawn from Spegazzini's type specimen (**11**, *pl. 28, f. 36*) show female branches in abundance, some of them with young calyptras. Massalongo's description, moreover, in spite of its brevity, brings out some of the most distinctive characters of the species. In the second place Naumann's "Gazelle" Expedition specimen of *M. decipiens* is equally well developed and also shows characteristic female branches. It even illustrates the variability of the species to a certain extent, some of the thalli being almost destitute of hairs. These two specimens, which the writer has carefully examined, agree in all essential respects and certainly form an adequate basis for the proposal of a new species.

Under the original description of *M. glaberrima*, which is here included among the synonyms of *M. decipiens*, Stephani cited specimens from the Straits of Magellan, "Chile," New Zealand and Australia. Apparently he afterwards changed his mind regarding the New Zealand and Australian material, for, in 1911, he restricted the range of the species to southern Chile, Tierra del Fuego, Falkland and Antipodes Islands (**24**, p. 10). The natural inference from this would be that *M. glaberrima* as originally defined was an aggregate. If this should be established the "type-specimen" should presumably be one of those from the Straits of Magellan, since these are mentioned first. As collectors of the Magellan specimens Stephani named Spegazzini, Dusén and the "Exped. Gazelle." The actual specimens of Spegazzini and Naumann cited have not been seen by the writer. Dusén's Port Gallant specimen, however, agrees fully with Spegazzini's specimen of *M. furcata* var. *decipiens* and Naumann's specimen of *M. decipiens;* and, since it agrees with none of the other specimens of *Metzgeria* collected by Spegazzini and Naumann, it would almost seem as if Stephani had based his *M. glaberrima*, at least in part, upon the very specimens utilized by Schiffner in his description of *M. decipiens*. In any case Dusén's Port Gallant specimen is referable to *M. decipiens*, and the same thing is true of at least two other specimens collected by Dusén and definitely listed by Stephani under the name *M. glaberrima*. It therefore seems justifiable to consider the latter a synonym of *M. decipiens*, even if the original *M. glaberrima* included other distinct forms. The writer regrets that the New Zealand and Australian specimens cited by Stephani have not been available for study.

The type specimen of *M. nuda*, likewise included as a synonym, is sterile and far more poorly developed than the original material of *M. decipiens*. The thalli are not invariably naked, as the description states, although the hairs even when present are scantily developed. They occur on the margin and also ventrally on the costa. The alar cells average about 33 × 21 μ but are not always as small as this, averaging in one area as much as 41 × 33 μ. Since the absence of cilia is the only important feature distinguishing *M. nuda* from *M. decipiens*, and since this feature has been proved inconstant, the two species are clearly synonymous.

8. **Metzgeria epiphylla,** sp. nov.

Yellowish or whitish green, not becoming bluish after drying, scattered or in thin depressed mats, loosely adherent to the substratum: thallus prostrate, repeatedly dichotomous but also with ventral vegetative branches, flat or slightly convex, well-developed thalli mostly 0.6–0.8 mm. wide and rarely as much as 1 mm., the forks mostly 0.6–2.4 mm. apart; costa bounded both dorsally and ventrally by two rows of cortical cells; wings mostly eight to thirteen cells broad, the cells mostly 37 × 30 μ, the walls thin or slightly and uniformly thickened, sometimes with indistinct trigones; hairs varying in abundance; marginal hairs usually occurring singly but not infrequently in pairs, sometimes branched at the apex and acting as rhizoids, mostly 0.1–0.15 mm. long and 8–10 μ wide; ventral hairs sometimes lacking, sometimes sparingly developed on the costa and still more sparingly on the wings, similar to the marginal hairs: inflorescence dioicous: ♂ branches subspherical, smooth, 0.3–0.4 mm. long and 0.25–0.35 mm. wide: ♀ branch broadly obovate, 0.3–0.35 mm. long and wide, hairs abundant on the margin and usually on the ventral surface; calyptra about 1 mm. long and 0.45 mm. wide, the hairs abundant above the middle, few and scattered below: capsule brown, oval, mostly 0.5–0.6 mm. long and 0.35–0.4 mm. wide, the valves (when spread out) 0.6–0.75 × 0.2–0.25 mm.; spores pale brownish yellow and very minutely punctulate, 16–18 μ in diameter; elaters 0.3–0.4 mm. long, 6 μ wide in the middle and with a single broad spiral band running the entire length: gemmae sometimes abundant, arising on more or less narrowed and specialized branches with limited growth, marginal or submarginal and dorsal in position, orbicular to oval, plane or slightly convex and bearing a few short marginal hairs slightly displaced to the concave surface.

Specimens examined: Corral, 1896, *Dusén 82, 191* (U., as *M. australis*, and listed under this name by Stephani, **20**, p. 19); same locality, 1905–06, *Thaxter 10a, 108, 140* (H., Y.). No. 10a, collected by Professor Roland Thaxter, may be designated the type.

In its vegetative features *M. epiphylla* resembles *M. decipiens* so closely that it would be difficult to tell them apart if these features alone were relied upon. Both species, for example, show a flat or nearly flat thallus, branching both dichotomously and ventrally, and having a costa bounded on each surface by two rows of cortical cells. The costae, moreover, although usually naked, occasionally develop a few ventral hairs; the alar cells are almost identical in size and in the characters derived from their walls; and the marginal hairs exhibit a similar range in abundance, being sometimes numerous and sometimes very few and occasionally showing a twinned arrangement although usually occurring singly.

On the whole *M. epiphylla* (Fig. 8, A) is slightly smaller than *M. decipiens* and prefers living leaves as a habitat, although it occasionally grows on bark. *M. decipiens*, on the contrary, is much more at home on bark and other substrata than on leaves. One other vegetative difference to be noted, although more observations are necessary to prove its constancy, is the occasional presence of ventral alar hairs in *M. epiphylla* and their complete absence in *M. decipiens*.

The most trustworthy differential characters, however, are those derived from the sexual branches and the capsules, and these are supplemented by the presence of gemmae in *M. epiphylla* and their absence in *M. decipiens*. The male branches are much alike in the two species, except that those of *M. epiphylla* are even more strongly incurved, so much so that the apex usually comes in contact with the base. The female branches when normally developed are distinguished by their greater hairiness, the ventral hairs not being restricted to the thickened median portion, as in *M. decipiens*, but scattered over the entire surface. The capsules are mainly distinguished by differences in the character and distribution of the local wall-thickenings of the valves. In *M. decipiens*, as has been shown, the median wall of the outer layer has two rows of local thickenings, these being largely restricted to the inner longitudinal walls of the valve-cells. In *M. epiphylla* the median wall has no local thickenings or very small ones (Fig. 8, B), approaching in this respect the condition described by Andreas. On each side of this median wall two rows (or more rarely three) have the thickenings restricted to the outer longitudinal walls, while the remaining cells have them on the inner walls, as in *M. decipiens*. Each valve thus has two longitudinal walls with double rows of local thickenings. In the inner layer of the valves, transverse bands, instead of being conspicuous, are either lacking or very indistinct (Fig. 8, C), although the prolongations of such bands on the radial walls are still apparent.

Even in the absence of sexual branches and capsules, the presence of gemmae will at once serve to distinguish *M. epiphylla* from *M. decipiens.* The gemmiparous branches seem to be always narrower than the normal vegetative thalli, the reduction in width being confined to the wings, but at first no other signs of differentiation are evident. With the appearance of the gemmae (FIG. 8, D) the wings become still narrower and the branches curve away from the substratum, their growth in length being sooner or later brought to an end. In extreme cases the wings become reduced to a width of only two or three cells, but the growth may cease while the wings are considerably broader. No cases have been observed where a gemmiparous branch had reverted to a vegetative condition, and yet it would not be surprising if such a change occasionally took place. The gemmiparous branches are flat or slightly convex when seen from the dorsal surface and sometimes branch after the production of gemmae has begun.

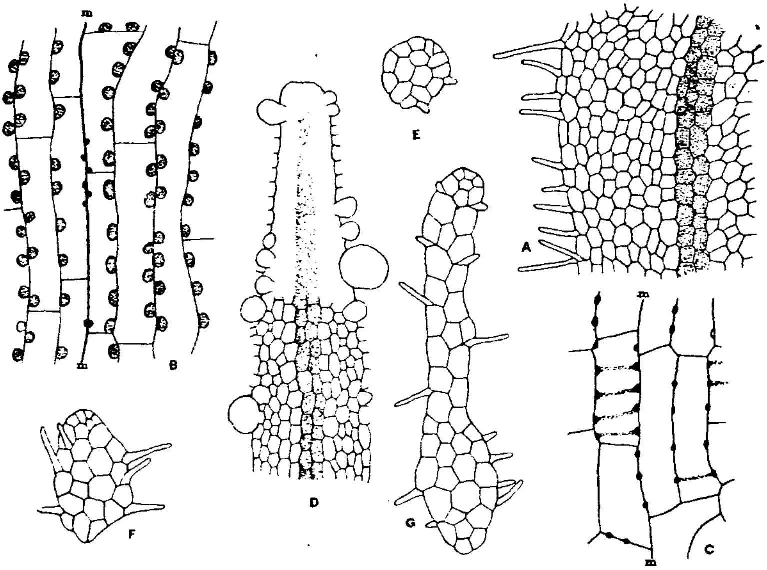

FIG. 8. METZGERIA EPIPHYLLA Evans.

A. Portion of a thallus, dorsal view, × 50. B. Cells from outer layer of a capsule-valve, × 300; *m*, median wall of valve. C. Cells from inner layer of a capsule-valve, × 300; *m*, median wall of valve. D. Gemmiparous branch, dorsal view, × 50. E, F. Gemmae about ready to separate, × 100. G. Germinating gemma, × 100. The figures were all drawn from the type material.

The gemmae which are first produced are strictly marginal, and in their development a marginal cell becomes directly the mother cell of the gemma. After the gemmae have been set free the empty mother cells with their ruptured outer walls can easily be demonstrated, and sometimes long and continuous stretches of such empty cells are present, indicating a copious formation of gemmae. Later on, with the narrowing of the gemmiparous branches, the cells of the submarginal row may in turn give rise to gemmae, setting them free dorsally. Apparently, however, the alar cells within this row and the cortical cells of the costa do not have this power.

The gemmae when set free are plane or slightly convex bodies, always small and relatively simple, yet varying somewhat in size and in the number of their component cells (FIG. 8, E, F). Most of them are orbicular to oval in outline, measuring 0.1–0.12 mm. in length by 0.09–0.1 mm. in width and being usually four cells across. A single apical cell is present and the stalk cells, although normally two, are often indistinct. On each side three or four short marginal hairs, borne singly and slightly displaced to the concave surface, can be detected; these are sometimes spreading and sometimes extend at right angles to the surface of the gemma. Upon germination the marginal hairs elongate, and the apical cell resumes its growth, giving rise to a flat thalloid extension only two or three cells wide and thus narrower than the gemma itself. In the case illustrated (FIG. 8, G) this extension had grown to more than twice the length of the gemma and had produced a series of scattered marginal hairs, slightly displaced to one surface. No later stages of germination have been observed.

9. METZGERIA VIOLACEA (Ach.) Dumort.

Jungermannia violacea Ach.; Weber & Mohr, Beitr. Naturk. **1**: 76. *pl. 1, f. 1–3.* 1805.
Fasciola violacea Dumort. Comm. Bot. 114. 1822 (in part).
Echinogyna violacea Dumort. Syll. Jung. 84. 1831 (in part).
Echinomitrium violaceum [*Echinomitrion violasceus*] Corda; Sturm, Deutschl. Flora **2**: 81. *pl. 22.* 1832 (in part).
Echinomitrium [*Echinomitrion*] *furcatum δ. violaceum* Hüben. Hep. Germ. 47. 1834 (in part).
Metzgeria violacea Dumort. Recueil d'Obs. sur les Jung. 26. 1835 (in part).
Metzgeria furcata δ. 2. *violacea* Nees, Naturg. Europ. Leberm. **3**: 489. 1838 (in part).
Metzgeria conjugata var. *β. violacea* Lindb. Acta Soc. F. et Fl. Fenn. **12**: 34. 1877.
Metzgeria angusta Steph. Bull. Herb. Boissier **7**: 944. 1899 (in part).
Metzgeria antarctica Steph. Sp. Hepat. **6**: 47. 1917.

SPECIMENS EXAMINED: near Arique, no date, *Lechler 633* (B., as *M. furcata* var. *violacea*, and listed as *M. furcata* by Stephani, **19**, p. 941); Corral, 1896, *Dusén*, mixed with *82* (U.); valley of the Aysen River, 1897, *Dusén 324* (Möll., St., as *M. hamata*, and listed under this name by Stephani, **20**, p. 20); Quellon, Chiloé, 1908, *Halle & Skottsberg 29* (St., as *M. chilensis*, and listed under this name by Stephani, **24**, p. 10); Punta Arenas, 1895, *Dusén 5* (U., as *Metzgeria* sp.); same locality and date, *Dusén*, no number (Möll., N. Y., as *M. angusta*, and listed under this name by Stephani, **21**, p. 10); same locality, 1905–06, *Thaxter 159* (H., Y.); same locality, 1907, *Von Schrenk* (B., type of *M. antarctica*); Isla di Navarino, Tierra del Fuego, 1902, *Skottsberg* (St.); Provenir, Tierra del Fuego, 1895, *Dusén 23* (U., as *M. angusta*, and listed under this name by Stephani, **21**, p. 10). The following specimens collected outside the boundaries of Chile may likewise be cited: without definite locality, Peru, *Lechler*, mixed with another species of *Metzgeria* (N. Y., as *M. furcata* var. *violacea*); San Carlos, Lake Nahuelhuapi, Argentina, 1897, *Dusén 456* (St., U., as *M. angusta*, and listed under this name by Stephani, **20**, p. 19); Dusky Bay, New Zealand, 1773, *Sparrmann* (Y., TYPE of *Jungermannia violacea*, specimen received from the Acharius Herbarium at Lund).

Specimens of *Metzgeria* showing a bluish or purplish coloration have long been familiar to students of the Hepaticae. As long ago as 1785 Dickson described the color of his *Riccia fruticulosa*, now known as *Metzgeria fruticulosa* (Dicks.) Evans (see **4**, p. 293), as "aeruginosus seu viridi-subcaeruleus," and Acharius, in 1805, stated that the present species had a "schöne Veilchenfarbe." The older writers evidently regarded these unusual hues as natural to the living plant, and Hübener, in 1834, associated the color with the presence of iron in the substratum (**9**, p. 47). A few years later, however, Funck showed that these ideas were untenable. In a letter addressed to Nees von Esenbeck (see **15**, p. 492) he discussed the blue color of *M. fruticulosa*, which he had found at Gefrees, in the Fichtel Mountains of Germany. His specimens grew on the young trunks of the Norway spruce and were distinctly green when he collected them. About six months after they were dried most of them had assumed a blue color. The soil where the trees grew was a disintegrated gneiss, without a trace of iron, and he suggested that there might be some connection between the color and the tannin in the bark. His opinion regarding the post mortem nature of the blue coloration has recently been confirmed by Miss Herzfelder (**7**, pp. 392–397), who worked mainly on *M. frutic*u-

losa. According to her statements no plants showing the blue or purple color are capable of revivication.

The coloration is by no means characteristic of the genus *Metzgeria* as a whole but is confined to certain definite species. It thus has a significance from the standpoint of taxonomy, even if lifeless specimens are the only ones that show it. So far as the writer's observations go the coloration is usually, if not invariably, associated with gemmiparous species and, in some cases at least, with species in which the gemmiparous branches show marked differentiation. The species may further be distinguished by a tendency toward reversion and by the long persistence of embryonic and juvenile stages of development. In extreme cases this persistence may be so pronounced that a large mat of plants will absolutely fail to show the normal features of the species to which it belongs. Most of the species in question are, moreover, usually sterile, and, even when male branches are present in some abundance, female branches are almost always extremely rare or absent altogether. On account of these various peculiarities the species of *Metzgeria* turning bluish or purplish have been the source of much confusion to students, and different observers have often reached divergent conclusions in regard to them.

The best known species in this category is *M. fruticulosa,* widely distributed in Europe and recently reported by the writer from the states of Washington and Oregon. *M. violacea* is a close relative of *M. fruticulosa,* so close that it can hardly be regarded as anything more than a "small" or "geographical" species. The original material of *Jungermannia violacea* was collected in 1773 at Dusky Bay, New Zealand, by A. Sparrmann, who accompanied Captain Cook on his second voyage. Strange to say there is no record of its having been collected there a second time, and most of the works dealing with the Hepaticae of New Zealand make no mention of it whatever. This is true, for example, of Hooker's well-known Handbook of the New Zealand Flora, published in 1867, and of Stephani's recent Species Hepaticarum.

The earlier writers, however, were more charitable toward the species. In 1815 its validity was recognized by Weber (**27**, p. 100), who regarded it as identical with Dickson's *Riccia fruticulosa,* reducing the latter to synonymy, in spite of its having been published earlier. For a while Weber's views prevailed to a certain extent, and European writers continued to use the name "*violacea,*" now in a specific and now in a varietal sense, always assuming as they did so that *J. violacea* and *Riccia fruticulosa* were one and the same thing. With the lapse of time the names "*violacea*" and "*fruticulosa*," as applied to *Metzgeriae*, gradually fell into disuse, and the plants to which they were applied came to be regarded as unimportant forms of *M. furcata*. Lindberg, fortunately, did not share these views. He considered that *R. fruticulosa* represented a distinct and well-marked variety of *M. furcata* and that it was amply distinct from *J. violacea*, which he regarded as a corresponding variety of *M. conjugata*. It remained for the writer to restore Dickson's plant to specific rank, under the name *M. fruticulosa*, and the same recognition, with considerable hesitation, is given to *M. violacea* in the present paper.

Through the kindness of Professor Nordstedt of Lund, the writer has had the privilege of studying a part of the type material of *Jungermannia violacea*. In spite of its long preservation it soaks up readily in water and retains the vivid coloration to which it owes its name. It lacks sexual organs completely, as the published descriptions emphasize, but shows pointed and highly differentiated gemmiparous branches, to which numerous gemmae still remain attached. These specimens have been carefully compared with the long series of Chilean specimens listed above and do not seem to differ from them in any important respect. *M. violacea*, as here understood, is exceedingly variable; it shows marked reversions, and many of the plants examined exhibit a juvenile or even embryonic stage of development. The following account is therefore somewhat composite in character.

The more typical vegetative thalli vary from flat to convex, with the margins of the wings more or less revolute. The width is usually 0.5–0.8 mm., but narrower thalli are not infrequent and some attain a width of 1 mm. or slightly more. Measured in cells the wings are mostly ten to twenty cells wide. Ventral branching occasionally occurs, although the usual method is by forking, the forks being mostly 0.6–1 mm. apart and the entire thallus rarely exceeding a length of 0.5–1 cm. The alar cells, taking the mean average from five specimens, measure about 31 × 25 μ, the highest average obtained being 34 × 27 μ and the lowest 27 × 21 μ. The walls are thin throughout and trigones are either absent altogether or minute and inconspicuous.

The variation in the number and distribution of the hairs is about as great as in *M. decipiens* and *M. epiphylla*. In other words a thallus may be hairless throughout the whole or the greater part of its extent, it may produce hairs in abundance, or it may present almost any intermediate condition between these extremes. The marginal hairs are not infrequently in pairs but usually arise singly. In fact, on some of the plants examined no twinned hairs could be discovered, although the hairs were fairly numerous. When the wings are distinctly revolute and the hairs abundant, a delicate weft is sometimes to be observed between the contiguous margins, much as in *M. decrescens*, but the hairs are just as likely to extend irregularly in all directions. Even when marginal hairs are present the thallus is often naked elsewhere; in other cases ventral hairs can be demonstrated on the costa and, still more rarely, on the wings. The hairs vary greatly in length, the majority measuring perhaps 0.08–0.12 μ in length; the diameter is mostly 10–12 μ. The costa, in ordinary well-developed thalli, is bounded both dorsally and ventrally by two rows of cortical cells.

In two specimens a few male branches were observed. They were smooth and almost spherical, the costa being so strongly incurved that the apex almost touched the base. Some of the branches were about 0.25 mm. in diameter but a few were somewhat larger, the largest one seen measuring 0.4–0.35 mm. The single female branch demonstrated was so disintegrated that its true features could not be determined. It bore a young calyptra with crowded hairs in the upper part and scattered hairs below the middle.

In most of the material gemmae are present in large numbers, and the gemmiparous branches (Fig. 9, A, B) show interesting modifications, comparable with those described under *M. epiphylla* but reaching a more advanced type of specialization and approaching in this respect the highly specialized gemmiparous branches of *M. fruticulosa*. Even when the vegetative branches are strongly convex and prostrate the gemmiparous branches are plane or nearly so and curve away from the substratum. At the same time the wings become narrower and narrower until, in extreme cases, they become reduced to a width of only two or three cells. No cases have been noted, however, in which the wings had entirely disappeared, and the growth of the gemmiparous branch often comes to an end while the wings are still four cells broad or more. With the reduction in the width of the wings, the cortical cells sometimes continue to show the usual arrangement in four rows, but the rows sometimes become increased to as many as six, both dorsally and ventrally, the rows under these circumstances being irregular and the cells themselves considerably reduced in size. Sometimes, especially when the formation of gemmae begins in a juvenile thallus, the gemmiparous branches may lose their costae and become reduced to narrow unistratose thalli only five or six cells broad; or, if the vegetative thallus itself lacks a costa, the gemmiparous branches may retain the same simple structure throughout their entire length. In other cases the gemmiparous branches may develop new costae (Fig. 9, A), sometimes retaining them as long as growth continues and sometimes losing them before growth is brought to an end. It will be seen from this account that the gemmiparous branches exhibit a wide range of variability.

The first gemmae to be produced are marginal and arise in acropetal succession, every marginal cell in extreme cases giving rise to a gemma. If the gemmiparous branch shows the more specialized features described above, some of the later gemmae may be given off dorsally from the submarginal alar cells and perhaps also from the cortical cells of the costa both dorsally and ventrally, the acropetal succession in such cases not persisting. In instances of extreme production the crowded gemmae extending in all directions almost conceal the tip of the slender gemmiparous branch, although even then the apical cell of the branch can usually be clearly distinguished.

At the time of separation the gemmae vary considerably in size, but an average example measures about 0.12-0.1 mm. and is five cells across. It is oblong in outline and strongly convex, the whole margin (including the single apical cell and the indistinct stalk) being revolute (Fig. 9, C). On each side three or four short rudiments of marginal hairs can be distinguished; these normally arise between every two marginal cells and may be borne singly or (more rarely) in pairs. Otherwise the gemmae show no cell-differentiation and are unistratose throughout.

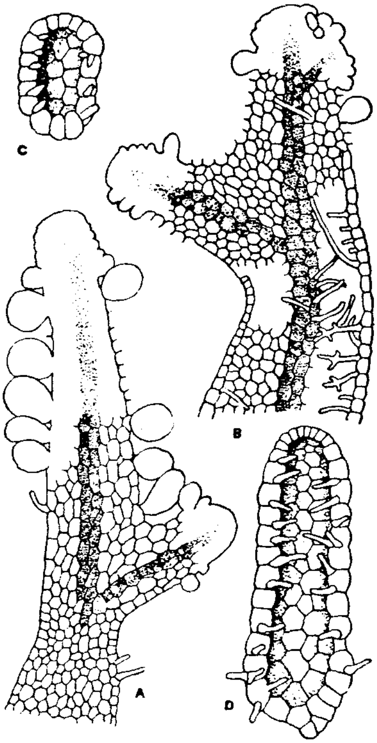

Fig. 9. Metzgeria violacea (Ach.) Dumort.

A, B. Gemmiparous branches, ventral view, × 50. C. Gemma about ready to separate, × 100. D. Germinating gemma, × 100. A was drawn from an unnumbered specimen labelled *M. angusta* and collected at Punta Arenas by Dusén; B–D, from a specimen collected at the same locality by Thaxter, No. 159.

In the early stages of germination the young plant simply repeats the features of the gemma and develops into a narrow strap-shaped thallus, strongly convex and bearing single or twinned hairs between every two marginal cells (FIG. 9, D). As the growth goes on the hairs of the gemma elongate and tend to equal those of the young plant. Even this early stage may be long-continued, and repeated dichotomies may take place before signs of further differentiation become apparent. The plant may, in fact, become gemmiparous almost immediately, and, in extreme cases, a gemma may form secondary gemmae before becoming detached. If differentiation proceeds normally the young plant gradually grows wider, develops a costa, and eventually shows the characteristic vegetative features of the species.

If the description just given is compared with the writer's earlier description of the gemmae and gemmiparous branches in *M. fruticulosa*, it will be seen that they correspond in most essential respects, and even in their vegetative features the two species are strikingly alike. In *M. fruticulosa*, however, the costa rather frequently shows three or four rows of ventral cortical cells, the gemmiparous branches sometimes lose their wings completely (becoming radial in character), and the gemmae are either plane or only slightly convex. In *M. violacea*, on the other hand, the costa of a vegetative thallus rarely shows more than two rows of ventral cortical cells (except just behind a dichotomy), the gemmiparous branches apparently never lose their wings completely (thus remaining dorsiventral), and the gemmae are distinctly convex. On the basis of these slight differences and the wide geographical separation of *M. fruticulosa* and *M. violacea*, it seems justifiable to admit the validity of both, at least provisionally.

Among the Chilean species *M. violacea* finds its closest allies in *M. decipiens*, which never produces gemmae, and *M. epiphylla*, which produces marginal and submarginal gemmae on specialized branches. The latter species is especially close, and the characters derived from the costa, the alar cells and the hairs are almost identical, except that the cell-measurements are a trifle higher. The following differential characters, however, suffice to distinguish the species under most circumstances. In *M. violacea* the plants usually grow on wood and develop a bluish coloration after being dried; the thallus is frequently convex; the gemmiparous branches reach a high stage of specialization, giving off gemmae from the cortical cells of the costa as well as from the alar cells; and the gemmae themselves are distinctly convex. In *M. epiphylla* the plants usually grow on leaves and do not develop a bluish coloration after being dried; the thallus is rarely distinctly convex; the gemmiparous branches, although specialized, develop gemmae only from alar cells; and the gemmae themselves are plane or only slightly convex.

Stephani's *M. angusta*, to which he referred some of Dusén's specimens of *M. violacea*, was based on material from Brazil, Venezuela, Trinidad, Mexico, Guatemala, Louisiana and Santo Domingo, as well as from Chile and Patagonia. He speaks of Dusén's specimens as exceedingly reduced, so that in all probability they are distinct from the other specimens cited. According to the description of *M. angusta* the wings are everywhere eight cells wide, the alar cells measure 54 × 37 μ, and the costa is setulose throughout on the ventral surface. It will be seen that this description does not apply very well to the specimens of *M. violacea*. The description of *M. antarctica* applies much better, except that the wings are usually much narrower than 0.7 mm., the measurement there given. It is unfortunate that Stephani made no mention in his description of the gemmae and gemmiparous branches, which certainly yield the most distinctive characters of the species.

10. METZGERIA MAGELLANICA Schiffn. & Gottsche.

Metzgeria magellanica Schiffn. & Gottsche in Schiffner, Forschungsreise "Gazelle" 4^4: 43. *pl. 8, f. 6.* 1890.

SPECIMEN EXAMINED: Tuesday Bay, Straits of Magellan, 1876, *Naumann* (S., TYPE); known with certainty only from the type locality.

The type material of this interesting species, kindly sent for examination by Professor Schiffner, shows that it is amply distinct from *M. nitida*, under which Stephani included it as a synonym (see p. 272). The plants are a dull whitish green and grew in loose mats in admixture with other bryophytes, including a trace of *M. decipiens*. The thallus is normally so strongly convex that it approaches a terete condition, the revolute wings almost meeting below, as shown in the published figure. Of course, as would be expected, the convexity is sometimes less pronounced than this, and the thallus may even approximate a plane condition. The width when explanate is about 1 mm.; in the natural state it is usually 0.6–0.8 mm., and the length rarely exceeds 2 cm. Measured in cells the wings are mostly ten to eighteen cells across. Ventral branching is not rare, but the normal branching is dichotomous, the successive forks being usually 1–3 mm. apart.

Hairs vary somewhat in abundance and are apparently restricted to the margin, the ventral surface being entirely naked throughout. In many cases the margin is likewise hairless for long stretches, but at the other extreme the hairs may occur between every two marginal cells. They usually extend in the same direction as the wings, rarely being numerous enough to form a weft in the space between the margins. According to the original description the hairs often occur in pairs or even in three's. The writer, however, has been unable to verify this statement. So far as his observations go the hairs are invariably borne singly, arising in the usual way from small cells cut off from the marginal cells. In rare instances the marginal cells themselves may project directly as hairs, especially if a rhizoidal function is assumed, and under these circumstances two hairs may be situated side by side, but this is very different from the usual paired condition. Sometimes the hairs are short and spine-like with strongly thickened walls, yet this type of hair is exceptional, most of them being of the usual slender type and measuring 0.09–0.15 mm. in length by about 10 μ in width. They are truly marginal in position.

The costa, as described by Schiffner, is uniform in structure, being bounded both dorsally and ventrally by two rows of cortical cells. The alar cells are unusually small, averaging about 28 × 24 μ, although somewhat larger cells are often interspersed among the others. The walls are distinctly thickened and show indistinct trigones and occasional intermediate thickenings.

The male plants bear sexual branches in some abundance. The latter are oval to globular in form, measuring usually 0.3–0.35 mm. in length by 0.25–0.35 mm. in width and their cells are but slightly smaller than those of the vegetative thallus. Although the costa is strongly incurved the apex of the branch does not usually approach the base very closely. No appendicular organs are present except the slime-papillae.

The female branches are broadly obovate and deeply indented at the apex. They are mostly 0.3–0.35 mm. long by 0.45–0.6 mm. wide, and their concave halves approach each other so closely before fertilization that their margins are almost in contact. Ventral hairs are usually completely absent, but in two instances a single such hair was seen growing out from the thickened median portion. Marginal hairs, on the contrary, are fairly abundant, and represent outgrowths of small cells, just as in a normal vegetative thallus. They are mostly short and spine-like, with strongly thickened walls. The calyptras at maturity are mostly 1.5–2.5 mm. long and 0.7–0.85 mm. in diameter.

Their relatively short hairs are densely crowded in the upper part but more scattered below the middle.

Although *M. magellanica* is undoubtedly a close relative of *M. hamata,* as Schiffner states, it bears a strong superficial resemblance to small forms of the variable *M. decrescens,* owing to its subterete thallus with marginal hairs borne singly. Differences in the costa will at once serve to separate the two species. In *M. magellanica* the cortical cells are definitely in two rows both dorsally and ventrally, and the boundary between the costa and the unistratose wings is abrupt; in *M. decrescens,* although the cortical cells may be in only two rows both dorsally and ventrally, this condition is exceptional, the number of rows being usually more than two, and the boundary between the costa and the unistratose wings is often gradual, the two being separated by a narrow band two or three cells thick. *M. decrescens* is further distinguished by its larger alar cells with thinner walls, and by the sharp contrast in size between the cells of the vegetative thalli and those of the male branches.

11. METZGERIA HAMATA Lindb.

Metzgeria linearis Lindb. Acta Soc. Sci. Fenn. **10**: 494. 1875. Not *M. linearis* (Sw.) Aust.
Metzgeria hamata Lindb. Acta Soc. F. et Fl. Fenn. **12**: 25. *f. 25.* 1877.
Metzgeria leptoneura Spruce, Trans. Bot. Soc. [Edinburgh] **15**: 555. 1885.
Metzgeria nitida Mitt. Jour. Linn. Soc. Bot. **22**: 243. 1887.
Metzgeria australis Steph. Hedwigia **28**: 267. 1889.

SPECIMENS EXAMINED: Corral, 1905, *Thaxter 34* (H., Y.); Huafo Island, 1908, *Skottsberg 253* (U., as *M. albinea,* and listed under this name by Stephani, **24**, p. 10); Newton Island, 1896, *Dusén 113* in part (B.); Punta Arenas and Tuesday Bay, Straits of Magellan, 1876, *Naumann* (S., listed as *M. linearis* by Schiffner, **16**, p. 42); Staten Island, 1882, *Spegazzini 65* in part (Massal., Y., **11**, p. 257). The material from Corral is mostly in a juvenile condition, many of the thalli being narrow and etiolated and lacking costae completely. In a few cases, however, the distinctive features of *M. hamata* are clearly apparent.

The following additional Chilean stations may be cited from the literature: near Puerto Varas, *Dusén* (**20**, p. 20); Wollaston Island, *Hariot* (**2**, p. 246); Hermite Island, *Hooker* (**10**, p. 27).

The geographical distribution of *M. hamata* is very extensive in both hemispheres. In Europe it seems to be restricted to Ireland and western Great Britain with an extension northward to the Faroe

Islands. In Asia it occurs abundantly in the Himalayas and is found also in Japan, Java and Sumatra. In America its northernmost station, so far as known, is in Alaska. It reappears in the Allegheny Mountains, although rarely collected there, and seems to attain its most vigorous development in Jamaica and other West Indian islands. Although not yet known from Mexico it occurs in Guatemala and Costa Rica and has been reported in South America from British Guiana and Brazil and along the chain of the Andes from Colombia to Bolivia. In Chile, as noted above, its range extends far into antarctic regions. It has been cited also from New Guinea and New Zealand.

The following characters of *M. hamata*, emphasized by Lindberg in his descriptions, are perhaps the most important: the dioicous inflorescence; the convex to subterete thallus with revolute wings; the crowded marginal hairs, usually arising in pairs; the wings otherwise destitute of hairs; the costa bounded both dorsally and ventrally by two rows of cortical cells and bearing hairs on its ventral surface. These characters, in spite of the wide distribution of the species, are found with remarkable constancy. There is, to be sure, a considerable range of variation in the convexity of the thallus and in the abundance of the hairs, especially those of the costa, but this would naturally be expected. When the hairs are considered in more detail they are found to vary in certain of their features. In the more typical plants the hairs of a marginal pair diverge widely and are more or less strongly curved, the concavities of the curves being directed downward or away from the edge of the wing. In subterete thalli a fairly dense weft of hairs may thus be formed, partially concealing the costa. This typical condition, however, is by no means constant; in many thalli the hairs are either straight or irregularly curved, or contorted and extend in various directions (FIG. 10, A–E). When the hairs are sparingly developed, some of them may arise singly; when they are unusually crowded, some of them may be in three's or four's (FIG. 10, E).

The cells of the wings have delicate walls, sometimes with minute and inconspicuous trigones, and vary a good deal in size. It is not unusual, in fact, for the cells in one part of a thallus to be considerably larger than those in other parts, just as in *M. decipiens* and other species. According to the writer's measurements $50 \times 37\,\mu$ would express the average size of the cells, although Stephani's figures, $65 \times 50\,\mu$, and Lindberg's, 50–$65\,\mu$, are both a little higher.

According to Lindberg the male branches, which seem to be rarely present, are smooth on the wings and bear a very few short hairs on

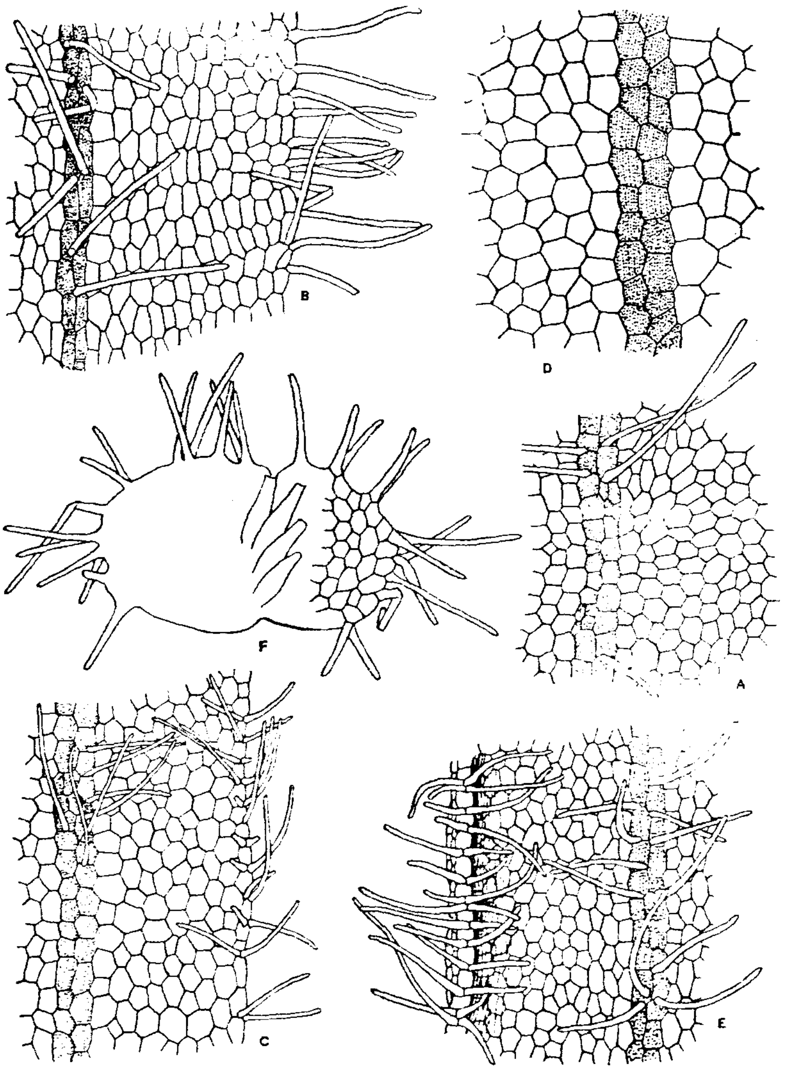

FIG. 10. METZGERIA HAMATA Lindb.

A–E. Portions of thalli, all except D ventral view, × 50. F. Female branch, × 50. A was drawn from a specimen collected at Punta Arenas by Naumann; B, from a specimen collected at Corral by Thaxter, No. 34; C, from the type material of *M. nitida;* D, from a New Zealand specimen of *M. nitida* collected by Colenso, No. 1100; E and F, from a specimen collected at Mabess River, Jamaica, by the writer, No. 306.

the costa. Stephani describes the branches as hairless throughout, and the writer has been equally unable to demonstrate hairs. It would be unwise, however, to draw definite conclusions from the relatively few male branches examined.

Female branches are much more common than male branches, and Lindberg's description brings out their most important features. He describes them as hairy along the margin and ventrally in the thickened median portion but as hairless elsewhere and adds that the marginal hairs occur singly and that they are very long, crowded, decurved and divaricate. The writer can confirm most of these statements but has demonstrated, in several cases, the occurrence of the hairs in pairs (FIG. 10, F). The calyptra, which is covered over with long hairs, presents few distinctive features; capsules have not yet been studied in detail; and gemmae are apparently never produced.

Stephani's record for the closely related species, *M. albinea,* was based on a specimen from Huafo Island, collected by Skottsberg. Since this specimen differs from *M. albinea* in being dioicous instead of autoicous, and since it shows the other distinctive features of *M. hamata*, it is included in the list of specimens given above. His record for *M. nitida* was based on a specimen collected by Naumann in the Straits of Magellan. In all probability it was an original specimen of Schiffner and Gottsche's *M. magellanica,* since this species is definitely cited as a synonym of *M. nitida,* but nothing to this effect is explicitly stated.

In the writer's opinion *M. nitida,* although recognized as valid by Stephani and others, is a synonym of *M. hamata.* Mitten's original description is brief and unsatisfactory, stating merely that the thallus is dichotomous; that the costa is bounded both dorsally and ventrally by two rows of cortical cells; that the margin bears a few cilia, arising singly or in pairs; and that the cells are hyaline, smooth, and four times as large as those of *M. furcata.* In a supplementary note the further information is given that the species is almost exactly like *M. furcata* in appearance, except that the larger and more translucent cells give it a shiny aspect.

The original description cites only two specimens: "Australia, Apollo Bay, *Sir F. von Mueller,*" and "New Zealand, *Rev. W. Colenso,* on a specimen of *Homalia pulchella,* a 279." In the Mitten Herbarium the *M. nitida* cover contains a series of specimens from Australia, Tasmania, and New Zealand and also a few where no locality is indicated. One of the Australian specimens is labeled, "*Jungermannia 87,* Apollo Bay," and is presumably the first of the specimens originally listed. It may therefore be regarded as the type of the species. This specimen is unfortunately fragmentary and is nearly destitute of sexual branches, the few present being poorly developed. The thallus is almost plane, and the costa shows the four rows of cortical cells — two dorsal and two ventral — as called for in the description (Fig. 10, C).

The alar cells are in most regions rather large, measuring mostly 65–70 μ in length by 50–55 μ in width, but cells as short as 55 μ (or even slightly shorter) are not infrequent and may occupy considerable areas in thalli where most of the cells show the higher measurements. It will be seen that these figures are appreciably higher than those of the writer for typical *M. hamata*. Trigones are scarcely discernible. When the marginal hairs are abundant they occur in pairs, slightly displaced to the ventral surface, and there may be a pair between every two marginal cells; when the marginal hairs are scattered they usually occur singly, and long stretches of the thallus may be wholly free from hairs. The costal hairs tend to be less numerous than the marginal hairs but are sometimes crowded. At their best development the hairs are long and flexuous, attaining a length of 0.3–0.4 mm.

Several of the New Zealand specimens in the Mitten Herbarium were collected by Colenso but No. 279 is not present. There is, however, in the herbarium of the New York Botanical Garden, a specimen collected by Colenso and received from Kew, that closely agrees with the Apollo Bay specimen. This bears the number 1100. It has a slightly convex thallus, the margins being subrevolute, and the marginal hairs in exceptional instances arise in three's or even in four's. The alar cells of this specimen average about 70 × 55 μ (Fig. 10, D).

If these two specimens were the only ones to be considered it might appear as if *M. nitida* could be separated from *M. hamata* by its larger leaf-cells. Other specimens, however, from Australia and New Zealand, show that this distinction is inconstant. Although agreeing with the Apollo Bay specimen and No. 1100 in other respects these specimens have distinctly smaller cells. In one Australian specimen, collected by Hartmann, for example, they average about 49 × 40 μ; in another, collected by Lucas, about 54 × 40 μ; in a New Zealand specimen without the collector's name, about 42 × 36 μ, etc. It is clear, therefore, that the distinction in the size of the cells breaks down, and since no other more important and constant distinction has been brought forward, the two species are evidently identical.

Stephani's *M. australis*, which he at one time regarded as a synonym of *M. nitida* and which is here included among the synonyms of *M. hamata,* was based on a series of five specimens, one from Lord Howe's Island, collected by De Camera, and the others from various parts of Australia. In the Mitten Herbarium none of the Australian specimens listed by Stephani (in his original description) are represented, but a specimen collected by De Camera on Lord Howe's Island is included in the *M. australis* cover. This specimen is fragmentary and hardly determinable, but its marginal hairs are borne singly. It disagrees therefore with *M. hamata* but it disagrees equally well with Stephani's description of *M. australis,* according to which the hairs are normally borne in pairs. The other characters brought out are the following: a dioicous inflorescence; a convex thallus with abruptly recurved margins; ventral hairs restricted to the costa; cortical cells of costa in two rows both dorsally and ventrally; alar cells averaging about 45 μ. These are all characters of *M. hamata,* as Lindberg's description clearly shows, and no differential characters of importance are indicated. Whether the specimen from Lord Howe's Island in the Mitten Herbarium is identical with the one listed by Stephani could only be determined by a comparison. If they should be identical it would simply prove that his original *M. australis* was an aggregate.

In its convex thallus and in the structure of its costa *M. hamata* bears a certain resemblance to the smaller *M. magellanica.* In *M. hamata,* however, the marginal hairs are normally borne in pairs, the alar cells average about 50 μ in length and the costa is hairy below; while in *M. magellanica* the marginal hairs are borne singly, the alar cells average about 28 μ in length and the costa is naked.

It will be seen from the preceding pages that the following valid species of *Metzgeria,* although reported from Chile by earlier writers, are not here recognized as members of the Chilean flora: *M. conjugata, M. furcata, M. albinea, M. Liebmanniana,* and *M. pubescens.* In the case of the last three species the Chilean records have all been carefully investigated, usually by means of the actual specimens upon which they were based, and found to rest on incorrect determinations. The same thing is true of most of the records for the first two species, but one record for *M. conjugata* and three for *M. furcata* remain to be further considered.

The record for *M. conjugata* is the following: Chile, *Hahn* (see **19**, p. 951). The specimen in the Boissier Herbarium, upon which this record was based, bears the name *M. conjugata β violacea* and was collected at Valdivia. It agrees in all essential respects with a specimen in the Schiffner Herbarium, likewise collected at Valdivia by Hahn and coming originally from the Jack Herbarium. Both specimens are sterile and, although hardly in a condition to be determined, are surely not *M. conjugata*. Their distinct bluish coloration might seem to indicate *M. violacea*, but the complete absence of gemmae does not support this idea. The wings of the thallus, moreover, are broader than is usual in *M. violacea* and their margins are scarcely if at all revolute. A specimen in the herbarium of the New York Botanical Garden, which is said to have been collected in Peru by Lechler, adds to the uncertainty. This specimen is a mixture of *M. violacea* and a species strongly resembling the Valdivia specimens. The material of this species, however, shows scattered dorsal gemmae on broad thallus-branches and is thus clearly distinct from *M. violacea*. Unfortunately the plants are not only too fragmentary for description but their identity with Hahn's specimens, which bear no gemmae, can not be regarded as definitely established.

The records for *M. furcata* are the following: Cape Horn, *Hooker* (see **8**, p. 480, as *Jungermannia furcata*); Basket Island, *Spegazzini* (see **11**, p. 257); and Chiloé, *Skottsberg* (see **19**, p. 10). Hooker's material of "*J. furcata*" in the Mitten Herbarium, representing a part of the original collection, is made up very largely of *M. decipiens* (see page 296), although a slight admixture of *M. decrescens* is present (see page 279). Since Lindberg found *M. hamata* in the same collection (see page 315) it is possible that still other species may have been included. This possibility, however, is rather remote, and it seems justifiable to conclude that Hooker and Taylor's record was wholly based on incorrect determinations. Regarding the Basket Island and Chiloé records the writer can make no statements, since the specimens involved have not been available for examination.

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SOME NEW FOSSIL PARASITIC HYMENOPTERA FROM BALTIC AMBER.

BY CHARLES T. BRUES.

SOME NEW FOSSIL PARASITIC HYMENOPTERA FROM BALTIC AMBER.[1]

By Charles T. Brues.

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The abundant occurrence of fossil Parasitic Hymenoptera in Baltic amber has long been known, but in spite of the great activity of entomologists during recent years in dealing with this interesting fauna scarcely any amber species of this group of insects have so far been described. Brischke [2] published a brief list of the genera which he had observed in amber in 1886, but described no species. Some years ago [3] the present writer listed the amber species as an appendix to an account of the Parasitic Hymenoptera of the miocene shales of Florissant, Colorado, and he later extended this list [4] and drew a comparison between the Florissant and Baltic amber faunæ.

The present paper deals with a small, but very interesting series belonging to the Zoölogical Museum of the University of Königsberg. They form a part of the extensive amber collections formerly belonging to the Physikalisch-ökonomische Gesellschaft, now in the custody of the University Museum. The great diversity of the amber fauna is very evident from the fact that among 14 species here described, seven families and 11 genera are represented, two of the genera being new. They are distributed as follows:

Evaniidæ	1 genus	1 species
Braconidæ	3 genera	3 species
Ichneumonidæ	1 genus	1 species
Bethylidæ	5 genera (2 new)	7 species
Dryinidæ	1 genus	1 species
Serphidæ	1 genus	1 species
Callimomidæ	1 genus	1 species

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[1] Contribution from the Entomological Laboratory of the Bussey Institution, Harvard University, No. 217.

[2] Die Hymenopteren des Bernsteins. Schrift. naturf. Gesellsch. Danzig, n. F. vol. 7, No. 6, pp. 278–279 (1886).

[3] Bull. Mus. Comp. Zool. Harvard, vol. 54, No. 1, pp. 108–122 (1910).

[4] Some Notes on the Geological History of the Parasitic Hymenoptera, Journ. New York Entom. Soc., vol. 18, No. 1, pp. 1–22 (1910).

In addition there are a large number of specimens which are not perfectly preserved and as these could be better studied in connection with more extensive material, they are not considered in the present paper. The types of the new species are in the University Museum, Königsberg.

FAMILY EVANIIDÆ.

Oleisoprister prævolans sp. nov.

♀. Length 12 mm. Ovipositor at least as long as the body, its tip not being preserved in the type. Head seen from above nearly quadrate, but distinctly broader than thick antero-posteriorly. Ocelli large, close together; the anterior one separated from the lateral ones by one-half its own diameter; lateral ones equidistant from each other and from the eye-margin. Eyes oval, one-third longer than broad. Malar space equal to one-third the eye-height. Antennæ inserted midway between the vertex and the tips of the closed mandibles; 14-jointed; scape stout, twice as long as thick, one-half longer and twice as thick as the pedicel which is nearly twice as long as thick; first flagellar joint a little more slender, and twice as long as the pedicel; two following each one-third longer; others growing shorter to the apex; the last being half as long as the first flagellar. Maxillary palpi 5-jointed; last three joints slender and equal, the second shorter and much thickened; labials 4-jointed, joints except the longer basal one, about as broad as long. Mandibles triangularly acute, edentate. Pronotum without lateral tooth or tubercle. Mesonotum slightly produced at the humeral angles into a very blunt tooth; coarsely transversely grooved, there being about nine ridges; parapsidal grooves very deep, converging at the deep basal scutellar fovea. Propodeum coarsely irregularly reticulate, strongly constricted on a level with the surface of the mesonotum and scutellum. Abdomen as long as the thorax; viewed from the side, the first segment is funnel-shaped, as long as high at apex; following segments growing shorter. Pleuræ more or less reticulated, the propleuræ and pronotum smoother, punctulate, except the anterior margin of the latter. Legs slender; longer spur of middle and hind tibiæ each as long as the breadth of its tibia at the tip. Tarsal claws with three teeth, including the apical one, and with a slight trace of another near the base of the claw. Wings hyaline (as preserved). Stigma oval-lanceolate, nearly as wide as the length of the first section of the radius; median and submedian cells of equal length; cubitus arising only a very little below the middle of the basal vein, the first discoidal cell therefore very large, its upper side twice as long as its basal one; first recurrent nervure received nearly one-half its length before the apex of the first cubital cell; second recurrent nervure received slightly but distinctly beyond the middle of the second cubital cell which is closed by a hyaline vein; second section of radius four times as long as the first, ending just before the wing tip. Posterior wing with one closed cell, the basal.

One finely preserved specimen No. XXB1697.

This species approaches closest to the genus *Oleisoprister* Bradley in general characters although the tarsal claws have only two long teeth before the apex; these however, are longer and much more prominent than in *Semenorius* and there is a trace of another nearer the base of the claw. It might thus be regarded as intermediate between the two. The first discoidal cell is very large.

Family BRACONIDÆ.

Brachistes normalis sp. nov.

♀. Length 2.5 mm. Probably black or very dark colored, with clear wings. Head transverse, twice as broad as thick, the temples narrow, viewed from the side, less than one-half the width of the eye; its surface smooth. Head completely and strongly margined behind on the occiput, temples and cheeks. Eyes oval, rather strongly protuberant, one-half longer than wide and nearly touching the base of the mandibles below. Ocelli large and prominent, the lateral ones as far from each other as from the eye margin. Antennæ fully three-fourths as long as the body; inserted just below the middle of the head, on a slight ledge, 18-jointed; scape twice as long and slightly thicker than the rounded pedicel; first flagellar joint narrower, four times as long as thick and as long as the scape and pedicel together; second to seventh joints growing shorter to one-half longer than broad, following quadrate moniliform, the apical one one-half longer and obtusely narrowed at the tip. Mesonotum slightly rugulose, but quite shining; very convex, with distinct but not very deep, crenulated furrows. Sides of scutellum irregularly rugose; metathorax reticulate, with a well defined basal median, and a petiolar areola, as well as a number of smaller, irregular lateral areolæ. Abdomen slightly longer than the thorax, sessile; first segment short, as long as the propodeum, elevated medially where it bears two convergent carinæ in addition to a lateral carina on each side, the surface between the carinæ finely rugulose; second segment smooth, twice as long as the first; third to seventh subequal, each half as long as the second. Ovipositor curving downward, as long as the abdomen. Propleuræ triangular, finely reticulate in a vertical band across the middle; mesopleuræ twice as high as broad, smooth and polished, with a crenulate line along the posterior margin. Legs slender, tibial spurs short. Wings with ovate stigma, the radius originating just beyond its middle, the latter strongly curved basally and straight apically, ending near the tip of the wing. First discoidal cell sessile; basal vein nearly straight; submedian cell slightly longer than the median; second discoidal cell closed, the discoidal vein broken near its posterior tip; first recurrent nervure received well before the transverse cubitus with which it is parallel. Anal cell with a slight stump of a vein above near the base.

On specimen XXB2159.

This is a very small, but typical representative of the genus. It was no doubt parasitic upon beetles occurring in or about the succiniferous trees.

Blacus multiarticulatus sp. nov.

♀. Length 3 mm. Ovipositor as long as the body. Head broadly transverse, much narrowed behind the eyes, the ocelli large and placed in a small triangle, the lateral ones being twice as far from the eye margin as from the median ocellus. Eyes oval, but not much longer than broad. Surface of head shining and smooth, except for a slight shagreened sculpture on the face. Antennæ about 27-jointed, inserted close together midway between the mouth and the vertex; face strongly convex in the median plane. Clypeus separated by a deep crenate suture; mandibles small, acute, their bases well removed from the eyes. Head completely margined behind. Scape short, oval, thick; first joint of flagellum elongate, four times as long as thick and nearly as long as the width of the eye; second joint nearly one-third shorter; following decreasing in length until those near the apex become quadrate. Occiput medially with a short impressed line which extends forward from the occipital margin toward the ocelli. Mesonotum scarcely two-thirds as wide as the head, much narrowed anteriorly; median lobe narrowed behind, the furrows meeting far before the base of the scutellum; tip of mesonotum before the scutellum with a deep depression that is longitudinally wrinkled and reaches forward nearly to the furrows. Scutellum narrow, convex. Propodeum rugose and, as well as can be seen, irregularly areolated. Pro- and mesopleuræ rugulose-reticulate, except the upper part of the mesopleura which is smooth. Abdomen subsessile, as long as the thorax; first segment of the same length, but much narrower than the second which is nearly twice as broad as long; third segment one-third shorter than the second, with an indistinct suture at the base; fourth as long as the third. Ovipositor as long as the body. Legs rather slender; spurs of four posterior tibiæ short. Wings probably entirely hyaline, stigma oval, distinctly broader than the length of the first section of the radius which issues a little beyond the middle of the stigma and is one-fourth as long as the nearly straight second to which it is almost perpendicular; first discoidal cell barely sessile, with an obsolete petiole above; submedian cell distinctly longer than the median; second discoidal cell open at apex; recurrent nervure received at the apical fourth of the first cubital cell, parallel with and one-fourth shorter than the transverse cubitus.

Male: Differs from the female by its somewhat more slender form, and longer antennæ. These are nearly a fourth longer than the body with the joints more elongate, the shortest ones being nearly twice as long as thick; basal joints of flagellum of about the same proportionate length as in the female.

Described from two specimens, male and female. Both are without number.

The species is here referred to Blacus since it does not agree in number of antennal joints with either Blacus *s. str.* or Ganychorus having considerably more than either of these genera. In the subpetiolate or subsessile form of the first discoidal cell it approaches Pygostolus and would on this account appear to be more generalized than any known recent species of the tribe Blacini.

Taphæus præcox sp. nov.

♂. Length 4 mm. Wings strongly tinged with brown, probably infuscated in life. Head broadly transverse, about three times as wide as thick. Ocelli large, in a small triangle, well removed from the occiput. Head sharply and distinctly margined behind, the occiput much depressed medially behind the ocelli. Antennæ 24-jointed, as long as the body; scape as long as the first flagellar joint; pedicel small, rounded; first flagellar joint the longest, four times as long as thick and nearly twice as long as those near the middle of the flagellum; following joints growing shorter and more slender; the penultimate one being twice as long as thick; each antenna inserted on a small tubercle. Surface of head smooth and polished. Thorax high, the mesopleura very large; above, it is quite distinctly wider than the head. Mesonotum trilobed, its crenulate furrows concurrent considerably before the base of the scutellum; surface of mesonotum shining, almost smooth. Propodeum rugose and partially areolated by a median carina at base that divides into two which enclose an area defining the posterior slope. Abdomen but little longer than the thorax, subpetiolate; spiracles of first segment placed a short distance before its middle; first segment with a pair of discal carinæ convergent behind, and a lateral carina including the spiracle. Second segment as long as the first, but much broader; its surface smooth; following segments each but little more than one-half as long as the second; tip of abdomen bent down. Legs slender, the posterior tibiæ widened toward the apex; hind tibial spurs equal, less than one-half as long as the metatarsus. Stigma lanceolate oval, the radius originating beyond its middle and extending to the tip of the wing, its second section straight; basal vein originating in the costa; submedian cell considerably longer than the median; recurrent nervure received beyond the middle of the first cubital cell; second cubital as long on its cubital side as the distance between the first transverse cubitus and the recurrent nervure, cell much narrowed above, its upper less than half as long as its lower side; second discoidal cell twice as long as high; discoidal nervure broken far below the middle.

One specimen No. B19319.

The species is a typical representative of this genus which is at the present time represented in Europe. Two related genera, Diospilus and Dyscoletes have been found in the American tertiaries at Florissant.

FAMILY ICHNEUMONIDÆ.

Astiphromma brischkei sp. nov.

♂. Length 9 mm. A large slender species with the antennæ distinctly thickened near the apex. Head two and one-half times as broad as thick, with the eyes prominently projecting above its surface. Ocelli large and prominent, the lateral ones closer together than to the eye-margin. Antennæ 30-jointed, almost reaching to the tip of the abdomen, quite slender basally, but very perceptibly thickened from the middle to near the tip; first joint much lengthened, as long as the width of the mesonotum; second to fifth joints each but little more than half as long as the first; following becoming shorter and thicker, till at the apical three-fourths they are quadrate from whence they remain of the same shape but grow smaller to the pointed apex of the antenna. Head very strongly contracted behind the eyes; margined behind, its surface entirely smooth above and behind. Face very much narrowed below to the clypeus which is oval, higher than broad, very convex, and separated from the face by a deep suture. Eyes ellipsoidal, very convex, nearly twice as long as broad; malar space short, shagreened. Maxillary palpi rather short, composed of four nearly equal joints. Mesonotum much elongated, strongly narrowed in front, with a distinct, raised margin around the sides and a deep, broad, foveate depression at the base of the scutellum; its surface shining, faintly punctulate. Scutellum convex, strongly carinate basally on each side of the basal depression. Propodeum gradually declivous from the base, completely and very distinctly areolated. Pleuræ more or less smooth, the mesopleura quite strongly convex medially; separated behind by a crenulate furrow from the short metapleural piece. Coxæ smooth, faintly punctate. Abdomen with a long petiole, claviform; fully twice as long as the head and thorax together. First segment nearly as long as the thorax, long stalk-like and but little widened toward the apex; its spiracles placed a short distance before its middle in a lateral carina which extends the entire length of the segment. Second segment half as long as the first, nearly twice as long as wide at the tip; third, fourth and fifth about equal, each one-third shorter than the second; following scarcely protruding. Wings nearly attaining the tip of the abdomen, apparently hyaline. Stigma narrowly ovate, the radius issuing from its middle, radial cell ending considerably before the wing-tip; first section of the radius one-third as long as the second; areolet rhombic, very large, the two sections of the radius forming straight lines with its two upper sides; median and submedian cells of equal length; cubito-discoidal cell as long as the greatest diagonal of the radial cell, the cubito-discoidal vein sharply bent, with a trace of a stump of a vein at this point. Legs slender, especially the anterior pair.

Two specimens. Type without number; paratype bearing the number K7554.

On account of the carinæ on the first segment of the abdomen, this species is better referred to *Astiphromma* than to *Mesochorus* although the structure of the tarsal claws is not to be seen.

The peculiar conformation of the antennæ which are very distinctly thickened near the apex raises some doubt whether the species may not be really worthy of generic rank, as I do not know of the occurrence of similar antennæ in any recent species of either genus, but as there appear to be no other differential characters, I have not thought such a procedure advisable.

I have dedicated this species to Brischke in recognition of his work on Amber Hymenoptera as it is quite possibly the one referred to by him ('88) as *Mesochorus*.

FAMILY BETHYLIDÆ.

Palæobethylus Gen. nov.

A characteristic bethylid type, showing however many resemblances to living members of the Ampulicidæ, particularly to Rhinopsis. Body much elongated, especially the prothorax and head, the former strongly flattened. Head large, flat; the occipital foramen placed very near the vertex. Eyes elongate, bare; ocelli in a triangle. Mandibles long, acute when seen from below, with rounded, edentate margins. Maxillary palpi with four nearly equal joints; labial palpi two-jointed, the basal joint more than twice as long as the apical one. Antennæ slightly thickened, 13-jointed; scape long. Prothorax very long, much depressed and narrowed anteriorly, with a sharp, median, grooved line; mesonotum shorter, with two pairs of furrows; scutellum without foveæ at the base; propodeum and all the pleuræ long. Abdomen of the usual form, obtusely pointed at the apex. Wings with a complete marginal cell, though the apical half of the radial vein is weakly marked; stigma lanceolate; submedian cell longer than the median; second discoidal cell closed, the first discoidal and also the first cubital cell indicated by streaks as in the genus Pristocera. Hind wing lobed at the base.

Type species, *P. longicollis* sp. nov.

Palæobethylus longicollis sp. nov.

♂. Length 5 mm. Apparently æneous throughout; wings with a distinct brownish cast. Head, viewed from the front, as wide as high, the malar space one-third as long as the eye-height. Antennæ inserted very close to the oral margin; 13-jointed; scape broad and flattened, as long as the four following joints together; pedicel contracted, as long as the broader first joint of the flagellum which is also narrowed at its base; second to sixth flagellar joints subequal, about quadrate and shorter than the first; joints beyond these of about the same length, but more slender. Surface of head punctate closely on the vertex, but becoming entirely smooth below the ocelli, except for a few very sparse delicate punctures. Ocelli in a triangle, the lateral ones as far from each other as from the eye-margin. Eyes over twice as high as broad. Prothorax above densely punctate, the median line complete; posterior edge strongly concave, the humeri produced into finger-like processes; propleuræ roughly shagreened below. Prosternum with a few sparse punctures, very long, the coxæ being far removed from the head. Mesonotum two-thirds as long as the prothorax and one-half broader than long; in front densely punctured, apically with larger, more scattered punctures; inner pair of furrows the deepest, placed far apart, and suddenly curved outwards at the anterior margin; external pair of furrows straight, close to the sides of the mesonotum. Scutellum twice as wide as long, smooth and polished, without foveæ at base. Propodeum medially above with a carina, on each side of which it is smooth and polished; posterior surface rounded off above, not sharply declivous; pleuræ with a carina above, their upper half reticulate, and their lower half in great part smooth and polished. Mesopleura elongated, polished, with large scattered punctures, abdomen rather short; second, third, and fourth segments of equal length, following ones not exserted beyond the fourth. Legs moderately thickened; anterior femora obclavate,

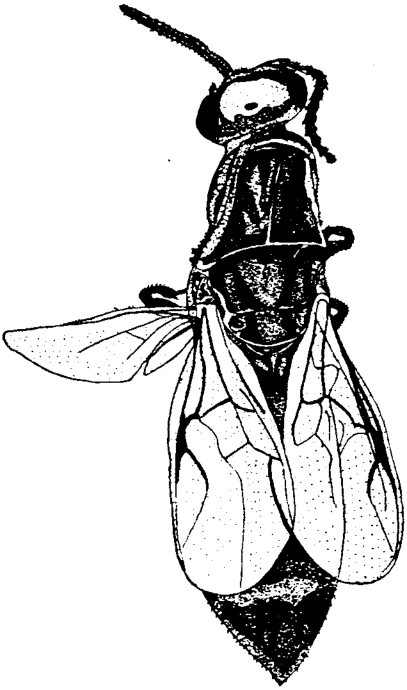

FIGURE 1. *Palæobethylus longicollis* sp. nov. ♂.

twice as long as their tibiæ; middle legs rather slender; hind femora greatly thickened and flattened, over half as broad as long and only two-thirds as long as their tibiæ, which are but little thickened. Tarsal claws apparently simple. Wings with an elongate, narrowly lanceolate stigma the tip of which reaches nearly to the middle of the radial cell; radial cell ovate, pointed at the tip; basal vein oblique, the cubitus arising near its upper third; submedian cell distinctly longer than the median; second discoidal cell closed, large, nearly twice as long as high; first cubital and first discoidal cells defined by hyaline thickenings, the cubital cell not closed externally.

Three specimens, type XXB1629; others not numbered.

This is a most remarkable form which combines certain characters of the Bethylidæ and Ampulicidæ. In its complete wing venation it resembles Pristocera, but the head is much more flattened and the legs much thicker, so thick indeed that one would suspect the specimens of being females. I am quite positive, however, that they are males, as there is no trace of a sting in any of the several finely preserved specimens at hand. The greatly elongated prothorax with a median furrow recalls members of the Ampulicidæ, as does also the form of the head, but the abdomen is not petiolate. The flattened pro- and mesothorax show it to be a highly modified form.

Calyoza longiceps sp. nov.

♀. Length 2.7 mm. Head very long, including the mandibles, fully as long as the entire thorax; these much elongated, seen from the side of the head fully two-thirds as long as the head. Eyes oval, placed near the anterior margin of the head, removed by one and one-half times their length from the posterior margin, their surface bare. Head finely shagreened, the ocelli in a small equilateral triangle near the vertex. Antennæ 13-jointed; short, reaching but little beyond the posterior margin of the head. Scape stout, twice as long as the pedicel, curved; first flagellar joint narrow, only about one-half the length of the pedicel; following of about equal length, widening and becoming quadrate; apical joint longer and obtusely pointed. Mandibles broad at tips, 5-dentate, the four posterior teeth short and equal, the anterior one very long and acute, about three times as long as the others. Pronotum one-half longer than the mesonotum, and about twice as broad as long, exclusive of the neck portion. Mesonotum with two furrows, convergent behind, which separate it into three nearly equal parts. Scutellum with no grooves or foveæ, except the lines separating the axillæ. Propodeum somewhat broader than long, margined laterally and at the tip; also with a complete median carina, and an oblique lateral groove on each side which connects the tip of the median carina with the middle of each lateral half. The included area thus formed is reticulate, while the rest of the propodeum and part of the pleuræ as well are shagreened. Mesopleura smooth. Abdomen long, fully equal in length to the head and thorax together; first segment short, quite sharply truncate at the base and sharply declivous; following segments about equal in length. Legs moderately thickened, the tibiæ all rather slender, except the anterior ones, and smooth, not at all spinulose. Wings rather short, but perfectly formed, reaching to the middle of the abdomen. Two basal cells, the posterior one (submedian cell) distinctly longer than the median on the externo-median vein. Second discoidal cell very distinctly marked except on the posterior side; basal vein attaining the submarginal vein far before the stigma which is small, about twice as long as wide at tip. Radial vein long, equalling the submarginal. The first discoidal cell indistinctly indicated, the transverse cubitus meeting the radius before its middle.

One specimen without number.

This species agrees very well with Calyoza Westwood, although the stigma is quite distinctly narrower and the head so extremely long. The striking form of the head will make it very readily recognizable.

Protopristocera Gen. nov.

♀. Head shaped as in Pristocera, but not coarsely punctate as in that genus; margined posteriorly. Ocelli large in a small triangle. Eyes oval, nearly contiguous with the base of the mandibles, pubescent. Mandibles apparently with several teeth, the apical one acute and much prolonged. Maxillary palpi 5-jointed; labials with two visible joints and possibly a third basal one. Antennæ long, 13-jointed. Pronotum nearly as long as broad, without any furrows or coarse punctures. Mesonotum nearly one-half as long as the pronotum, with two rather approximated parapsidal grooves which converge posteriorly and a second, less distinct pair, laterally. Propodeum margined and with several longitudinal carinæ, sharply declivous posteriorly. Abdomen subsessile, acuminate; second segment the longest, wings fully developed. Marginal cell open at the apex; submedian cell much longer than the median; second discoidal large, oblong; closed by a weak vein at the tip. Legs moderately thickened.

While the type species is very much like Pristocera it is remarkable in having the female winged and like the male, not entirely apterous and so different from the male as in the recent genus. On this account it appears to be a stem form close to those from which the recent Pristocera has developed with its greatly degenerate female. Recently Kieffer (Bull. Soc. Hist. Nat. Metz XII, p. 98 (1905)) has described a genus Propristocera from the Indian region which is known only in the male sex. It is of course possible that this genus may be found to have winged females and approach the present one although this appears unlikely. If such be the case the present one may not be distinct although it differs by its pubescent eyes and quite distinct second discoidal cell.

Protopristocera sucini sp. nov.

♀. Length 4.5–5.25 mm. Apparently uniformly dark-colored, with hyaline wings. Head from tip of mandibles about twice as long as high. Eyes close to the mandibles, but not contiguous; separated from the hind margin of the head by a little more than their own length. Head above finely punctulate and clothed with a fine, close, pubescence; on the underside with very sparse, delicate, setigerous punctures. Antennæ inserted close together on a frontal tubercle. Antennal scape as long as the following three joints, curved, and slightly thickened apically; pedicel and following joints of nearly equal length, thickening to the second flagellar, then becoming more slender to the tip; penultimate and last longer and shorter than the others, respectively; tip of antennæ reaching about to the tip of the propodeum. Pro- and mesonotum finely punctulate and thinly hairy. Parapsidal furrows very delicate, the lateral pair more deeply cut anteriorly. Mesopleura smooth, margined by a foveate line; with a foveate depression medially, above which it is divided by a carina nearly perpendicular to its hind margin. Upper surface of propodeum margined laterally and at the tip, with a median carina which extends to the tip of the posterior slope; with two other less distinct carinæ above on each side of the median one. Each posterior angle with a small areola. Pleuræ and posterior slope irregularly wrinkled. Abdomen conic-ovate. Legs, except anterior femora, only slightly thickened; four posterior tibiæ each with two spurs; tarsal claws slender, apparently entire. Wings hyaline, marginal cell almost closed, the marginal vein as long as the submedian cell; the latter above longer than the median by two-thirds the length of the basal vein. Stigma long, narrow.

Two specimens, B1064 (type) and B1480.

Epyris inhabilis sp. nov.

♂. Length 5 mm. Head, seen from above, fully one-fourth wider than long; surface above finely rugulose without any coarse sparse punctures. Ocelli in an equilateral triangle, the lateral ones equidistant from each other and the eye-margin. Eyes bare, separated by over one-third their length from the hind margin of the head; head narrowing behind the eyes, the posterior angles broadly rounded and the occiput rounded and regularly emarginate. Head behind, and the cheeks, shagreened. Antennæ 13-jointed scarcely longer than the width of the head, inserted far apart, at their bases separated by one-fourth the width of the head. Scape short, stout, strongly curved; pedicel long, two-thirds as long as the scape; first three flagellar joints of about equal length, growing thicker, the first a little shorter; following joints distinctly transverse, of about the same width as the third joint of the flagellum; apical joint obtusely narrowed and somewhat longer. Maxillary palpi 5-jointed, including the shorter basal joint; labial palpi 3-jointed. Thorax slightly over twice as long as broad. Pronotum, exclusive of the anterior neck-like portion, rounded anteriorly and less than half as long as broad behind; posterior margin slightly curved forward medially. Mesonotum half as long as the pronotum, sculptured like it and the head, except behind medially where it is more nearly smooth; parapsidal furrows deep, complete, strongly converging behind; posterior margin carinate, the carina curving forward a short distance on each side. Scutellum with an impressed line at the base, which is curved backward and foveate on the sides; much depressed laterally and with a deep pit on each side at the bottom of the depression; its surface shagreened. Propodeum quadrate, margined laterally and at the tip; with a sharp median carina; elsewhere finely reticulate. Pleuræ, as nearly as they can be seen, finely punctate. Abdomen short, but little longer than the thorax, ovate; the base sharply constricted and with an extremely short but quite distinct petiole; second segment the longest, occupying one-third the length of the abdomen. Legs, except the anterior femora, but little thickened; four posterior tibiæ each with two spurs.

Wings hyaline. Radial vein long, nearly equal to the distance from the root of the wing to the origin of the basal vein, which arises some distance before the small linear stigma; the latter narrow and twice as long as broad; basal vein short, nearly straight; transverse median vein strongly curved, one-third longer than the basal vein.

One specimen, without number.

This species may possibly belong to one of the groups recently segregated from *Epyris*, agreeing well with *Monepyris* Kieffer except that there is no third closed basal cell in the wing. As it does not agree perfectly with any of these I have preferred to let it remain in the old genus.

Sierola Cameron.

The three amber species all have a small closed discoidal cell, and may possibly represent a distinct genus.

Key to the Amber Species.

1. Prothorax more than one and one-half times as long as the mesonotum; antennae stout; joints 4–10 of the flagellum quadrate; head above without bristly hairs .... **S. simplex.**
   Prothorax and mesonotum of more nearly equal length; antennae more slender; head above with a series of long bristly hairs .... 2.
2. Slender species; eyes oval, nearly twice as long as broad; mesonotum distinctly longer than the prothorax .... **S. setigera.**
   Stout species; eyes nearly circular, but little longer than broad; mesonotum distinctly shorter than the pronotum .... **S. crastina.**

Sierola simplex sp. nov.

♀. Length 3 mm. Body apparently entirely dark colored, probably black or aeneous, the head and thorax thinly clothed with very short hair. Head evenly convex in front, its surface finely punctulate. Clypeus with a very prominent, but short median carina which extends on to the front above the clypeal suture for a distance equal to the length of the pedicel of the antenna. Eyes oval, nearly twice as long as broad, thinly pubescent. Ocelli in a small triangle, close to the occipital margin of the head. Mandibles moderately stout, their apices not visible. Maxillary palpi 5-jointed. Antennæ 13-jointed, of nearly equal thickness throughout, slightly stouter before the middle of the flagellum; scape stout, bent, broadened at the apex; pedicel and first two flagellar joints each of nearly equal length, the second one broader; third stouter, fourth and following quadrate, becoming slightly moniliform; last joint oval, one-half longer than the preceding. Thorax above shagreened, but little convex; the propleuræ below the suture smooth and shining. Pronotum almost twice as long as the mesonotum, its pleural piece with slight longitudinal striate sculpture. Mesonotum about twice as wide as long, its anterior and posterior margins nearly straight; with two fine but very distinct parapsidal furrows, which are interior apically to the small foveæ at the base of the scutellum; the latter as long as the mesonotum, with a delicate impressed line at the base connecting the foveæ. Propodeum margined behind and at the sides in addition to a median carina; its pleuræ microscopically reticulated and its posterior face shagreened. Mesopleural piece very convex; its discal fovea small, distinct, equidistant from the upper and hind margins of the mesopleura, nearly twice as far from its oblique anterior edge. Abdomen rather slender, one-fourth longer than the elongate thorax; second segment the longest, the following gradually decreasing in length. Legs very much thickened; anterior femora nearly one-half as broad as long. Tarsal claws apparently with a single tooth near the base. Wings fully developed, quite strongly infuscated. Radial cell complete, broadly rounded at the apex. Median cell nearly as long as the submedian; closed discoidal cell small, much narrowed basally, being nearly ovate in form; closed cubital cell large, two-thirds as long and nearly as broad as the radial.

One finely preserved specimen, without number.

Sierola setigera sp. nov.

♀. Length 3.2 mm. Head in front but little convex, nearly flat on the front below the vertex. Median carina of clypeus very strongly elevated anteriorly, extending but little on to the front above the clypeal suture. Eyes oval, nearly twice as long as wide, not, or very indistinctly pubescent. Ocelli large, in a moderately small triangle close to the occipital margin. Head seen in side view, quite strongly triangularly prolonged behind the eye for a distance nearly equal to the width of the eye. Mandibles rather narrow, indistinctly bidentate at the tips; palpi 5-jointed. Antennæ slender; scape broad and flattened toward the apex; 13-jointed, all the joints longer than thick. Pedicel slender, a trifle longer than the first flagellar joint; following joints a little shorter; those near the middle of the flagellum about twice as long as thick; near the tip one-third longer than thick;

apical joint twice as long as the preceding. Occipital margin with about six long bristly hairs across the vertex and a couple along the inner eye-margin near the top; also with three close to the outer eye-margin near the top. Pronotum distinctly shorter than the mesonotum; polished and faintly shagreened above, on the sides faintly irregularly striated; prosternum smooth and polished on the sides. Mesonotum very faintly shagreened, with very faint traces of parapsidal furrows external to the scutellar foveæ; the latter rather large, connected by a very distinct transverse groove. Scutellum shorter than the mesonotum. Mesopleura with its discal fovea large and somewhat nearer to the posterior than upper margin. Propodeum margined on the sides, but not behind, where its surface rounds off to the posterior slope. Mesopleura smooth; metapleura shagreened except at the base which is smooth; metathorax both above and behind very faintly sculptured, nearly smooth. Abdomen slender, nearly as long as the head and thorax together; second segment longest; third nearly as long; following rapidly growing shorter. Legs rather slender except the femora which are thickened, particularly the anterior ones which are fully one-third as broad as long. Wings fully developed, hyaline; radial cell broad, broadly rounded at the tip, scarcely longer, but one-third wider than the closed cubital cell; median cell indistinctly shorter than the submedian; discoidal cell diamond-shaped, the two proximal sides equal, each one-half longer than the equal distal sides; stigma elongate, narrow, with nearly parallel sides.

One specimen, without number.

Sierola crastina sp. nov.

♀. Length 3 mm. Stout, seemingly entirely aeneous, with somewhat infuscated wings. Head large and stout, thickened antero-posteriorly. Eyes oval, but much more nearly circular in outline than in the preceding species. Head seen from the front as broad as high, the clypeal carina strongly elevated, three-fourths as long as the height of the eye. Eyes pubescent; front shining, scarcely at all shagreened. Ocelli in a rather large triangle, the lateral ones one-half closer to each other than to the eye margin. Occipital margin above with six long bristly hairs, and with several more on each side behind the upper part of the eyes. Antennæ short and slender, but little exceeding the head in length. Scape stout, two and one-half times as long as thick; following joints to the fifth flagellar of nearly equal length, almost twice as long as thick; following narrower and shorter, tenth and eleventh submoniliform; apical joint narrower and one-half longer. Thorax above strongly convex, the pronotum shagreened, but the mesonotum and scutellum nearly smooth. Seen from above the mesonotum is slightly longer than the pronotum. Parapsidal furrows distinct posteriorly, but indistinctly marked on the anterior half of the mesonotum, meeting the scutellum a considerable distance external to the basal foveæ of the latter. Scutellar foveæ deep, but not large; connected by an impressed line along the base of the scutellum. Metanotum short, rounded on its posterior crest; with a lateral carina both above and on the posterior slope, but otherwise without sculpture, except for a faint transverse or oblique aciculation. Mesopleura large, sparsely punctulate, its discal fovea large and deep, placed near its upper posterior angle. Pro- and metapleuræ shagreened or faintly aciculated. Abdomen short and stout; second segment longest, the following gradually shorter. Legs stout, the anterior femora twice as long as broad; middle and posterior tibiæ not thickened, smooth. Tarsal claws apparently each with a single long tooth beneath near the base. Wings fully developed. Stigma two and one-half times as long as broad; median cell slightly shorter than the submedian; discoidal cell trapezoidal, the proximal lower side fully twice as long as the apical lower side; cubital cell but little shorter and narrower than the radial which is broadly rounded at the tip as in the preceding two species.

One specimen, without number.

FAMILY DRYINIDÆ.

Dryinus filicornis sp. nov.

♀. Length 6 mm. Head large, with the front slightly concave, seen from above somewhat more than twice as wide as thick. Ocelli large, very close together in a small triangle. Occiput with a strongly carinate marginal line which extends to the temples. Eyes bare. Palpi moderately elongate, but not preserved so as to show the number of joints present. Antennæ 10-jointed, very slender and greatly elongated, somewhat exceeding the body in length. Scape short and thick, one-half longer than the second joint which is three times as long as thick; third very long, six times the length of the second and equalling the width of the head; fourth almost half longer than the third; fifth and sixth growing shorter, each little more than half the length of the fourth and fifth respectively; next three decreasing in about the same ratio; terminal joint one-half longer than the penultimate. Prothorax much narrowed anteriorly and strongly elevated posteriorly, but without any transverse depressions. Mesonotum somewhat shorter than the pronotum, with strong, moderately convergent parapsidal furrows; its surface indistinctly punctate. Scutellum as broad as long, with a shallow transverse depression across its base. Metanotum gradually declivous and tapering apically, with reticulations and some carinæ, the exact position of which cannot be made out in the type specimen. Abdomen as long as the head and thorax together, lanceolate, acute apically. Wings fully developed, appearing hyaline; stigma very narrowly lanceolate. Veins piceous, only the radial, basal, transverse, median and basal section of the submedian vein thickly chitinized, the first basal and first cubital cell defined by nearly hyaline veins. Radial cell almost closed by the heavy part of the radius, and completely by its thinner, weakly chitinized tip. First discoidal cell above with a petiole fully one-third as long as the entire basal vein; first cubital cell receiving the recurrent nervure at its apical third; median and submedian cells of equal length. Legs very long; anterior coxæ and trochanters extremely elongated, the coxæ nearly as long as the femora; the trochanters equally long and strongly arcuate. Femora and tibiæ of equal length, the former strongly obclavate and the latter slender, straight. First tarsal joint four times as long as the short second; third much swollen; fourth very long; two-thirds as long as the tibia; fifth very short, quadrate; forceps denticulate on both arms, the inner one reaching the base of the second tarsal joint, at the tip with a large tuft of denticles. Middle and posterior legs elongate and slender, but otherwise of the usual form.

Described from a female XXB1547.

This species will be easily recognized by its extremely elongate and slender antennæ.

FAMILY SERPHIDÆ.

Serphus cellularis sp. nov.

♂. Length 3 mm. Apparently black, with hyaline wings and brownish legs. Head transverse, about twice as broad as thick antero-posteriorly, the lateral ocelli as far from the median one as from the eye-margin. Surface of head smooth and shining above, faintly punctulate on the cheeks. Antennæ 13-jointed, the second to fifth flagellar joints dentate below, the teeth small, but sharply indicated; scape thickened toward the apex, about one-half longer than wide; pedicel small, very short, rounded; first joint of flagellum a little over twice as long as thick, cylindrical; second joint three-fourths as long, following to the apex of about equal length, each nearly twice as long as thick; last joint thicker, one-half longer and acute at the apex. Mesonotum elongate, with indications of parapsidal depressions. Scutellum but little convex medially. Propodeum very gradually rounded off behind, and not strongly decumbent; above irregularly rugose-reticulate, but without indications of a longitudinal arrangement, except possibly near the median line which is very much obscured in the type. Pleuræ not sculptured except on the metathorax; mesopleura with a horizontal impression extending for nearly its entire length and a punctate frenum posteriorly. Abdomen as long as the thorax, its second segment longer than all the following taken together, striate at its extreme base. Legs slightly, but quite distinctly thickened; inner spur of hind tibiæ barely one-half the length of the metatarsus. Wings with an oval stigma which is more than twice as long as broad; radius originating at the apical third of the stigma; radial cell well-developed; fully as long on the costa as the width of the stigma; its costal side twice as long as the first section of the radius; discoidal veins not well indicated except the upper part of the discocubital vein.

One specimen, No. 12,072.

This species resembles very closely some of the living members of this cosmopolitan genus. The radial cell is, however, unusually large, although not unparalleled among recent species.

FAMILY CALLIMOMIDÆ.

Monodontomerus primævus sp. nov.

♀. Length 4 mm. Ovipositor exserted to nearly the length of the abdomen, its tip removed, but the sheaths are a little shorter than the part preserved, being about three-fourths as long as the abdomen. Head smooth below, above on the front slightly shagreened. Antennæ 13-jointed, scape, pedicel, one ring joint, seven funicle joints and a 3-jointed club. Scape inserted well below the middle of the eye and reaching as far as the anterior ocellus; pedicel as long as thick; ring joint small, but very distinct; first flagellar joint quadrate; second twice as broad as long; following to the club of the same form but becoming wider; club with the joints compact, considerably shorter than the three preceding joints together. Malar line very distinct, one-fourth as long as the eye-height. Head behind more coarsely shagreened, distinctly margined. Pro- and mesonotum shining, shagreened, of about equal length. Scutellum shagreened and with a few scattered punctures anterior to the cross furrow which is at the apical third, tip smooth, impunctate. Metanotum with some irregular areolæ defined by very strong carinæ. Abdomen elongate ovate; distinctly longer than the thorax; second, third and fourth segments of nearly equal length. Mesopleura large, polished with a deep fovea near the middle of its posterior margin; femoral furrow extending halfway up the pleura, transversely aciculate. Legs thickened, the posterior femora enlarged to near the tip and sharp below, with a blunt rounded tooth at the apical fourth. Wings apparently infuscated; basal vein very oblique; marginal vein two-thirds as long as the submarginal and nearly twice as long as the postmarginal; stigmal vein nearly half as long as the marginal, with a small knob at the tip and a short slender stump of a vein extending obliquely toward the costal margin of the wing.

One specimen, without number.

This species is either a *Monodontomerus* or a member of the closely allied genus *Diomorus*. It is impossible to make out the shape of the posterior margin of the second abdominal segment, but the sculpture of the head and thorax is not coarse as in *Diomorus*. Except for the typical monodontomerine form of the hind femora, the species looks much like those of the fossil genus *Palæotorymus* known from the Miocene of America.

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TEXT OF THE CHARTER OF THE ACADEMIE ROYALE DE BELGIQUE, TRANSLATED FROM THE ORIGINAL IN THE ARCHIVES OF THE ACADEMIE AT BRUSSELS.

BY A. E. KENNELLY.

TEXT OF THE CHARTER OF THE ACADEMIE ROYALE DE BELGIQUE, TRANSLATED FROM THE ORIGINAL IN THE ARCHIVES OF THE ACADEMIE AT BRUSSELS.

By A. E. Kennelly.

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As one of the delegates of the American Academy of Arts and Sciences to the celebration in Brussels, on May 23–24, 1922, of the 150th anniversary of the founding of the Academie Royale de Belgique, the writer had the opportunity of seeing the charter displayed on that occasion, under a glass case, in the Belgian Academy Building. This royal charter, with its appended great red wax seal 14 cm. in diameter, is a remarkable document in many ways, and possesses a wonderful history. It is written in beautiful French script. It is perhaps the only charter granted to a learned society, which conveys to the members of the same the rights of nobility, by virtue of their election. This prerogative seems to have been enjoyed by the academicians of Brussels until shortly after the French revolution.

By the kindness of the courteous Academy Secretary, the writer was permitted to examine the charter in detail and to make a copy of its main text, a translation of which is here offered. It will be remembered that the empress Maria Theresa, who bestowed this charter, was one of the most remarkable royal personages of the eighteenth century. One of her daughters, the ill-fated Marie Antoinette, a princess seventeen years old at the Court of Vienna when this charter was signed, became Queen of France and perished under the guillotine during the Reign of Terror, on what is now the Place de la Concorde at Paris.

Maria Theresa, by the Grace of God, Dowager Empress of the Romans, Queen of Hungary, of Bohemia, of Dalmatia, of Croatia, of Esclavonia, etc., Archduchess of Austria, Duchess of Burgundy, of Lothier, of Brabant, of Limburg, of Luxemburg, of Guelders, of Milan, of Styria, of Carinthia, of Carniola, of Mantua, of Parma and Plaisance, of Guastalle, of Wirtemburg, of High and Low Silesia, etc., Sovereign Princess of Transylvania, Princess of Suabia, Marquise of the Holy Roman Empire, of Burgovia, of Moravia, of High and Low Lusacia, Countess of Hapsburg, of Flanders, of Artois, of Tyrol, of Hainault, of Namur, of Ferrete, of Kyburg, of Goricia, and of Gradiscia, Landgrave of Alsace, Lady of the Marches of Esclavonia, of Port Naon, of Salins and of Malines, Duchess of Lorraine and of Bar, Grand Duchess of Tuscany. To all to whom these presents shall come, GREETING. Our attention having been called to the present condition of the Literary Society, which, according to our good will, was established, in 1769, in Our city of Bruxelle, it has been represented to Us that, for the better fulfilment of the purpose of this society, it would be advantageous to give it stable and legal form; and since We ever adopt with pleasure all that tends to stimulate, foster and expand the taste for, and the study of, useful sciences and of good literature; so We have founded and instituted, as by these Presents We do here found and institute the said Society as a permanent body, under the title of THE IMPERIAL AND ROYAL ACADEMY OF SCIENCES AND LETTERS, assigning to it, for its meetings, the hall of Our Royal Library, which we have recently adapted and opened for public use. It is Our wish that the members of this Academy rigorously conform to the appended Rules, attached under Our seal to these presents, such as we have decided on, for determining more particularly the objects, Order and form of their assemblies, conferences and exercises. We grant, by virtue of the confidence we place in the wisdom and in the spirit of the members of this Academy, that they shall have the right to print, without applying for the approval of the Book Censors, all the writings and literary productions which they themselves compose; likewise the Memoirs, which after having contested for the annual premiums shall be judged worthy of publication; provided that these writings, productions and Memoirs shall have been examined and approved by the Academy. We grant that the said Academy may select, for the printing of its various works, a printing office to which we will transmit the necessary privileges; We assign to this Academy the right to use in all its proper affairs, a special seal, consisting of the Arms of Burgundy with the legend — Sigillum Caesareae Regiae Scientarum et Litterarum Academiae — of which the permanent Secretary shall have charge. Finally, to give an enduring token of the special esteem in which We hold useful abilities, and those who know how to apply them successfully, We ordain that the title of Academician shall convey to all those who shall become endowed with it, and who have not already been raised to the nobility, or are not already nobles by birth,— the distinctions and prerogatives attaching to the rank of nobility in person, and this by virtue of the act of their admission into this Company. It is Our wish that the Registration of these Presents, so far as occasion demands, shall be made free of charge, where and whenever may be pertinent. We charge his Royal Highness the Duke Charles Alexander of Lorraine and of Bar, our trusty and well beloved Brother-in-Law and Cousin, Administrator of the Grand Mastery in Prussia, Grand Master of the Teutonic Order in Germany and Italy, Our Lieutenant Governor and Captain General of the Low Countries and We likewise command all Our Councillors, Justices, Officers and Subjects hereby involved or affected, as also the Kings and Heralds of Arms in Our Belgian Provinces, that they shall cause, and fully and freely permit Our said Academy of Sciences and Letters, as well as all the members of the same, to secure and enjoy all the Privileges, Prerogatives and Distinctions which it has pleased Us to attach thereunto, and to the full extent of these Presents, in cessation of all let or hindrance to the contrary; FOR SO IT PLEASETH US. In testimony whereof We have signed and We have attached hereto Our Great Seal.

Handed down at Vienna, the 16th. December of the year of Grace, one thousand seven hundred and seventy two, and of Our reign the Thirty Third.

K. R. V$^{\underline{ds}}$

(signed) MARIA THERESE.

By the Dowager Empress and Queen
(signed) A. Go. LEDERER

Letters Patent of the Founding of the
Imperial And Royal Academy of
Sciences and Letters at Bruxelles.

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ON DOUBLE POLYADICS, WITH APPLICATION TO THE LINEAR MATRIX EQUATION.

By Frank L. Hitchcock.

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By Frank L. Hitchcock.

DEPARTMENT OF MATHEMATICS, MASSACHUSETTS INSTITUTE OF TECHNOLOGY.

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1. GIBBS' CONCEPT OF DYADICS AND POLYADICS.

A SYSTEM of Vector Algebra in N dimensions following Gibbs is based on a set of N unit vectors $\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_N$ such that the dot products $\mathbf{e}_i \cdot \mathbf{e}_k$ are unity when subscripts are equal, otherwise zero. Any other vector $\mathbf{a}$ may be expressed in terms of the fundamental units,

$$\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + \cdots + a_N\mathbf{e}_N. \tag{1}$$

Two vectors $\mathbf{a}$ and $\mathbf{b}$ written together with no dot or other sign between them constitute a dyad $\mathbf{ab}$ called the indeterminate product of $\mathbf{a}$ into $\mathbf{b}$. A dyad $\mathbf{e}_i\mathbf{e}_k$ will be called a fundamental dyad or dyad unit. If we have any dyad $\mathbf{ab}$ the first vector $\mathbf{a}$ is called the antecedent, the second the consequent. Both $\mathbf{a}$ and $\mathbf{b}$ will be called factors of the dyad. A dyad is zero when and only when all products of scalar elements a_ib_k are zero, that is, when one of the factors vanishes in all its elements.

A dyadic is a sum of dyads, or, with no increase in generality, a sum of dyads each multiplied by a scalar factor. The *elements* of a dyadic are the sums $\Sigma a_i b_k$ of corresponding products taken from each dyad term, where, for an element A_{ik} of a dyadic $\mathbf{A}$ the subscripts i and k are constant and the summation is over the various dyad terms. Thus if

$$\mathbf{A} = \mathbf{a}_1\mathbf{b}_1 + \mathbf{a}_2\mathbf{b}_2 + \cdots + \mathbf{a}_h\mathbf{b}_h \tag{2}$$

we have

$$A_{ik} = a_{1i}b_{1k} + a_{2i}b_{2k} + \cdots + a_{hi}b_{hk}. \tag{3}$$

It is frequently of use to write the dyad $\mathbf{ab}$ as symbolical of the dyadic and $a_i b_k$ as symbolical of the element A_{ik}, omitting the subscript h and the sign of summation with respect to h.[1]

A dyadic may be written in terms of the dyad units $\mathbf{e}_i\mathbf{e}_k$ *thus*

$$\mathbf{A} = \Sigma A_{ik}\mathbf{e}_i\mathbf{e}_k \tag{4}$$

where both subscripts run from 1 to N.

In a similar manner several vectors $\mathbf{abc}\cdots\mathbf{g}$ written with no sign between them constitute a polyad. When we wish to indicate that the polyad is of order K, that is, it is the indeterminate product of K vectors, we may call it a K-ad. The vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, etc., in order, will be called the first, second, third, etc. factors of the polyad. A polyadic is a sum of polyads, all of the same order. The polyads $\mathbf{e}_p\mathbf{e}_q\mathbf{e}_r\cdots\mathbf{e}_s$ to K factors will be called the fundamental polyads of order K or fundamental K-ads. It is evident that any polyadic may be expanded in terms of the fundamental polyads of its own order. The elements of a polyadic are the scalar coefficients in this expansion. Thus if we have K subscripts $p, q, \cdots, s$, all of which run from 1 to N, we may write a K-adic $\mathbf{A}$ as

$$\mathbf{A} = \Sigma A_{pq\cdots s}\mathbf{e}_p\mathbf{e}_q\cdots\mathbf{e}_s. \tag{5}$$

It is thus apparent that in general a K-adic depends on N^K scalar elements. These may be primarily regarded as forming a K-dimensional block, and most conveniently written by means of adjacent squares when special values have to be assigned. For example if $N = 2$ and $K = 4$ we should have the scheme of elements

$$\begin{array}{ll|ll}
A_{1111}, & A_{1112} & A_{1211}, & A_{1212} \\
A_{1121}, & A_{1122} & A_{1221}, & A_{1222} \\
\hline
A_{2111}, & A_{2112} & A_{2211}, & A_{2212} \\
A_{2121}, & A_{2122} & A_{2221}, & A_{2222}
\end{array}$$

as one possible way of representing a four-dimensional assemblage of elements on a flat surface.

A polyadic is defined to be zero when and only when all its elements are zero. Since our fundamental conception of a polyadic is as a sum of polyads rather than as a mere aggregation of scalars, the equation $\mathbf{A} = 0$ is to be thought of as equivalent to N^K equations of the form

$$a_{1p}b_{1q}\cdots g_{1s} + a_{2p}b_{2q}\cdots g_{2s} + \cdots + a_{hp}b_{hq}\cdots g_{hs} = 0\,. \tag{7}$$

It is frequently useful to indicate such a set of equations symbolically as $\mathbf{ab}\cdots\mathbf{g} = 0$, where, as before, the polyad is written symbolically for the polyadic.

2. Polyadics as Vectors in Space of Higher Dimensions.

Investigations on polyadics may be distinguished according to whether the K-dimensional character is important or not. If we agree on some definite order among the fundamental polyads we may write

$$\mathbf{E}_r = \mathbf{e}_p\mathbf{e}_q\cdots\mathbf{e}_s \tag{8}$$

and let r run from 1 to N^K while each of the K subscripts on the right runs from 1 to N. With $A_r = A_{pq\,\cdots\,s}$ we shall then have

$$\mathbf{A} = A_1\mathbf{E}_1 + A_2\mathbf{E}_2 + \cdots + A_n\mathbf{E}_n \tag{9}$$

where, for convenience, n has been put for N^K.

The polyadic $\mathbf{A}$ thus takes the form of a vector in space of n dimensions. This concept is justified if we introduce the multiple dot product defined by

$$(\mathbf{e}_a\mathbf{e}_b\cdots\mathbf{e}_g):(\mathbf{e}_p\mathbf{e}_q\cdots\mathbf{e}_s) = (\mathbf{e}_a\cdot\mathbf{e}_p)\,(\mathbf{e}_b\cdot\mathbf{e}_q)\cdots(\mathbf{e}_g\cdot\mathbf{e}_s) \tag{10}$$

where the colon in every case indicates K-tuple dot product. In words

Definition. The K-tuple dot product of two K-ads is the product of dot products of corresponding factors.

The K-tuple dot product of two K-adics is the sum of K-tuple dot products term by term. By virtue of the distributive law of multiplication the relation (10) is symbolical of the K-tuple dot multiplication of two K-adics. When $K = 2$ we have Gibbs' double dot multiplication.

By (8) and (10) it is evident that $\mathbf{E}_i : \mathbf{E}_k$ is unity when $i = k$, otherwise zero. Thus the fundamental polyads behave with respect to multiple dot product as do unit vectors $\mathbf{e}_1, \mathbf{e}_2, \cdots \mathbf{e}_n$ in space of n

dimensions with respect to ordinary dot product. Therefore all properties of such vectors which pertain to dot multiplication alone will go over into properties of polyadics and multiple dot product.

To say the same thing in another way, the fundamental polyads $\mathbf{e}_p\mathbf{e}_q\cdots\mathbf{e}_s$ form a normal orthogonal system with respect to multiple dot multiplication, in the sense that the product of two unlike polyads of order K is zero while the product of a polyad by itself is unity. The notation $\mathbf{E}_r$ for the polyad is convenient but not essential. The analogy with $\mathbf{e}_r$, while of much value in a formal sense, is temporary in character, to be employed or laid aside according as we wish to forget or to emphasize the K-tuple nature of the polyads.

3. The Fundamental Identity of Dot Multiplication.

Let there now be a set of N^K or n polyadics of order K as $\mathbf{A}_1, \mathbf{A}_2, \cdots \mathbf{A}_n$, arbitrarily chosen. Any one of these, as $\mathbf{A}_i$, may be expanded in the manner of (9), thus

$$\mathbf{A}_i = a_{i1}\mathbf{E}_1 + a_{i2}\mathbf{E}_2 + \cdots + a_{in}\mathbf{E}_n. \tag{11}$$

Let there also be another polyadic which we may on occasion call $\mathbf{A}_{n+1}$ but which it will be more convenient at present to call $\mathbf{M}$ and to expand in the form

$$\mathbf{M} = M_1\mathbf{E}_1 + M_2\mathbf{E}_2 + \cdots + M_n\mathbf{E}_n. \tag{12}$$

If we now take the multiple dot product $\mathbf{M}:\mathbf{A}_i$, remembering the principle of orthogonality pointed out in the last article, we shall have

$$\mathbf{M}:\mathbf{A}_i = M_1a_{i1} + M_2a_{i2} + \cdots + M_na_{in}. \tag{13}$$

There are n equations of this form, one for each of the polyadics $\mathbf{A}_1\cdots\mathbf{A}_n$. Together with (12) we thus have $n+1$ equations linear in the scalars $M_1\cdots M_n$. It is true that (12) is a K-adic equation and is itself equivalent to n scalar equations. But since all the equations (13) are merely scalar equations, it is not hard to see that the determinant

$$\begin{vmatrix} \mathbf{M} & , \mathbf{E}_1, & \mathbf{E}_2, \cdots, & \mathbf{E}_n \\ \mathbf{M}:\mathbf{A}_1, & a_{11}, & a_{12}, \cdots, & a_{1n} \\ \mathbf{M}:\mathbf{A}_2, & a_{21}, & a_{22}, \cdots, & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \mathbf{M}:\mathbf{A}_n, & a_{n1}, & a_{n2}, \cdots, & a_{nn} \end{vmatrix} \tag{14}$$

must vanish; because when polyadics have to be multiplied by scalars only, all the laws of ordinary algebra are obeyed.

We have now to develop this determinant by the elements of the first column, yielding the identity

$$C_0\mathbf{M} + \mathbf{C}_1\mathbf{M}:\mathbf{A}_1 + \mathbf{C}_2\mathbf{M}:\mathbf{A}_2 + \cdots + \mathbf{C}_n\mathbf{M}:\mathbf{A}_n = 0 \tag{15}$$

where the scalar C_0 and the polyadics $\mathbf{C}_1 \cdots \mathbf{C}_n$ are the cofactors of the elements into which they are multiplied. Evidently C_0 is the determinant of the n^2 elements a_{ik}. To exhibit the character of $\mathbf{C}_1$, $\mathbf{C}_2$, etc., let c_{ik} denote the cofactor of a_{ik} from the n-rowed determinant C_0. On developing (14) it appears that

$$\mathbf{C}_i = -(c_{i1}\mathbf{E}_1 + c_{i2}\mathbf{E}_2 + \cdots + c_{in}\mathbf{E}_n) \tag{16}$$

whence the $\mathbf{C}_i$ are polyadics whose elements are the *negatives* of the $n - 1$ rowed cofactors from C_0.

So far the $n + 1$ polyadics $\mathbf{M}, \mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_n$ have been quite arbitrary. If, however, the $\mathbf{A}_i$ are linearly independent, their determinant C_0 is not zero and we may introduce the important new set of polyadics $\mathbf{A}'_1, \mathbf{A}'_2 \cdots, \mathbf{A}'_n$ defined by

$$C_0\mathbf{A}'_i = -\mathbf{C}_i \tag{17}$$

or by (16)

$$\mathbf{A}'_i = \frac{1}{C_0}(c_{i1}\mathbf{E}_1 + c_{i2}\mathbf{E}_2 + \cdots + c_{in}\mathbf{E}_n). \tag{18}$$

The set $\mathbf{A}'_i$ is said to be reciprocal to the set $\mathbf{A}_i$.

By multiplying the expansions (11) and (18), remembering that c_{ik} is the cofactor of a_{ik} from the determinant C_0, we have the relation $\mathbf{A}'_i:\mathbf{A}_k = 1$ when subscripts are equal, otherwise zero, which we may also express by saying that a set of n polyadics and its reciprocal set with respect to multiple dot product are bi-orthogonal.

The C_i may now be eliminated by the aid of (17); and the fundamental identity (15), (on the hypothesis that C_0 does not vanish), rearranged in the form

$$\mathbf{M} = (\mathbf{A}'_1\mathbf{A}_1 + \mathbf{A}'_2\mathbf{A}_2 + \cdots + \mathbf{A}'_n\mathbf{A}_n):\mathbf{M}. \tag{19}$$

The expression in parentheses will be denoted by I; it is an idemfactor in the sense that its K-tuple dot product into an arbitrary K-adic leaves that K-adic unaltered; and it is the most important special case of a *double polyadic,— a dyadic whose antecedents and consequents are polyadics.*

Let the expression in parentheses be now transformed by putting in the values of all the polyadics $\mathbf{A}$ and $\mathbf{A}'$ from the expansions (11) and (18). Collecting the scalar coefficient of $\mathbf{E}_k\mathbf{E}_k$ we find it to be $\frac{1}{C_0}\Sigma c_{ik}a_{ik}$ *summed on* i, which is unity. Again, the coefficient of $\mathbf{E}_j\mathbf{E}_k$ when j and k are unequal is $\frac{1}{C_0}\Sigma c_{ij}a_{ik}$ *summed on* i, which is zero. Hence $I = \Sigma\mathbf{E}_k\mathbf{E}_k$ *summed on* k, and this latter expression for the idemfactor is symmetrical in the sense that it is unaltered by interchanging antecedents and consequents. It follows that the reciprocal relation between two sets of polyadics is a mutual one, and we have

$$I = \Sigma\mathbf{E}_k\mathbf{E}_k = \Sigma\mathbf{A}'_k\mathbf{A}_k = \Sigma\mathbf{A}_k\mathbf{A}'_k \tag{20}$$

and also

$$\mathbf{M} = I:\mathbf{M} = \mathbf{M}:I \tag{21}$$

or the idemfactor may be used either as prefactor or as postfactor.

To illustrate, if we take $K = 1$ and $N = 3$, our polyadics reduce to vectors in ordinary space; and if our unit vectors be the usual **i**, **j**, and **k** the idemfactor becomes the usual idemfactor of Gibbs, namely **ii** + **jj** + **kk**. If $K = 2$ and $N = 3$ with the same notation, our polyadics are ordinary dyadics; the idemfactor (20) is a double dyadic, having nine terms obtained by doubling the nine fundamental dyads, viz.

$$I = \mathbf{iiii} + \mathbf{jjjj} + \mathbf{kkkk} + \mathbf{ijij} + \mathbf{jiji} + \mathbf{jkjk} + \mathbf{kjkj} + \mathbf{kiki} + \mathbf{ikik} \tag{22}$$

and it may easily be verified that the double dot product of any of the nine fundamental dyads **ii**, **ij**, etc., either *by* or *into* this expression gives the dyad unchanged; whence the same is true for any dyadic.

4. Double Polyadics.

Two polyadics **A** and **M** written together with no sign between them will constitute a double polyad **AM**. A sum of double polyads is a double polyadic, assuming always that the terms of the sum are of like character. In the present investigation it will always be assumed that the factors **A** and **M** of all double polyads are polyadics of the same order K. When desirable, all the antecedents **A** may be expanded in terms of the fundamental K-ads $\mathbf{E}_i$, and the consequents collected so that any double polyadic φ may be written in the form

$$\varphi = \mathbf{E}_1\mathbf{M}_1 + \mathbf{E}_2\mathbf{M}_2 + \cdots + \mathbf{E}_n\mathbf{M}_n. \tag{23}$$

Each of the consequents $\mathbf{M}_i$ may be expanded in terms of the $\mathbf{E}_i$ by aid of the notation

$$\mathbf{M}_i = m_{i1}\mathbf{E}_1 + m_{i2}\mathbf{E}_2 + \cdots + m_{in}\mathbf{E}_n. \tag{24}$$

If the $\mathbf{M}_i$ of (23) be expanded by (24) the complete development of φ will be in terms of the double polyad units $\mathbf{E}_i\mathbf{E}_k$ or

$$\varphi = \Sigma m_{ik}\mathbf{E}_i\mathbf{E}_k, \quad [i, k, = 1, 2, \cdots n]. \tag{25}$$

It is clear that a double polyadic depends on n^2, that is N^{2K} scalar elements. Thus double dyadics in ordinary space, of which the idemfactor (22) is a special case, would yield in general 81 terms if it were necessary to expand completely, which fortunately it is not. A double dyadic is of course a tetradic. If need arose to emphasize the $2K$-adic character of a double polyadic and at the same time to depict the scalar elements, use could be made of a system of adjacent squares like (6). In the applications which follow, however, the scalar elements of a double polyadic are to be thought of as a binary assemblage or square array. For example if $K = 2$ and $N = 2$, with **i** and **j** for unit vectors, the order of the fundamental dyads may be agreed upon by taking $\mathbf{E}_1 = \mathbf{ii}$, $\mathbf{E}_2 = \mathbf{ij}$, $\mathbf{E}_3 = \mathbf{ji}$ and $\mathbf{E}_4 = \mathbf{jj}$. The corresponding scalars m_{pq} will then be arranged as

$$\begin{array}{llll} m_{11}, & m_{12}, & m_{13}, & m_{14} \\ m_{21}, & m_{22}, & m_{23}, & m_{24} \\ m_{31}, & m_{32}, & m_{33}, & m_{34} \\ m_{41}, & m_{42}, & m_{43}, & m_{44} \end{array} \tag{26}$$

where m_{pq} is the coefficient of $\mathbf{E}_p\mathbf{E}_q$ in the double dyadic $\Sigma m_{pq}\mathbf{E}_p\mathbf{E}_q$. Comparing with the four-subscript arrangement (6), we see that the upper left hand square of (6) would correspond to the first row of (26), the upper right hand square of (6) to the second row of (26), and so on.

Returning to our fundamental concept of a double polyadic as a sum of dyads whose antecedents and consequents are polyadics, we next define the K-tuple dot product $\varphi_1 : \varphi_2$ of two double polyadics, (in precise analogy with Gibbs' dot product of two dyadics), to be the sum of all terms of the form $\mathbf{AM} : \mathbf{BN}$ or $(\mathbf{M} : \mathbf{B})\,\mathbf{AN}$ obtained by multiplying out the two double polyadics and taking the K-tuple dot product of each consequent of φ_1 into each antecedent of φ_2. Thus $\varphi_1 : \varphi_2$ is also a double K-adic. We abbreviate $\varphi : \varphi$ as φ^2, and $\varphi : \varphi : \varphi$ as φ^3, etc. This multiplication is easily seen to be associative, as well as distributive, but of course not generally commutative.

5. THE HAMILTON-CAYLEY EQUATION FOR DOUBLE POLYADICS.

Let φ be a double K-adic expanded as in (23) so that the antecedents are the fundamental K-ads $\mathbf{E}_1\cdot\cdot\mathbf{E}_n$. Let Ψ be a second double K-adic $\Sigma\mathbf{BN}$ and let its consequents first be expanded in terms of $\mathbf{E}_1\cdot\cdot\mathbf{E}_n$ and terms collected so that Ψ may take the form

$$\Psi = \mathbf{B}_1\mathbf{E}_1 + \mathbf{B}_2\mathbf{E}_2 + \cdots + \mathbf{B}_n\mathbf{E}_n \qquad (27)$$

so that the consequents are the fundamental K-ads. If we now form the K-tuple dot product $\varphi{:}\Psi$ it is clear that the scalar coefficient of $\mathbf{E}_1\mathbf{E}_k$ will be $\mathbf{M}_i{:}\mathbf{B}_k$ or, in terms of the scalar elements of the polyadics, will be $\Sigma m_{is}b_{sk}$ summed on s. This agrees in form with the law of multiplication of matrices of order n, a result which might have anticipated; for if vectors are analogous to polyadics we should expect matrices to behave like double polyadics, just as matrices of order N behave like ordinary dyadics. The analogy holds of course only so long as we are concerned with formal laws possessed in common by the two algorisms.

By virtue of this analogy, however, it is evident without further proof that any double polyadic φ must satisfy an identity of the same form as the Hamilton-Cayley equation belonging to a square matrix of order n, namely

$$\varphi^n - m_1\varphi^{n-1} + m_2\varphi^{n-2} - \cdots + (-1)^n m_n I = 0 \qquad (28)$$

where the coefficients $m_1\cdot\cdot m_n$ are scalars. The determinant of the matrix or double K-adic $\varphi - gI$, where g is a scalar, is

$$\begin{vmatrix} m_{11}-g, & m_{12}\quad, & m_{13}, & \cdots, & m_{1n} \\ m_{21}\quad, & m_{22}-g, & m_{23}, & \cdots, & m_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ m_{n1}\quad, & m_{n2}\quad, & m_{n3}, & \cdots, & m_{nn}-g \end{vmatrix} \qquad (29)$$

and by writing φ in place of g and equating to zero we have the Hamilton-Cayley equation.[2]

It is by no means necessary, however, to expand in terms of the unit polyads in order to form the Hamilton-Cayley equation. Consider for a moment the case $K = 1$, $N = 3$, when the double polyadic becomes Gibbs' dyadic in ordinary space. The coefficient m_1 in the Hamilton-Cayley equation becomes Gibbs' φ_S which, if the dyadic be $\Sigma\mathbf{ab}$ is $\Sigma\mathbf{a}\cdot\mathbf{b}$, or symbolically $m_1 = \mathbf{a}\cdot\mathbf{b}$. Again, the coefficient m_2

becomes half $(\varphi{}^{\times}_{\times}\varphi)$ or φ_{2S} which, if **ab** and **cd** are any two terms of the dyadic, is the sum of all terms of the form $\mathbf{a}\cdot\mathbf{b}\mathbf{c}\cdot\mathbf{d} - \mathbf{a}\cdot\mathbf{d}\mathbf{c}\cdot\mathbf{b}$, or symbolically

$$2m_2 = \mathbf{a}\cdot\mathbf{b}\mathbf{c}\cdot\mathbf{d} - \mathbf{a}\cdot\mathbf{d}\mathbf{c}\cdot\mathbf{b}. \tag{30}$$

The case of dyadics whose antecedents and consequents are combinatorial products of vectors in N-space, the sum of the classes of antecedent and consequent being equal to the dimensionality of the space, has also been treated;[3] and by applying multiple algebra to a projective point space of three dimensions, various types of dyadics occur, with corresponding Hamilton-Cayley equations.[4]

By analogy it is plain we shall expect m_1 for the double polyadic to be the sum of $\mathbf{A}:\mathbf{M}$, the K-tuple dot product of each antecedent into its own consequent. Using the expansion (23) and noting that $\mathbf{E}_i:\mathbf{M}_i = m_{ii}$ we have $\Sigma\mathbf{A}:\mathbf{M} = \Sigma m_{ii}$ which is the same as m_1 by the usual matrix theory.[5] If we agree to write, by analogy with Gibbs, $\Sigma\mathbf{A}:\mathbf{M} = \varphi_S$, called the scalar of φ, we shall have

$$m_1 = \Sigma\mathbf{A}:\mathbf{M} = \varphi_S. \tag{31}$$

By analogy with Gibbs' $(\varphi{}^{\times}_{\times}\theta)_S$ we may now define the following scalar:

$$((\mathbf{AM},\ \mathbf{BN}))_S = \mathbf{A}:\mathbf{MB}:\mathbf{N} - \mathbf{A}:\mathbf{NB}:\mathbf{M} \tag{32}$$

where **A**, **M**, **B**, and **N** are any four polyadics whatever. If φ and θ are any two double polyadics the scalar $((\varphi, \theta))_S$ is to be found by multiplying the double polyadics term by term and adding the scalars of all the terms. This scalar will be treated later on as a kind of product of the two double polyadics. For the present it will suffice to note that if $\theta = \varphi$ the scalar becomes twice the coefficient of φ^{n-2} in the Hamilton-Cayley equation, or

$$2m_2 = ((\varphi, \varphi))_S. \tag{33}$$

The proof is simple. Let φ be expanded in the form (23), and note that by (24) we have

$$\mathbf{E}_i:\mathbf{M}_k = m_{ki}. \tag{34}$$

Multiplying φ into itself we shall have terms of the form $\mathbf{E}_i\mathbf{M}_i\mathbf{E}_i\mathbf{M}$, whose scalar on developing by (32) vanishes, and terms of the form $\mathbf{E}_i\mathbf{M}_i\mathbf{E}_k\mathbf{M}_k$, whose scalar is $m_{ii}m_{kk} - m_{ik}m_{ki}$ a two-row minor from the determinant of the n^2 elements m_{pq} of φ. Each of the latter terms will occur twice, with all possible selections of pairs of subscripts.

Hence the required scalar is twice the sum of two-row minors whose main diagonals coincide with the main diagonal of the determinant of φ. That is, (by the usual matrix theory), it equals $2m_2$, as we see by developing (29).

More generally, let $\mathbf{A}_1, \mathbf{A}_2, \cdots \mathbf{A}_p$ and $\mathbf{B}_1, \mathbf{B}_2, \cdots \mathbf{B}_p$ be any $2p$ polyadics. **Definition.** We define the scalar

$$((\mathbf{A}_1\mathbf{B}_1, \mathbf{A}_2\mathbf{B}_2, \cdots, \mathbf{A}_p\mathbf{B}_p))_S \tag{35}$$

to be the sum of terms computed as follows: the leading term is the product of K-tuple dot products $\mathbf{A}_1 : \mathbf{B}_1\mathbf{A}_2 : \mathbf{B}_2 \cdots \mathbf{A}_p : \mathbf{B}_p$; the other terms are of like form, and are obtained from the leading term by keeping the antecedents $\mathbf{A}_1 \cdots \mathbf{A}_p$ fixed in position while the consequents $\mathbf{B}_1 \cdots \mathbf{B}_p$ are permuted in all ways; the sign of any term is positive or negative according as the number of simple interchanges needed to form the term is even or odd.

If $\varphi_1, \varphi_2, \cdots, \varphi_p$ are any p double polyadics, the scalar

$$((\varphi_1, \varphi_2, \cdots, \varphi_p))_S \tag{36}$$

is defined to be the result of expanding each φ as in (23), multiplying out, and adding the scalars formed from each term by the above definition. We may now suppose these p double polyadics to be all equal. The scalar (36) becomes $((\varphi, \varphi, \cdots$ to p factors$))_S$ and may be abbreviated $((\varphi^p))_S$. It is now easy to see that

$$((\varphi^p))_S = p!m_p \tag{37}$$

Proof. Expand φ as in (23). The terms of the product $\varphi, \varphi, \cdots,$ to p factors are of the form

$$\mathbf{E}_i\mathbf{M}_i, \mathbf{E}_j\mathbf{M}_j, \mathbf{E}_k\mathbf{M}_k, \cdots, \mathbf{E}_r\mathbf{M}_r \tag{38}$$

where subscripts may have any values from 1 to n; we see by (34) that the scalar of this expression has for its leading term $m_{ii}m_{jj}m_{kk} \cdots m_{rr}$, while, by the definition, the other terms are formed by interchanging the subscripts corresponding to the consequents. Therefore the scalar obtained from each expression (38) is a determinant of order p whose main diagonal lies on the main diagonal of the determinant of φ, provided the p subscripts $i, j, k, \cdots, r$ are all different; if they are not all different the scalar of the expression vanishes because it is a determinant with two or more identical rows as well as columns. Each choice of subscripts of non-vanishing terms will occur $p!$ times. The sum of the scalars of all expressions (38) is then $p!$ times the sum of p- row minors along the main diagonal of the determinant of the m_{ik}. By the usual matrix theory, it is therefore $p! m_p$.

In the above demonstration it is assumed that φ has been expanded in the form (23). It follows from the distributive character of all the steps involved in calculating these scalars that their value is independent of the particular form of φ, precisely as in the case of similar scalars occurring in the usual vector analysis. This is a property of fundamental importance. For, when we have to perform operations into which these quantities enter, it is frequently sufficient to treat a single expression such as (35) in place of the sum of expressions obtained by expansion of (36); and we may develop φ, when necessary, as $\Sigma\mathbf{AM}$ in any desired manner.

To calculate any coefficient m_p in the Hamilton-Cayley equation, we have then merely to express φ as $\Sigma\mathbf{AM}$ in any convenient way, make all possible selections of p different terms, form the scalar (35) for each selection, and add the results, dividing by $p!$.

Many other analogies with ordinary dyadics will naturally suggest themselves. It will be sufficient at this point to note briefly two of the more significant of these.

By solving the Hamilton-Cayley equation as if φ were a scalar we may find roots $g_1, g_2, \cdots, g_n$ and corresponding polyadics $\mathbf{R}_1, \mathbf{R}_2, \cdots, \mathbf{R}_n$ such that

$$(\varphi - g_i I) : \mathbf{R}_i = 0, \quad (i = 1, 2, \cdots, n). \tag{39}$$

The polyadics $\mathbf{R}_1 \cdots \mathbf{R}_n$ may by analogy be called the axes of the double polyadic φ with respect to dot product.

Again, we may say that a double polyadic which can be expressed as the sum of l (and no fewer) properly chosen dyads $\mathbf{AM}$ has $n-l$ degrees of nullity. To get the simplest criterion of the number of degrees of nullity, we may, following Gibbs and Wilson,[6] introduce cross products of polyadics by the law

$$\mathbf{A} X \mathbf{B} = -\mathbf{B} X \mathbf{A} \tag{40}$$

and it is then possible to follow reasoning parallel to that of the paper referred to in Note 3, so far as concerns the number of degrees of nullity, by use of double powers of φ. Evidently a product defined by (40) is not itself a *multiple* cross product in the sense that it would reduce to double cross product when $K = 2$. For in Gibbs' system the product by double cross of two dyadics with one common factor vanishes, thus $\mathbf{ab} {}^{\times}_{\times} \mathbf{ac} = 0$, while by (40) the cross product of two polyadics vanishes only when they are scalar multiples of one another.

It need hardly be emphasized that formal analogies, when they exist, are of much value, but do not in themselves imply more than certain common laws of operation.

6. THE SCALARS (36) ARE SUMS OF CUBIC DETERMINANTS.

The quantities denoted by (36) are more general than the coefficients in the Hamilton-Cayley equation. When p has any value from 2 to n such a scalar is a function of all the elements of each of the double polyadics which enter. Suppose $p = 2$ and let φ_1 and φ_2 be expanded in the form (23). The scalar elements of φ_1 and φ_2 may be designated respectively by m_{1ij} and m_{2ij}. We shall have in the indeterminate product $\varphi_1\varphi_2$ pairs of terms of the form $\mathbf{E}_i\mathbf{M}_{1i}, \mathbf{E}_j\mathbf{M}_{2j} + \mathbf{E}_j\mathbf{M}_{1j}, \mathbf{E}_i\mathbf{M}_{2i}$ which, by forming the scalar by the definition (32), gives $m_{1ii}m_{2jj} - m_{1ij}m_{2ji} + m_{1jj}m_{2ii} - m_{1ji}m_{2ij}$. These four scalar terms are the same as the cubic determinant

$$\left|\begin{array}{cc|cc} m_{1ii}, & m_{1ij} & m_{2ii}, & m_{2ij} \\ & & & \\ m_{1ji}, & m_{1jj} & m_{2ji}, & m_{2jj} \end{array}\right| \qquad (41)$$

where the indices i and j are signant,[7] but the first index is non-signant as might be expected because $((\varphi_1, \varphi_2))_S = ((\varphi_2, \varphi_1))_S$. We see that if the respective determinants of φ_1 and φ_2 be placed as non-signant layers of a cubic matrix, the cubic determinant (41) has for its own non-signant layers the two-row minors whose main diagonals occupy corresponding positions on the main diagonals of φ_1 and φ_2. As a verification, if $\varphi_1 = \varphi_2$ the cubic determinant does not vanish, but becomes twice one of these minors, in agreement with the preceding discussion of the coefficients in the Hamilton-Cayley equation for φ.

Since i and j have all pairs of values from 1 to n, it is clear that $((\varphi_1\varphi_2))_S$ is the sum of all cubic determinants of the second order whose main diagonals occupy corresponding positions on the main diagonals of the determinants of φ_1 and φ_2.

An analogous proof holds for all values of p up to n. Thus (36) can be written as the sum of all the cubic determinants of order p whose main diagonals are corresponding elements from the determinants of $\varphi_1, \varphi_2, \cdots, \varphi_p$.

In particular, when $p = n$, that is, when we have any n double polyadics, the scalar (36) is the cubic determinant of order n of which the non-signant layers are formed by the square matrices m_{1ij}, m_{2ij}, $\cdots$, m_{nij} of these double polyadics.

7. Scalars formed in Combination with the Idemfactor.

It may happen that one or more of the double polyadics of the scalar (36) is the idemfactor. The scalar formed from any p double polyadics together with the idemfactor taken q times is the same as the scalar formed from the p double polyadics alone, aside from a numerical factor f that is

$$((\varphi_1, \varphi_2, \cdots, \varphi_p, I, I, \cdots))_S = f\,((\varphi_1, \varphi_2, \cdots, \varphi_p))_S \tag{42}$$

where, however, $p + q$ is not to be greater than n. It is sufficient to prove the proposition for one idemfactor and any p double polyadics, assuming $p + 1$ not greater than n; the general form of (42) then follows by letting some of the polyadics equal the idemfactor. I shall in fact prove the identity for one idemfactor with $p = 2$; the proof for other values of p is quite similar. The scalar $((\varphi_1, \varphi_2, I))_S$ is by Art 6 the sum of all cubic determinants of the third order of the form

$$\left|\begin{array}{ccc|ccc|ccc} 1, & 0, & 0 & m_{1ii}, & m_{1ij}, & m_{1ik} & m_{2ii}, & m_{2ij}, & m_{2ik} \\ 0, & 1, & 0 & m_{1ji}, & m_{1jj}, & m_{1jk} & m_{2ji}, & m_{2jj}, & m_{2jk} \\ 0, & 0, & 1 & m_{1ki}, & m_{1kj}, & m_{1kk} & m_{2ki}, & m_{2kj}, & m_{2kk} \end{array}\right| \tag{43}$$

which on development yields the sum of three cubic determinants of the second order having the same form as (41), and corresponding to the three pairs of subscripts ij, jk, ki. Consider some particular one of these cubic determinants of the second order, say that one which involves the subscripts ij. It will evidently occur once in the expansion of every cubic determinant of the form (43), provided the subscripts are i, j and any other subscript not greater than n. That is, the cubic determinant (41) will occur $n - 2$ times in the development of $((\varphi_1, \varphi_2, I))_S$. But i and j are any pair of different subscripts. Hence

$$((\varphi_1, \varphi_2, I))_S = (n - 2)\,((\varphi_1, \varphi_2))_S \tag{44}$$

The same reasoning applies for any value of p. Hence we have

$$((\varphi_1, \varphi_2, \cdots, \varphi_p, I))_S = (n - p)\,((\varphi_1, \varphi_2, \cdots, \varphi_p))_S \tag{44a}$$

where p has any value from 1 to $n - 1$ and $\varphi_1 \cdots \varphi_p$ are any double K-adics whatever. By letting several of these K-adics successively become equal to the idemfactor we obtain (42) and see that the numerical factor f is given by

$$f = (n - p - q + 1)\,(n - p - q + 2) \cdots (n - p) \tag{45}$$

We might also have obtained (44) by a count of the number of terms obtained by developing either side. If ${}_nC_{p+1}$ denote the number of combinations of n things $p + 1$ at a time, this will be the number of cubic determinants of order $p + 1$ obtained on developing the left side of (44). Each of these cubic determinants yields $p + 1$ cubic determinants of order p, because one layer is taken from the idemfactor, hence contains $p + 1$ non-vanishing elements. Thus the left side of (44) may be written as the sum of $(p + 1){}_nC_{p+1}$ cubic determinants of order p. On the right the scalar $((\varphi_1, \varphi_2, \cdots, \varphi_p))_S$ is the sum of ${}_nC_p$ cubic determinants of order p. But we have identically

$$(p + 1){}_nC_{p+1} = (n - p){}_nC_p \tag{46}$$

agreeing with (44).

In particular it follows from (42) that any scalar (36) can be written as a single cubic determinant of order n by adjoining the idemfactor $n - p$ times, so that $p + q = n$ and the numerical factor becomes $(n - p)!$. Thus

$$((\varphi_1, \varphi_2 \cdots, \varphi_p, I, I, \cdots))_S = (n - p)!\ ((\varphi_1, \varphi_2 \cdots, \varphi_p))_S \tag{47}$$

in the case where the idemfactor occurs just $n - p$ times.

It also follows that any coefficient in the Hamilton-Cayley equation for φ can be written as a single cubic determinant of order n. For by letting the p arbitrary double polyadics all equal φ, we have in virtue of (37)

$$((\varphi, \varphi, \cdots, \varphi, I, I, \cdots I))_S = p!\ (n - p)!\ m_p \tag{48}$$

provided φ enters p times and I enters $n - p$ times.

8. SCALARS EXPRESSED AS SQUARE DETERMINANTS WHOSE ELEMENTS ARE K-ADICS.

By virtue of the result of Art. 7, it becomes possible to express any scalar of the form (36) without the use of cubic determinants, if we are willing instead to use determinants whose elements are not scalars but polyadics, and to make certain slight but needful modifications in the usual rules for determinate expansions.

Suppose, as before, that there are n arbitrary double polyadics $\varphi_1 \cdots \varphi_n$ and let each of them be expanded in the manner of (27), thus

$$\varphi_i = \mathbf{B}_{i1}\mathbf{E}_1 + \mathbf{B}_{i2}\mathbf{E}_2 + \cdots + \mathbf{B}_{in}\mathbf{E}_n \tag{49}$$

where the consequents $\mathbf{E}_1 \cdots \mathbf{E}_n$ are the fundamental K-ads and the antecedents are any arbitrary K-adics. Let $\mathbf{E}_j$ be any one of the fundamental K-ads. We have

$$\varphi_i : \mathbf{E}_j = \mathbf{B}_{ij}. \tag{50}$$

The binary assemblage of n^2 arbitrary polyadics $\mathbf{B}_{ij}$ are now to be formed into a determinant

$$\begin{vmatrix} \mathbf{B}_{11}, & \mathbf{B}_{12}, & \cdots, & \mathbf{B}_{1n} \\ \mathbf{B}_{21}, & \mathbf{B}_{22}, & \cdots, & \mathbf{B}_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \mathbf{B}_{n1}, & \mathbf{B}_{n2}, & \cdots, & \mathbf{B}_{nn} \end{vmatrix} \tag{51}$$

where the law of expansion is defined to be as follows: the leading term is written $[\mathbf{B}_{11}\mathbf{B}_{22}\cdots\mathbf{B}_{nn}]$ and signifies the determinant in the ordinary sense whose elements are the scalar elements of the polyadics which enter: namely, if we expand as

$$\mathbf{B}_{ij} = b_{ij1}\mathbf{E}_1 + b_{ij2}\mathbf{E}_2 + \cdots + b_{ijn}\mathbf{E}_n \tag{52}$$

the leading term in expanding (51) is the determinant of ordinary character

$$\begin{vmatrix} b_{111}, & b_{112}, & \cdots, & b_{11n} \\ b_{221}, & b_{222}, & \cdots, & b_{22n} \\ \cdots & \cdots & \cdots & \cdots \\ b_{nn1}, & b_{nn2}, & \cdots, & b_{nnn} \end{vmatrix} \tag{53}$$

the other terms in the expansion of (51) are of similar character, and are formed by writing out the determinant (51) as if the elements were scalars, enclosing each term in square brackets to signify the determinant formed from it as in (53). The signs will occur as in ordinary determinants if we adopt the rule that, in every term, *the order of the polyadics shall be the order of the rows in the original determinant.* The meaning of the determinant (51) is thus defined without ambiguity. We note that by this rule the order of rows in (53) corresponds to the order of rows in (51), in the sense that each row of (53) consists of the elements of a single polyadic from the corresponding row of (51).

It is then apparent that (51) is the sum of $n!$ ordinary determinants, hence the sum of $(n!)^2$ terms on complete expansion. The order of the rows is non-signant, in the sense that, if two rows in (51) be interchanged, the value of the expression is quite unchanged; for on expanding, first, into a sum of ordinary determinants or square bracket expressions, the sign before every square bracket is changed because the signs behave as if (51) were an ordinary determinant; but, since the development follows the rows, the sign of every determinant (53) will have been changed due to the interchange of rows; thus each square bracket has its sign changed, we may say, both from within and from without. Hence the whole expression is unaltered. On the other hand, an interchange of two columns in (51) changes the sign before every square bracket but leaves the brackets themselves unaffected, for the development is according to rows; hence the sign of the whole is changed.

It is now not hard to see that the polyadic determinant (51) is the same as $((\varphi_1, \varphi_2, \cdots, \varphi_n))_S$. For we have seen that the first index is non-signant, the second signant; the third is obviously signant since it governs the columns of the ordinary determinants of the form (53) but does not otherwise enter. Furthermore, on complete expansion of (51) we shall have each term a product of n elements, one from each of the double polyadics, one from each row of (51), and one from each column of (51). But this is precisely the law of formation of a cubic determinant having the elements of the polyadics as non-signant layers. By Art. 6 this is the scalar in question and we may write the identity

$$\begin{vmatrix} \varphi_1 : \mathbf{E}_1, & \varphi_1 : \mathbf{E}_2, & \cdots, & \varphi_1 : \mathbf{E}_n \\ \varphi_2 : \mathbf{E}_1, & \varphi_2 : \mathbf{E}_2, & \cdots, & \varphi_2 : \mathbf{E}_n \\ \cdots & \cdots & \cdots & \cdots \\ \varphi_n : \mathbf{E}_1, & \varphi_n : \mathbf{E}_2, & \cdots, & \varphi_n : \mathbf{E}_n \end{vmatrix} = ((\varphi_1, \varphi_2, \cdots, \varphi_n))_S \qquad (54)$$

where the polyadic determinant on the left is defined as above.

To take the simplest possible illustration, let $K = 1$, $N = 2$, $n = 2$. Our K-adics become vectors in two dimensions. Take $\mathbf{E}_1 = \mathbf{i}$, $\mathbf{E}_2 = \mathbf{j}$. Let there be two dyadics $\varphi_1 = \mathbf{b}_{11}\mathbf{i} + \mathbf{b}_{12}\mathbf{j} = b_{111}\mathbf{ii} + b_{112}\mathbf{ji} + b_{121}\mathbf{ij} + b_{122}\mathbf{jj}$ and similarly for φ_2. The determinant on the left of (54) becomes

$$\begin{vmatrix} b_{111}\mathbf{i} + b_{112}\mathbf{j}, & b_{121}\mathbf{i} + b_{122}\mathbf{j} \\ b_{211}\mathbf{i} + b_{212}\mathbf{j}, & b_{221}\mathbf{i} + b_{222}\mathbf{j} \end{vmatrix}$$

whose elements are vectors. Expanding by the rule we have

$$[b_{111}\mathbf{i} + b_{112}\mathbf{j}, b_{221}\mathbf{i} + b_{222}\mathbf{j}] - [b_{121}\mathbf{i} + b_{122}\mathbf{j}, b_{211}\mathbf{i} + b_{212}\mathbf{j}]$$

which is the same as the difference of two determinants

$$(b_{111}b_{222} - b_{112}b_{221}) - (b_{121}b_{212} - b_{122}b_{211})$$

and this, again, agrees with the cubic determinant formed with the element of the two dyadics as layers, namely

$$\left|\begin{array}{cc|cc} b_{111}, & b_{121} & b_{211}, & b_{221} \\ b_{112}, & b_{122} & b_{212}, & b_{222} \end{array}\right| .$$

A number of results follow immediately. We might have formed the left side of (54) by taking the multiple dot product of $\mathbf{E}_1$, $\mathbf{E}_2$, etc. into each of the arbitrary double polyadics, as $\mathbf{E}_1 \colon \varphi_1$, $\mathbf{E}_2 \colon \varphi_1$, etc. in the first row, and similarly for the other rows. Instead of (51) we should have had the determinant of the $\mathbf{M}_{ij}$ by developing after the manner (23) instead of (27). The rest of our reasoning would have been unchanged, however, hence the value of the expression would have remained the same. The change would in fact be equivalent to changing rows into columns and columns into rows in all the non-signant layers of the cubic determinant, which, since the second and third indices are signant, leaves the result unchanged. This is illustrated by the example just given, where the last index refers to the rows instead of to the columns as in (41). If we let φ' stand for $\Sigma\mathbf{MA}$ when φ stands for $\Sigma\mathbf{AM}$, and say that the conjugate φ' of a double polyadic φ is formed from it by interchanging antecedent and consequent, it is clear that $\mathbf{E} \colon \varphi = \varphi' \colon \mathbf{E}$. We therefore have the identity

$$((\varphi_1, \varphi_2, \cdots, \varphi_n))_S = ((\varphi_1', \varphi_2', \cdots, \varphi_n'))_S \qquad (55)$$

which may also be deduced directly from the definition of these scalars. The well known fact that the Hamilton-Cayley equation for any dyadic or matrix is identical with that of its conjugate appears as a special case of this identity.

Also if $n - p$ out of the arbitrary double polyadics are allowed to become equal to the idemfactor, the identity (54) becomes in virtue of (47),

$$\left|\begin{array}{cccc} \varphi_1 \colon \mathbf{E}_1, & \varphi_1 \colon \mathbf{E}_2, & \cdots, & \varphi_1 \colon \mathbf{E}_n \\ \varphi_2 \colon \mathbf{E}_1, & \varphi_2 \colon \mathbf{E}_2, & \cdots, & \varphi_2 \colon \mathbf{E}_n \\ \cdots & \cdots & \cdots & \cdots \\ \varphi_p \colon \mathbf{E}_1, & \varphi_p \colon \mathbf{E}_2, & \cdots, & \varphi_p \colon \mathbf{E}_n \\ \mathbf{E}_1 \ , & \mathbf{E}_2 \ , & \cdots, & \mathbf{E}_n \\ \mathbf{E}_1 \ , & \mathbf{E}_2 \ , & \cdots, & \mathbf{E}_n \\ \cdots & \cdots & \cdots & \cdots \\ \mathbf{E}_1 \ , & \mathbf{E}_2 \ , & \cdots, & \mathbf{E}_n \end{array}\right| = (n-p)!((\varphi_1, \varphi_2, \cdots, \varphi_p))_S \qquad (56)$$

where the last $n - p$ rows are alike, a circumstance which need cause no surprise, since, as already pointed out, the rows are non-signant in the very nature of the definition of these polyadic determinants.[8]

If we still further specialize (56) by letting the remaining p double polyadics all become equal to φ we shall have, by (48),

$$\begin{vmatrix} \varphi:\mathbf{E}_1, & \varphi:\mathbf{E}_2, & \cdots, & \varphi:\mathbf{E}_n \\ \varphi:\mathbf{E}_1, & \varphi:\mathbf{E}_2, & \cdots, & \varphi:\mathbf{E}_n \\ \cdots & \cdots & \cdots & \cdots \\ \varphi:\mathbf{E}_1, & \varphi:\mathbf{E}_2, & \cdots, & \varphi:\mathbf{E}_n \\ \mathbf{E}_1, & \mathbf{E}_2, & \cdots, & \mathbf{E}_n \\ \mathbf{E}_1, & \mathbf{E}_2, & \cdots, & \mathbf{E}_n \\ \cdots & \cdots & \cdots & \cdots \\ \mathbf{E}_1, & \mathbf{E}_2, & \cdots, & \mathbf{E}_n \end{vmatrix} = p!\,(n-p)!\,m_p \qquad (57)$$

where, on the left, p rows are like the first row, and $(n - p)$ rows are like the last row.

9. AN INVARIANT PROPERTY OF THE SCALARS (36).

From the form of the definition of the scalars of the present discussion, they are independent of development in terms of a particular set of polyadics. Throughout the argument the fundamental polyads $\mathbf{E}_1 \cdots \mathbf{E}_n$ may be replaced by any other set of n polyadics forming a normal, orthogonal system with respect to K-tuple dot product, for no other properties of these polyads have been so far used. These scalars have another kind of invariance of a somewhat wider sort, with respect to a reference system which need no longer be normal nor orthogonal but only linearly independent. The properties in question are essentially included in the following theorem:—

Theorem. The scalar $((\varphi_1, \varphi_2, \cdots, \varphi_n))_S$ formed from a set of n arbitrary double polyadics is unaltered when each of these double polyadics is multiplied by (or into) the same double polyadic θ, (by K-tuple dot product), except for a factor which is the determinant of θ.

This theorem is equivalent to the identity

$$((\varphi_1, \varphi_2, \cdots, \varphi_n))_S\,((\theta^n))_S = n!\,((\varphi_1:\theta, \varphi_2:\theta, \cdots, \varphi_n:\theta))_S \qquad (58)$$

for, by (37), $((\theta^n))_S$ equals $n!$ times the coefficient of θ^n in the Hamilton-Cayley equation for θ, which is the determinant of θ; and $\theta:\varphi_1 = \varphi_1':\theta$, while by (55) the left side of (58) is unchanged if we change every φ into φ'; hence it is sufficient to prove (58) as written.

Proof. Let every φ be expanded as in (23). The scalar $((\varphi_1 \cdot\cdot, \varphi_n))_S$ is equal to the polyadic determinant of the $\mathbf{M}_{ij}$ as shown in Art. 8. To multiply every φ into a double polyadic θ is equivalent to performing the same linear transformation upon every $\mathbf{M}$. But the polyadic determinant is the sum of ordinary determinants whose rows consist of elements of the respective polyadics $\mathbf{M}$. It is well known that when every row of a determinant is affected by the same linear transformation the determinant is merely multiplied by the determinant of the transformation. Thus every square bracket term in the expansion of the polyadic determinant of the $\mathbf{M}_{ij}$ is multiplied by the determinant of θ. The theorem is therefore proved.

As an immediate consequence, any polyadic element $\varphi_i : \mathbf{E}_j$ on the left of the identity (54) may be replaced by $\varphi_i : \theta : \mathbf{E}_j$, provided we multiply the right side by the determinant of θ, which may be called m_θ and is the same as the determinant of the n polyadics $\theta : \mathbf{E}_1$, $\theta : \mathbf{E}_2 \cdot\cdot, \theta : \mathbf{E}_n$. By a proper choice of the arbitrary double polyadic θ, these n polyadics may be made to be any n polyadics whatever, $\mathbf{A}_1, \cdots, \mathbf{A}_n$. If we agree to write $[\varphi_i : \mathbf{A}_j]$ for the polyadic determinant whose elements have the form indicated, and $[a_{ij}]$ for the determinant (in the ordinary sense), whose elements are the scalar elements of the polyadics $\mathbf{A}_1 \cdot\cdot \mathbf{A}_n$ resolved according to (11), we shall now have the identity

$$((\varphi_1, \varphi_2, \cdots, \varphi_n))_S = \frac{[\varphi_i : \mathbf{A}_j]}{[a_{ij}]} \tag{59}$$

assuming the polyadics $\mathbf{A}_i$ linearly independent. Thus the scalar on the left is invariant of the particular choice of these polyadics.[9]

In a similar manner and with similar notation we may derive from (56)

$$[\varphi_1 : \mathbf{A}_j, \varphi_2 : \mathbf{A}_j, \cdots, \varphi_p : \mathbf{A}_j, \mathbf{A}_j, \mathbf{A}_j, \cdots, \mathbf{A}_j] = [a_{ij}]\,(n-p)!\,((\varphi_1, \varphi_2, \cdots, \varphi_p))_S \tag{60}$$

and from (57)

$$[(\varphi : \mathbf{A}_j)^p\,(\mathbf{A}_j)^{n-p}] = [a_{ij}]\,p!\,(n-p)!\,m_p \tag{61}$$

In employing this abbreviated notation for polyadic determinants, if there are two subscripts the first always refers to the rows, the second to the columns, as in (59) and (60). Where a series of n polyadics is written in brackets as in (60) and (61) these refer to the rows. We pick out an element of a particular column by giving a particular value to the common subscript j. If several rows are alike the fact may be denoted by an exponent, as in (61). Thus in (60) the left side might have been written $[\varphi_1 : \mathbf{A}_j, \cdots, \varphi_p : \mathbf{A}_j, (\mathbf{A}_j)^{n-p}]$. Where, however, the notation implies merely n polyadics (instead of n^2) the square bracket signifies the ordinary determinant of the n^2 elements by the scheme of (11). Thus instead of $[a_{ij}]$ we might write $[\mathbf{A}_j]$ without change of meaning. Unless the contrary is stated, all subscripts are to run from 1 to n.

10. FORMS OF THE SCALARS (36) WHICH SHOW THEIR POLYADIC CHARACTER.

Up to this point have been noted three distinct methods of actually calculating a scalar of the form (36), namely

1°. Multiply the double polyadics term by term and take scalars according to definition.

2°. *Form a sum of cubic determinants.*

3°. Form a polyadic determinant by operating on any set of linearly independent polyadics and divide by the determinant of the set.

In all these cases the double polyadics have behaved formally as if their antecedents and consequents were vectors in space of n dimensions, instead of polyadics each of which is a sum of products of K vectors in space of N dimensions, where $n = N^K$. Each of these methods applies with no change whatever if $K = 1$ so that $n = N$ and the polyadics become actually vectors,— with the obvious exception that the multiple dot product, indicated by a colon, becomes ordinary or single dot product. While I am not aware that any of these three methods has actually been employed by writers on vector analysis in N dimensions, yet it can hardly be said that the individual steps do not occur, in principle at least, in the writings of Hamilton, Gibbs, or Grassmann. For example, the definition of these scalars is equivalent to a double indeterminate product followed by inner product. In the case K = 1, $N = 3$ they were introduced by Gibbs as scalars of double cross products. In the same case they were expressed by Hamilton both by summations which are equivalent to double multiplication,[10] (though never so regarded by him), and also as quotients equivalent to our method 3°. It is true that none of these writers expressed the scalars as sums of determinants whose elements are extensive magnitudes, use of which was first made by Cayley, nor by cubic determinants, also due to Cayley, but these two equivalent formulations differ more in appearance than in reality from such expressions as Hamilton's

$$-m_2 = Si\varphi j\varphi k + Si\varphi k\varphi i + Sk\varphi i\varphi j$$

which in Gibbs' notation would be

$$m_2 = \Sigma(\mathbf{i}\varphi\cdot\mathbf{j}\varphi\cdot\mathbf{k})$$

where i, j, k denote rectangular unit vectors.

It will be of some value for the sake of later applications if we take account of these scalars as they appear when our polyadics are determined by K-dimensional matrices. Consider for a moment the case $((\mathbf{AM},\ \mathbf{BN}))_S$ where **A**, **M**, **B**, and **N** are dyadics in two dimensions,— evidently the simplest case which can be regarded as a generalization of the usual vector analysis. With unit vectors **i** and **j** we may either write

$$\mathbf{A} = a_{11}\mathbf{ii} + a_{12}\mathbf{ij} + a_{21}\mathbf{ji} + a_{22}\mathbf{jj} \tag{62}$$

or may regard **A** as an abbreviation for the matrix

$$\left\| \begin{matrix} a_{11}, & a_{12} \\ a_{21}, & a_{22} \end{matrix} \right\| \tag{63}$$

with similar notation for the other three dyadics. If we form the required scalar directly from the definition, namely as $\mathbf{A}:\mathbf{MB}:\mathbf{N} - \mathbf{A}:\mathbf{NB}:\mathbf{M}$ we obtain

$$\begin{aligned} ((\mathbf{AM},\ \mathbf{BM}))_S = {} & (a_{11}m_{11} + a_{12}m_{12} + a_{21}m_{21} + a_{22}m_{22})\ (b_{11}n_{11} + b_{12}n_{12} \\ & + b_{21}n_{21} + b_{22}n_{22}) \\ & - (a_{11}n_{11} + a_{12}n_{12} + a_{21}n_{21} + a_{22}n_{22})\ (b_{11}m_{11}b_{12}m_{12} + \\ & b_{21}m_{21} + b_{22}m_{22}) \end{aligned} \tag{64}$$

If instead we use the method of cubic determinants we shall first form the square array of the fourth order belonging to the double dyadic **AM**, which is

$$\begin{matrix} a_{11}m_{11}, & a_{11}m_{12}, & a_{11}m_{21}, & a_{11}m_{22} \\ a_{12}m_{11}, & a_{12}m_{12}, & a_{12}m_{21}, & a_{12}m_{22} \\ a_{21}m_{11}, & a_{21}m_{12}, & a_{21}m_{21}, & a_{21}m_{22} \\ a_{22}m_{11}, & a_{22}m_{12}, & a_{22}m_{21}, & a_{22}m_{22} \end{matrix} \tag{65}$$

with a similar array for **BN**. We then take the sum of all cubic determinants along the main diagonal, using the two arrays as similarly placed layers. The first of these will be

$$\left|\begin{array}{cc|cc} a_{11}m_{11}, & a_{11}m_{12} & b_{11}n_{11}, & b_{11}n_{12} \\ a_{12}m_{11}, & a_{12}m_{12} & b_{12}n_{11}, & b_{12}n_{12} \end{array}\right| \tag{66}$$

which develops into

$$a_{11}m_{11}b_{12}n_{12} - a_{12}m_{11}b_{11}n_{12} - a_{11}m_{12}b_{12}n_{11} + a_{12}m_{12}b_{11}n_{11} \tag{67}$$

There are six of these groups of terms, or 24 in all. The scheme of each group is given by

$$\left|\begin{array}{cc|cc} a_{pq}m_{pq}, & a_{pq}m_{rs} & b_{pq}n_{pq}, & b_{pq}n_{rs} \\ a_{rs}m_{pq}, & a_{rs}m_{rs} & b_{rs}n_{pq}, & b_{rs}n_{rs} \end{array}\right| \tag{68}$$

which develops into

$$a_{pq}m_{pq}b_{rs}n_{rs} - a_{rs}m_{pq}b_{pq}n_{rs} - a_{pq}m_{rs}b_{rs}n_{pq} + a_{rs}m_{rs}b_{pq}n_{pq} \tag{69}$$

and the summation is performed by giving to the number-pairs pq and rs unlike values chosen from the pairs 11, 12, 21, and 22. Comparing with (64) it is easy to verify that the two methods agree.

If we use the method of polyadic determinants and adopt the dyads **ii**, **ij**, etc. as operands we first form the dyadics **AM** : **ii**, **AM** : **ij**, etc. treating **BN** in a similar manner. We thus have two rows of a determinant of the fourth order whose elements are dyadics. The other two rows are alike and given by I : **ii**, I : **ij**, etc. that is by **ii**, **ij**, etc. The result

$$\left|\begin{array}{cccc} \mathbf{A}m_{11}, & \mathbf{A}m_{12}, & \mathbf{A}m_{21}, & \mathbf{A}m_{22} \\ \mathbf{B}n_{11}, & \mathbf{B}n_{12}, & \mathbf{B}n_{21}, & \mathbf{B}n_{22} \\ \mathbf{ii}\ , & \mathbf{ij}\ , & \mathbf{ji}\ , & \mathbf{jj} \\ \mathbf{ii} & \mathbf{ij} & \mathbf{ji} & \mathbf{jj} \end{array}\right| \tag{70}$$

should by (56) be the double of the required scalar. Developing by the definition of Art. 8 the leading term is $[\mathbf{A}m_{11}, \mathbf{B}n_{12}, \mathbf{ji}, \mathbf{jj}]$ which denotes the ordinary determinant of the fourth order

$$\left|\begin{array}{cccc} a_{11}m_{11}, & a_{12}m_{11}, & a_{21}m_{11}, & a_{22}m_{11} \\ b_{11}n_{12}\ , & b_{12}n_{12}\ , & b_{21}n_{12}\ , & b_{22}n_{22} \\ 0\ , & 0\ , & 1\ , & 0 \\ 0\ , & 0\ , & 0\ , & 1 \end{array}\right| \tag{71}$$

which develops into $a_{11}m_{11}b_{12}n_{12} - a_{12}m_{11}b_{11}n_{12}$. The next term may be taken as $-[\mathbf{A}m_{11}, \mathbf{B}n_{12}, \mathbf{jj}, \mathbf{ji}]$ which doubles the two scalar terms already found. It is easy to see that developing (70) by two-row minors after Laplace's method yields the scalar terms in a fashion similar to the method of cubic determinants.

The fact that any scalar (36) is unaltered by changing all the double dyadics into their conjugates, and is also unaltered by changing the order of double dyadics, appears in the double symmetry of (64). A further and quite distinct fact could not have been brought out so long as our notation represented polyadics as formal vectors: a scalar formed from double dyadics, like (64), is unaltered if all the dyadics **A**, **M**, **B**, **N** are replaced by their conjugates. This is equivalent to interchanging the two subscripts throughout the expression on the right of (64); whereas changing the double polyadics into their conjugates is equivalent to everywhere interchanging a with m and at the same time b with n; and changing the order of the polyadics is equivalent to everywhere interchanging a with b and at the same time m with n.

More generally a polyadic which is a sum of terms of the form $\mathbf{a}_1\mathbf{a}_2\cdots\mathbf{a}_k$ where the K factors are vectors in space of N dimensions may be regarded as one of a set of $K!$ polyadics obtained from one another by making like permutations of factors in every term. A glance at the definition (10) of multiple dot product is sufficient to show that all scalars (36) are unaltered when all the antecedents and consequents of every double polyadic have the factors of all their terms permuted in the same manner. If each polyadic consists of a single polyad and we apply the definition of Art. 5, the truth of the proposition is evident, hence by the distributive principle holds universally.

Examples like (64) or (69) may be generalized in three ways, according as we increase N, the dimensions of the space, K, the order of the polyadics, or p, the order of the scalars.

If N increases we still have double dyadics and the form of (69) is unaltered, but the summation is performed over a larger group of number-pairs, namely 11, 12, 21, 13, 31, $\cdots$, NN.

If K increases the number of subscripts increases but (69) still consists of 4 scalar terms and the summation is over pairs of unlike number-triplets, quadruplets, etc.

If p increases the order of the determinant (68) increases, likewise the number of terms in (69). Thus if we keep $N = 2$, $K = 2$, but make $p = 3$ there are 4 cubic determinants whose sum is $((\mathbf{AL}, \mathbf{BM}, \mathbf{CN}))_S$ or 72 scalar terms in all.

If we have double dyadics in 3 dimensions with $p = 2$, $((\mathbf{AM}, \mathbf{BN}))_S$ will be the sum of 36 expressions of the same form as (69) or 144 scalar terms.

It is usually possible to neglect the dimensionality of the space and the number of terms in the double polyadic, frequently also the order of the polyadics. Thus $((\mathbf{AM},\ \mathbf{BN}))_S$ is symbolic of $((\varphi_1, \varphi_2))_S$. When the order of the polyadic is given we may also treat with polyads in many operations as symbolic of polyadics. Thus if we have double dyadics in any space we may often write $\mathbf{aa}'$ for $\mathbf{A}$, $\mathbf{mm}'$ for $\mathbf{M}$, etc. in which case a bar may be used to separate antecedent from consequent in the double polyads, or $((\mathbf{aa}' \,|\, \mathbf{mm}',\ \mathbf{bb}' \,|\, \mathbf{nn}'))_S$ may be taken symbolically for the much more general $((\varphi_1, \varphi_2))_S$. Hence if φ_1 and φ_2 are double dyadics we may write

$$((\varphi_1, \varphi_2))_S = \Sigma(\mathbf{a}\cdot\mathbf{ma}'\cdot\mathbf{m}'\mathbf{b}\cdot\mathbf{nb}'\cdot\mathbf{n}' - \mathbf{a}\cdot\mathbf{na}'\cdot\mathbf{n}'\mathbf{b}\cdot\mathbf{mb}'\cdot\mathbf{m}') \qquad (72)$$

and may even omit the summation sign in many operations. It is of interest to compare this vector expression with (64) which was based on the notion of the dyadic as a matrix.

11. Invariants regarded as Products.

The examples of the last article are sufficient to indicate that the scalars (36) are invariants of polyadic systems and that they are subject to a considerable variety of transformations. There is one more method by which they may be calculated, based on a concept of these scalars which may serve to widen considerably our view of the formal laws to which double polyadics are amenable. Returning to the earlier conception of a double polyadic as a sum of unit double polyads $\mathbf{E}_i\mathbf{E}_k$ each multiplied by a scalar, as in (25), consider scalars of the form $((\mathbf{E}_i\mathbf{E}_k,\ \mathbf{E}_r\mathbf{E}_s))_S$, that is $\mathbf{E}_i\!:\!\mathbf{E}_k\mathbf{E}_r\!:\!\mathbf{E}_s - \mathbf{E}_i\!:\!\mathbf{E}_s\mathbf{E}_r\!:\!\mathbf{E}_k$. The unit polyads $\mathbf{E}_i$ and $\mathbf{E}_r$ are antecedents while $\mathbf{E}_k$ and $\mathbf{E}_s$ are consequents. We see that the scalar is equal to $+1$ when each antecedent is equal to its own consequent, but unlike the other antecedent and consequent; is equal to -1 when each antecedent differs from its own consequent but is equal to the other consequent; and is otherwise zero. That is

$$\begin{gathered}((\mathbf{E}_i\mathbf{E}_i,\ \mathbf{E}_k\mathbf{E}_k))_S = +1,\quad ((\mathbf{E}_i\mathbf{E}_k,\ \mathbf{E}_k\mathbf{E}_i))_S = -1,\quad ((\mathbf{E}_i\mathbf{E}_i,\ \mathbf{E}_r\mathbf{E}_s))_S = 0\\ ((\mathbf{E}_i\mathbf{E}_i,\ \mathbf{E}_i\mathbf{E}_i))_S = 0,\quad ((\mathbf{E}_i\mathbf{E}_k,\ \mathbf{E}_r\mathbf{E}_r))_S = 0,\quad ((\mathbf{E}_i\mathbf{E}_i,\ \mathbf{E}_i\mathbf{E}_k))_S = 0\\ ((\mathbf{E}_i\mathbf{E}_k,\ \mathbf{E}_k\mathbf{E}_k))_S = 0,\quad ((\mathbf{E}_i\mathbf{E}_k,\ \mathbf{E}_r\mathbf{E}_s))_S = 0\end{gathered} \qquad (73)$$

on the understanding that subscripts are unequal unless denoted by the same letter.

In the same way it is easy to see that any scalar derived from a set of p unit double polyads, namely

$$((\mathbf{E}_a\mathbf{E}_i,\ \mathbf{E}_b\mathbf{E}_k,\ \cdots,\ \mathbf{E}_g\mathbf{E}_s))_S \tag{74}$$

where subscripts run from 1 to n, will vanish unless, by interchanges among the consequents, it can be brought to the form

$$((\mathbf{E}_a\mathbf{E}_a,\ \mathbf{E}_b\mathbf{E}_b,\ \cdots,\ \mathbf{E}_g\mathbf{E}_g))_S \tag{75}$$

wherein each antecedent is equal to its own consequent; and moreover vanishes unless all antecedents are unequal. The sign is plus or minus according as the number of interchanges needed to give the form (75) is even or odd.

We may now regard the scalars (36) as products of double polyadics formed according to this law, and, if each double polyadic is developed as in (25), can evidently calculate these scalars with no more labor than would be needed to multiply algebraic polynomials.

12. Star Products.

It is especially in the case $p = 2$ that the concept of the last article appears useful. In keeping with the idea that $((\varphi_1, \varphi_2))_S$, the calculation of which has been exemplified in some detail, is a kind of product of the two double polyadics, we may introduce the notation

$$((\varphi_1, \varphi_2))_S = \varphi_1{}^*\varphi_2 \tag{76}$$

and speak of this scalar as the star product of the two double polyadics. It is essentially a kind of *double* product, but since there is nothing analogous for mere vector factors it is not necessary to write two stars to emphasize the distinction. It may properly be compared, or rather contrasted, with the double dot product of Gibbs. To illustrate, take again the simplest case, $K = 1$, $N = 2$, so that the factors are dyadics in space of two dimensions. By Art. 11 the multiplication table for unit dyads will be

$$\begin{array}{ccccc} & \mathbf{ii}, & \mathbf{ij}, & \mathbf{ji}, & \mathbf{jj} \\ \mathbf{ii} & 0, & 0, & 0, & +1 \\ \mathbf{ij} & 0, & 0, & -1, & 0 \\ \mathbf{ji} & 0, & -1, & 0, & 0 \\ \mathbf{jj} & +1, & 0, & 0, & 0 \end{array} \tag{77}$$

which brings out the "skew" character of the star product: it is formed by the multiplication of unlike elements. If the scalar elements of φ_1 and φ_2 are a_{rs} and b_{rs} in the style of (63), we shall have

$$\begin{aligned}\varphi_1{}^*\varphi_2 &= (a_{11}\mathbf{ii} + a_{12}\mathbf{ij} + a_{21}\mathbf{ji} + a_{22}\mathbf{jj})^*(b_{11}\mathbf{ii} + b_{12}\mathbf{ij} + b_{21}\mathbf{ji} + b_{22}\mathbf{jj}) \\ &= a_{11}b_{22} - a_{12}b_{21} - a_{21}b_{12} + a_{22}b_{11} \qquad (78)\end{aligned}$$

The result is of course the same as if the dyadics had been expanded in any other manner and we had applied one of the three methods of Art. 10.

By contrast the double dot product $\varphi_1 : \varphi_2$ yields a multiplication table having $+1$ along the main diagonal and elsewhere zero, giving

$$\varphi_1 : \varphi_2 = a_{11}b_{11} + a_{12}b_{12} + a_{21}b_{21} + a_{22}b_{22} \qquad (79)$$

The concept of $\varphi_1{}^*\varphi_2$ as a product is justified by the fact that it is in every case possible to set up systems of double polyadics φ_1, φ_2, $\cdot\cdot$, φ_ν such that $\varphi_r{}^*\varphi_s = 1$ if subscripts are equal, otherwise zero. Here $\nu = n^2 = N^{2K}$ for this is the number of linearly independent double polyadics in any case. In the example (77) if we make

$$\mathbf{F}_1 = \frac{(\mathbf{ii} + \mathbf{jj})}{\sqrt{+2}},\ \mathbf{F}_2 = \frac{(\mathbf{ii} - \mathbf{jj})}{\sqrt{-2}},\ \mathbf{F}_3 = \frac{(\mathbf{ij} + \mathbf{ji})}{\sqrt{-2}},\ \mathbf{F}_4 = \frac{(\mathbf{ij} - \mathbf{ji})}{\sqrt{+2}} \qquad (80)$$

we shall have $\mathbf{F}_r{}^*\mathbf{F}_s = 1$ if subscripts are equal, otherwise zero.

It is clear that, just as the dyads **ii**, **ij**, **ji**, **jj** could be formally treated with reference to double dot product as would be vectors in 4-space with reference to ordinary dot product, so the dyadics **F** can be similarly treated with respect to star product.

A correspondence may be thus set up between invariant properties of the dyadics or polyadics or double polyadics and the geometry of the space in which these magnitudes behave formally like vectors.

A special case of much interest has been treated in great detail, with some differences of terminology, by E. Waelsch.[11] By his very ingenious system the whole of the well-worked theory of invariants of binary forms is employed as algorism for phenomena of various sorts in ordinary space. In the language of the present discussion his system would arise by making up three fundamental units

$$\mathbf{E}_1 = \mathbf{ii},\ \mathbf{E}_2 = \frac{(\mathbf{ij} + \mathbf{ji})}{\sqrt{2}},\ \mathbf{E}_3 = \mathbf{jj} \qquad (81)$$

so that $\mathbf{E}_1$, $\mathbf{E}_2$, $\mathbf{E}_3$ form a normal orthogonal system with respect to double dot product. A binary quadric may be written

$$\alpha_{11}x_1^2 + 2\,\alpha_{12}x_1x_2 + \alpha_{22}x_2^2 \tag{82}$$

and a variable vector may be written

$$\mathbf{x} = x_1\mathbf{i} + x_2\mathbf{j} \tag{83}$$

but this vector is regarded as merely formal. We have identically

$$[\alpha_{11}\mathbf{ii} + \alpha_{12}(\mathbf{ij} + \mathbf{ji}) + \alpha_{22}\mathbf{jj}] : \mathbf{xx} = \alpha_{11}x_1^2 + 2\alpha_{12}x_1x_2 + \alpha_{22}x_2^2 \tag{84}$$

so that any binary quadric is the double dot product of a dyadic into the dyad $\mathbf{xx}$. We now make up the new system, orthogonal with respect to star product.

$$\mathbf{F}_1 = \frac{\mathbf{ii} + \mathbf{jj}}{\sqrt{+2}},\quad \mathbf{F}_2 = \frac{\mathbf{ii} - \mathbf{jj}}{\sqrt{-2}},\quad \mathbf{F}_3 = \frac{\mathbf{ij} + \mathbf{ji}}{\sqrt{-2}} \tag{85}$$

and also write

$$x = \frac{\alpha_{11} + \alpha_{22}}{\sqrt{+2}},\quad y = \frac{\alpha_{22} - \alpha_{11}}{\sqrt{-2}},\quad z = \alpha_{12}\sqrt{-2} \tag{86}$$

when we shall have identically

$$\alpha_{11}\mathbf{ii} + \alpha_{12}\,(\mathbf{ij} + \mathbf{ji}) + \alpha_{22}\mathbf{jj} = x\mathbf{F}_1 + y\mathbf{F}_2 + z\mathbf{F}_3 = \varphi \tag{87}$$

The system $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$ is now taken as a set of rectangular unit vectors in ordinary space. Thus any binary quadric corresponds to a vector drawn from a stipulated origin in space. As a single but important example of very numerous correspondences which arise, we have

$$\varphi{*}\varphi = x^2 + y^2 + z^2 = 2\,(\alpha_{11}\alpha_{22} - \alpha_{12}^2) \tag{88}$$

This brief outline renders scant justice to the very beautiful system of Waelsch, from which it differs in slight details, owing to difference in definitions and point of view. It differs in particular by identification of the dyadic with the desired vector through the star product and the relation (87). The essential thing is the correspondence set up between geometry and invariant theory by relations of the type (88).

More generally, if we take the ordinary dyadics of Gibbs, making $K = 1$, $N = 3$, with unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, it is easy to verify that the system of nine dyadics given by

$$\mathbf{F}_1\sqrt{2} = \mathbf{ii}+\mathbf{jj},\quad \mathbf{F}_2\sqrt{-4} = \mathbf{kk}-2\mathbf{ii},\quad \mathbf{F}_3\sqrt{-4} = 2\mathbf{jj}-\mathbf{kk},\quad \mathbf{F}_4\sqrt{-2} = \mathbf{ij}+\mathbf{ji}$$

$$\mathbf{F}_5\sqrt{-2} = \mathbf{jk}+\mathbf{kj},\quad \mathbf{F}_6\sqrt{-2} = \mathbf{ki}+\mathbf{ik},\quad \mathbf{F}_7\sqrt{2} = \mathbf{ij}-\mathbf{ji},\quad \mathbf{F}_8\sqrt{2} = \mathbf{jk}-\mathbf{kj},\quad \mathbf{F}_9\sqrt{2} = \mathbf{ki}-\mathbf{ik}$$

are normal and orthogonal under star product. As already noted, the star product in this special case of two dyadics φ and θ is the scalar $(\varphi \overset{\times}{\times} \theta)_S$ of Gibbs. If we wish to restrict ourselves to self-conjugate dyadics we have merely to drop $\mathbf{F}_7$, $\mathbf{F}_8$, and $\mathbf{F}_9$ from the list. It would be interesting to work out the analogy between the scalars called above $((\varphi^p))_S$, which might be symbolically written $((\mathbf{ab}))^p{}_S$, and the invariants of ternary quadrics written in the usual symbolic language.

We see however that systems orthogonal under star product will always be possible, because to every double polyad $\mathbf{E}_r\mathbf{E}_s$ corresponds always at least one other double polyad not orthogonal to the first.

13. Scalars formed by Star Multiplication.

In what follows I shall, for simplicity of language, take $K = 1$, so that double polyadics become dyadics, and polyadics become vectors. This imposes no formal limitation, and can be extended to the most general case with slight verbal change.

Supposing N to have any value, we assume $\mathbf{F}_1, \mathbf{F}_2, \cdots \mathbf{F}_\nu$ a set of linearly independent dyadics orthogonal under star product. Any other dyadics **A**, **M**, **B**, or **N** can be expressed in terms of these. Let φ_1 and φ_2 as before be a pair of double dyadics $\Sigma\mathbf{AM}$ and $\Sigma\mathbf{BN}$. The star product of **AM** into **BN** is defined to be $\mathbf{A}(\mathbf{M}{*}\mathbf{B})\mathbf{N}$ which is the same as $\mathbf{ANM}{*}\mathbf{B}$. Thus $\varphi_1{*}\varphi_2$ is a new double dyadic. If we regard $\varphi{*}\varphi$ and $\varphi{*}\varphi{*}\varphi$ etc. as a new type of power abbreviated φ^{*2}, φ^{*3}, etc. no further proof is required to show that φ^* satisfies a Hamilton-Cayley equation similar in form to (28),

$$\varphi^{*\nu} - m_1{}^* \varphi^{*(\nu-1)} + \cdots + (-1)\, m^*{}_\nu I'' = 0 \tag{89}$$

nor to show that the star idemfactor I'' is identical with $\Sigma\mathbf{F}_i\mathbf{F}_i$. The whole of Art. 3 might be repeated at this point, replacing multiple dot product by star product and **E** by **F**. The coefficients $m^*{}_p$ in (89) will be formed as in Art. 5, replacing multiple dot by star product. We shall also have a set of scalars

$$(\varphi_1, \varphi_2, \cdots, \varphi_p)_S{}^* \tag{90}$$

analogous with (36). For example

$$(\mathbf{AM}, \mathbf{BN})_S{}^* = \mathbf{A}{*}\mathbf{MB}{*}\mathbf{N} - \mathbf{A}{*}\mathbf{NB}{*}\mathbf{M} \tag{91}$$

which differs from (32) in having star in place of colon. It also differs in having no analogue when **A**, **M**, **B**, **N** are mere vectors, whereas

(32) would still hold if single dot be written for colon. As before remarked, the star product is essentially double.

For further comparison, suppose $N = 2$, with the notation of Art. 10.

$$\begin{aligned}(\mathbf{AM}, \mathbf{BN})^*_S = {} & (a_{11}m_{22} - a_{12}m_{21} - a_{21}m_{12} + a_{22}m_{11})\,(b_{11}n_{22} - b_{12}n_{21} - b_{21}n_{12} \\ & + b_{22}n_{11}) \\ & - (a_{11}n_{22} - a_{12}n_{21} - a_{21}n_{12} + a_{22}n_{11})\,(b_{11}m_{22} - b_{12}m_{21} \\ & - b_{21}m_{12} + b_{22}m_{11}) \qquad (92)\end{aligned}$$

which, like (64), reduces to 24 scalar terms by expanding.

The method of cubic determinants might be applied to star scalars, but with less ease, because a system of dyadics orthogonal under star product is in general imaginary.

Identities like (44), (42), (45), (47), (48), and (54) to (61) inclusive may be proved in quite parallel steps, with appropriate change of notation. With exception of (54), (56), and (57), these identities do not imply the presence of $\sqrt{-1}$ even when we change to star product. While the theory of invariants lies outside the scope of this paper, it may be noted that star scalars, like those of (36), are by nature of their definition, invariant under transformation of the fundamental units, and some of them are invariant in a much wider sense.

14. IDENTITIES WITH THE STAR IDEMFACTOR.

If we have a system of linearly independent dyadics $\mathbf{A}_1, \mathbf{A}_2, \cdots \mathbf{A}_\nu$ their reciprocals with respect to star product may be called $\mathbf{A}''_1, \mathbf{A}''_2, \cdots, \mathbf{A}''_\nu$. By steps as in Art. (3) we may show that $\Sigma\mathbf{AA}''$ is identically the idemfactor $\Sigma\mathbf{FF}$. It is evident by Art. 12 that the star idemfactor is real. Taking $N = 3$ with unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, it is easy to verify that the star idemfactor

$$I'' = \Sigma\left[\frac{1}{2}\,\mathbf{ii}(\mathbf{jj} + \mathbf{kk} - \mathbf{ii}) - (\mathbf{ijji} + \mathbf{jiij})\right] \qquad (93)$$

because $I''{}^*\,\mathbf{ii} = \mathbf{ii}$, $I''{}^*\,\mathbf{ij} = \mathbf{ij}$, etc. In fact since, as operators,

$$I''{}^* = I: \qquad (94)$$

both being identical operations, we must have

$$\mathbf{A}''_r{}^* = \mathbf{A}'_r: \qquad (95)$$

where $\mathbf{A}''_r$ and $\mathbf{A}'_r$ are respectively the star and dot reciprocals of $\mathbf{A}_r$ in the set of linearly independent dyadics $\mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_\nu$. Operating with both sides of (95) on the dot idemfactor gives

$$\mathbf{A}''_r{}^* I = \mathbf{A}'_r \tag{96}$$

or in words: the dot reciprocal is the star product of star reciprocal and dot idemfactor. Similarly by operating on the star idemfactor,

$$\mathbf{A}''_r = \mathbf{A}'_r : I'' \tag{97}$$

or: the star reciprocal is the dot product of dot reciprocal and star idemfactor. Thus the star reciprocal of $\mathbf{ii}$ is $\mathbf{ii} : I''$ or $\frac{1}{2}\,(\mathbf{jj} + \mathbf{kk} - \mathbf{ii})$, and the star reciprocal of $\mathbf{ij}$ is $-\mathbf{ji}$ etc.

Star multiplication may always be replaced by dot multiplication by introducing the double dyadic I^*I between the factors. For if $\mathbf{M}$ and $\mathbf{N}$ are any two dyadics we have $\mathbf{M} : I = \mathbf{M}$ and $I : \mathbf{N} = \mathbf{N}$, hence

$$\mathbf{M}^*\mathbf{N} = \mathbf{M} : I^*I : \mathbf{N} \tag{98}$$

We also observe that I^*I is the reciprocal (double dyadic) of I'' with respect to double dot multiplication, that is

$$I'' : I^*I = I''^*I = I \tag{99}$$

In a similar manner double dot multiplication can be changed to star by the operation $I'' : I''$ for

$$\mathbf{M} : \mathbf{N} = \mathbf{M}^*I'' : I''^*\mathbf{N} \tag{100}$$

and $I'' : I''$ is the reciprocal of I with respect to star multiplication,

$$I^*I'' : I'' = I : I'' = I'' \tag{101}$$

It is thus plain that, in spite of the imaginary character of star orthogonality, the reciprocals of a set of real dyadics are real, and the reciprocal, (or inverse), of a real double dyadic, if it exist, is also real.

15. ILLUSTRATION OF VARIOUS TYPES OF MULTIPLICATION.

We may close this discussion of the general properties of double dyadics by a few examples of the types of product considered. Consider the two tetrads $\mathbf{ijji}$ and $\mathbf{jiij}$. If they be taken as polyadics of order 4 and multiplied by multiple dot product we have

$$(\mathbf{ijji}) : (\mathbf{jiij}) = 0 \tag{102}$$

where colon means multiple dot product.

If they be taken as a pair of double dyadics the scalar (36) is

$$((\mathbf{ij} \mid \mathbf{ji}, \mathbf{ji} \mid \mathbf{ij}))_S = \mathbf{ij} : \mathbf{jiji} : \mathbf{ij} - \mathbf{ij} : \mathbf{ijji} : \mathbf{ji} = -1 \tag{103}$$

where the colon stands for double dot product, for we assume the case $K = 2$, hence might also have written $(\mathbf{ij} \mid \mathbf{ji})^*(\mathbf{ji} \mid \mathbf{ij})$ for this scalar.

If we assume $K = 1$ we return to the case of Art. 13 and 14 and have

$$(\mathbf{ij} \mid \mathbf{ji}, \mathbf{ji} \mid \mathbf{ij})^*_S = \mathbf{ij}^*\mathbf{jiji}^*\mathbf{ij} - \mathbf{ij}^*\mathbf{ijji}^*\mathbf{ji} = +1 \tag{104}$$

and finally if we had merely four dyads **ij**, **ji**, **ji**, and **ij**, the scalar

$$((\mathbf{ij}, \mathbf{ji}, \mathbf{ji}, \mathbf{ij}))_S = 0 \tag{105}$$

since all such scalars vanish unless the antecedents are distinct.

Again, we may form double and mixed products of double dyadics. Assuming $K = 1$ the star product written singly will refer to the dyadic factors. A double star will refer to antecedents and consequents as in the usual double product, thus if **A**, **M**, **B**, **N** are dyadics

$$\mathbf{AM}^{**}\mathbf{BN} = \mathbf{A}^*\mathbf{BM}^*\mathbf{N} \tag{106}$$

and we may have

$$\mathbf{AM}^*{:}\,\mathbf{BN} = \mathbf{A}^*\mathbf{BM}{:}\,\mathbf{N} \tag{107}$$

and

$$\mathbf{AM}{:}^*\mathbf{BN} = \mathbf{A}{:}\,\mathbf{BM}^*\mathbf{N} \tag{108}$$

while

$$\mathbf{AM}{:}\,{:}\,\mathbf{BN} = \mathbf{A}{:}\,\mathbf{BM}{:}\,\mathbf{N} \tag{109}$$

It is evident that, regarding the double dyadic as an element, we could start a fresh investigation of each of these types of product with respect to orthogonality, idemfactors, and so on ad inf. But we have now ample tools for the applications it is desired to make.

PART II. THE SOLUTION OF THE LINEAR MATRIX EQUATION.

16. Transformation from Matrices to Double Dyadics.

Any matrix A of order N is defined as a binary assemblage of elements a_{ik} where subscripts run from 1 to N. We add two matrices by adding corresponding elements. We multiply by the rule

$$(AB)_{ik} = \Sigma a_{is} b_{sk} \quad (s = 1, 2, \cdots, N) \tag{110}$$

If $A_1, A_2, \cdots, A_h$ and $B_1, B_2 \cdots, B_h$ and C are known matrices while x is a required matrix, the linear matrix equation may be written

$$\theta(x) = A_1 x B_1 + A_2 x B_2 + \cdots + A_h x B_h = C. \tag{111}$$

The number of terms h on the left is said to denote the *extent* of the operation θ or of the equation.

It is known that any matrix A is equivalent to a dyadic $\mathbf{A}$ or $\Sigma\mathbf{aa}'$ where $\mathbf{a}$ and $\mathbf{a}'$ are vectors. Any term $A\,x\,B$ is therefore equivalent to a sum of terms of the form $\mathbf{aa}'\cdot\mathbf{X}\cdot\mathbf{bb}'$ where $\mathbf{a}$, $\mathbf{a}'$, $\mathbf{b}$, and $\mathbf{b}'$ are vectors in space of N dimensions and $\mathbf{X}$ is a required dyadic. Since by the definition of double dot product we have $\mathbf{ab}:\mathbf{xy} = \mathbf{a}\cdot\mathbf{xb}\cdot\mathbf{y}$, and since $\mathbf{X}$ might be written $\Sigma\mathbf{xy}$ we shall have

$$\mathbf{aa}'\,\mathbf{X}\,\mathbf{bb}' = \mathbf{ab}'\mathbf{a}'\mathbf{b}:\mathbf{X} \tag{112}$$

which is the transformation at the basis of the method to be exhibited. Thus the linear matrix equation is equivalent to

$$\varphi:\mathbf{X} = \Sigma\mathbf{MN}:\mathbf{X} = C \tag{113}$$

where φ is a dyadic whose antecedents and consequents are themselves dyadics, that is a double dyadic.

Since $\varphi:\mathbf{X} = \theta(x)$, θ and φ satisfy the same Hamilton-Cayley equation. One method of solving the linear matrix equation is therefore by putting

$$x = \theta^{-1}C = \frac{1}{m_n}[\theta^{n-1} - m_1\theta^{n-2} + \cdots + (-1)^{n-1}m_{n-1}I]\,(-1)^{n-1}C \tag{114}$$

where $m_1\cdots m_n$ are the same coefficients as in (28) and where m_n is assumed not zero.

17. TRANSFORMATION BACK TO MATRICES.

If the matrices A and B are known in such form that transformation to (113) is convenient, any of the methods exhibited in Part I may be used to calculate the coefficients m. A general solution, however, demands ability to compute them directly from $A_1\cdots A_h$ and $B_1\cdots B_h$ without the necessity of first forming φ by the rule (112). In other words we need to transform the scalars (36) from the language of double dyadics to that of matrices, taking account of (112). This may be conveniently accomplished by a partly symbolic notation, omitting subscripts and summation signs which refer to the terms $1, 2, \cdots h$ of the matrix equation but occasionally preserving those which refer to the $p!$ terms of the expansion of (35). We thus write symbolically

$$\varphi = \mathbf{ab}' \mid \mathbf{a}'\mathbf{b} = \mathbf{MN} \tag{115}$$

and

$$\begin{aligned} m_1 &= \mathbf{ab}' : \mathbf{a}'\mathbf{b} \\ &= \mathbf{a} \cdot \mathbf{a}'\mathbf{b} \cdot \mathbf{b}' \\ &= A_S B_S \end{aligned} \tag{116}$$

where A_S and B_S are respectively the first coefficients in the Hamilton-Cayley equations for A and B; or, what is the same thing, they are the sums of elements along the main diagonal of these matrices. Stated in complete form the result is

$$m_1 = A_{1S}B_{1S} + A_{2S}B_{2S} + \cdots + A_{hS}B_{hS} \tag{117}$$

As remarked earlier, the symbolic process is valid by virtue of the distributive character of all the steps involved. Thus for m_2 we may put, in abbreviated language,

$$\begin{aligned} 2m_2 &= ((\varphi, \varphi))_S = ((\mathbf{ab}' \,|\, \mathbf{a}'\mathbf{b}, \mathbf{cd}' \,|\, \mathbf{c}'\mathbf{d}))_S \\ &= \mathbf{ab}' : \mathbf{a}'\mathbf{bcd}' : \mathbf{c}'\mathbf{d} - \mathbf{ab}' : \mathbf{c}'\mathbf{dcd}' : \mathbf{a}'\mathbf{b} \\ &= \mathbf{a} \cdot \mathbf{a}'\mathbf{b} \cdot \mathbf{b}'\mathbf{c} \cdot \mathbf{c}'\mathbf{d} \cdot \mathbf{d}' - \mathbf{a} \cdot \mathbf{c}'\mathbf{b}' \cdot \mathbf{dc} \cdot \mathbf{a}'\mathbf{d}' \cdot \mathbf{b} \end{aligned} \tag{118}$$

Now the first term on the right is the product of the scalars of the four dyadics or matrices which enter; that is, it is symbolically $A_S B_S C_S D_S$ or $A_{iS} B_{iS} A_{jS} B_{jS}$. The second term may be further transformed

$$\begin{aligned} \mathbf{a} \cdot \mathbf{c}'\mathbf{b}' \cdot \mathbf{dc} \cdot \mathbf{a}'\mathbf{d}' \cdot \mathbf{b} &= [\mathbf{a} \cdot \mathbf{c}'\mathbf{a}' \cdot \mathbf{c}]\,[\mathbf{b} \cdot \mathbf{d}'\mathbf{b}' \cdot \mathbf{d}] \\ &= [\mathbf{aa}' \cdot \mathbf{cc}']_S\,[\mathbf{bb}' \cdot \mathbf{dd}']_S \\ &= [AC]_S\,[BD]_S \\ &= [A_i A_j]_S\,[B_i B_j]_S \end{aligned} \tag{119}$$

where $[A_i A_j]_S$ is the first coefficient in the Hamilton-Cayley equation for the matrix $A_i A_j$, that is, for the product of A_i and A_j, it is the sum of elements on the main diagonal of this product; and similarly for $[B_i B_j]_S$. Stated in complete form the result is

$$2m_2 = \Sigma[A_{iS}B_{iS}A_{jS}B_{jS} - (A_i A_j)_S (B_i B_j)_S], \quad [i, j = 1, 2, \cdot \cdot, h] \tag{120}$$

the summation being now with respect to all possible pairs of terms selected from the left side of the original equation (111). It is to be particularly noted that the summation includes the cases $i = j$; in other words, by "pairs of terms" is meant not merely different terms but also the same term taken twice. The reason is not far to seek. By the transformation (112) any term AxB was changed into a sum of terms of the form **MN**: **X**, but not, in general, into a single such term. It follows that, although $((\mathbf{MN}, \mathbf{MN}))_S$, which is the star product of a single term **MN** into itself, must vanish, yet a scalar $A^2{}_S B^2{}_S - (A^2)_S (B^2)_S$ *symbolically derived from it* does not in general vanish.

This fact will be fully exemplified below. A symbolic equation implies, not the equality of individual terms each to each, but equality of the sums of all possible terms of the equated forms.

Following a similar procedure to find m_3 we have

$$\begin{aligned}6m_3 &= ((\varphi, \varphi, \varphi))_S = ((\mathbf{ab}' \mid \mathbf{a}'\mathbf{b}, \mathbf{cd}' \mid \mathbf{c}'\mathbf{d}, \mathbf{ef}' \mid \mathbf{e}'\mathbf{f}))_S, \text{ symbolically,}\\ &= \mathbf{a}\cdot\mathbf{a}'\mathbf{b}\cdot\mathbf{b}'\mathbf{c}\cdot\mathbf{c}'\mathbf{d}\cdot\mathbf{d}'\mathbf{e}\cdot\mathbf{e}'\mathbf{f}\cdot\mathbf{f}' - \mathbf{a}\cdot\mathbf{a}'\mathbf{b}\cdot\mathbf{b}'[\mathbf{c}\cdot\mathbf{e}'\mathbf{c}'\cdot\mathbf{e}]\,[\mathbf{d}'\cdot\mathbf{fd}\cdot\mathbf{f}']\\ &\quad - \mathbf{c}\cdot\mathbf{c}'\mathbf{d}\cdot\mathbf{d}'[\mathbf{e}\cdot\mathbf{a}'\mathbf{e}'\cdot\mathbf{a}]\,[\mathbf{f}'\cdot\mathbf{bf}\cdot\mathbf{b}'] - \mathbf{e}\cdot\mathbf{e}'\mathbf{f}\cdot\mathbf{f}'[\mathbf{a}\cdot\mathbf{c}'\mathbf{a}'\cdot\mathbf{c}]\,[\mathbf{b}'\cdot\mathbf{db}\cdot\mathbf{d}']\\ &\quad + (\mathbf{ab}':\mathbf{c}'\mathbf{d})\,(\mathbf{cd}':\mathbf{e}'\mathbf{f})\,(\mathbf{ef}':\mathbf{a}'\mathbf{b}) + (\mathbf{ab}':\mathbf{e}'\mathbf{f})\,(\mathbf{cd}':\mathbf{a}'\mathbf{b})\,(\mathbf{ef}':\mathbf{c}'\mathbf{d})\end{aligned} \tag{121}$$

by direct application of the definition of Art. 5, and by grouping the dot products in the second, third, and fourth terms after the manner of (119). The first or leading term is evidently the same symbolically as $A_{iS}B_{iS}A_{jS}B_{jS}A_{kS}B_{kS}$. The next three terms taken together are symbolically $-\Sigma A_{iS}B_{iS}(A_jA_k)_S(B_jB_k)_S$. In transforming the fifth term the procedure, as in all cases, will be to group into one factor those vectors which correspond to prefactors A in the original equation, and into another factor those which correspond to the postfactors B. Thus

$$\begin{aligned}(\mathbf{ab}':\mathbf{c}'\mathbf{d})\,(\mathbf{cd}':\mathbf{e}'\mathbf{f})\,(\mathbf{ef}':\mathbf{a}'\mathbf{b}) &= (\mathbf{a}\cdot\mathbf{c}'\mathbf{c}\cdot\mathbf{e}'\mathbf{e}\cdot\mathbf{a}')\,(\mathbf{b}'\cdot\mathbf{dd}'\cdot\mathbf{ff}'\cdot\mathbf{b})\\ &= (\mathbf{aa}'\cdot\mathbf{ee}'\cdot\mathbf{cc}')_S(\mathbf{bb}'\cdot\mathbf{dd}'\cdot\mathbf{ff}')_S\\ &= (A_iA_kA_j)_S(B_iB_jB_k)_S\end{aligned} \tag{122}$$

It is especially worthy of remark that the order of the matrices in one factor is the reverse of that in the other, a consequence of the transformation (112). When there are only two matrices in each factor, as in (119), it was of no importance whether we wrote $(AC)_S$ or $(CA)_S$, for these are equal: the scalar of the product of two matrices is independent of their order, because $(\mathbf{aa}'\cdot\mathbf{cc}')_S = \mathbf{a}\cdot\mathbf{c}'\mathbf{a}'\cdot\mathbf{c} = (\mathbf{cc}'\cdot\mathbf{aa}')$. But by similar reasoning we see that the scalar of the product of several matrices depends on their cyclic order, as is well known.

In the same way the last term of (121) becomes $(A_iA_jA_k)_S(B_iB_kB_j)_S$. Collecting results, the complete statement for m_3 is

$$\begin{aligned}6m_3 = A_{iS}A_{jS}A_{kS}B_{iS}B_{jS}B_{kS} &- A_{iS}B_{iS}(A_jA_k)_S(B_jB_k)_S -\\ &A_{jS}B_{jS}(A_kA_i)_S(B_kB_i)_S\\ - A_{kS}B_{kS}(A_iA_j)_S(B_iB_j)_S &+ (A_iA_kA_j)_S(B_iB_jB_k)_S +\\ &(A_iA_jA_k)_S(B_iB_kB_j)_S\end{aligned} \tag{123}$$

which is to be summed over all possible sets of three terms from the h terms of the original equation (111), allowing repetitions for the reason already noted under m_2; that is $i, j, k = 1, 2, \cdot\cdot, h$. Such a summation will hereafter be referred to as *summation over the extent of the equation*,[12] and is implied in every symbolic equation.

In general we may express m_p symbolically by the equation

$$p!m_p = ((\mathbf{a}_1\mathbf{b}'_1 \mid \mathbf{a}'_1\mathbf{b}_1, \mathbf{a}_2\mathbf{b}'_2 \mid \mathbf{a}'_2\mathbf{b}_2, \cdots, \mathbf{a}_p\mathbf{b}'_p \mid \mathbf{a}'_p\mathbf{b}_p))_S \tag{124}$$

Expanding by the definition of Art. 5, the leading term is the product of p factors of the form $\mathbf{a}_r\mathbf{b}'_r : \mathbf{a}'_r\mathbf{b}_r$, because the leading term is made without interchanges among the consequents. These factors are the same as $\mathbf{a}_r\cdot\mathbf{a}'_r\mathbf{b}_r\cdot\mathbf{b}'_r$ and symbolically the same as $A_{rS}B_{rS}$. Thus if we have a set of p subscripts $i, j, k, \cdots, r, s, t$, the leading term in the development of $p!m_p$ is

$$A_{iS}B_{iS}A_{jS}B_{jS}A_{kS}B_{kS}\cdots A_{rS}B_{rS}A_{sS}B_{sS}A_{tS}B_{tS} \tag{125}$$

which is to be summed over the extent of the equation (11).

The other terms in the expansion of (124) are obtained by making all possible interchanges among the consequents, according to the definition of Art. 5. Therefore so long as we maintain the polyadic notation any term is a product of factors of the form $\mathbf{a}_r\mathbf{b}'_r : \mathbf{a}'_s\mathbf{b}_s$, which is the same as $\mathbf{a}_r\cdot\mathbf{a}'_s\mathbf{b}'_r\cdot\mathbf{b}_s$. The vectors $\mathbf{a}_r$ and $\mathbf{a}'_s$ might correspond to different matrices, likewise $\mathbf{b}'_r$ and $\mathbf{b}_s$. Hence such a factor cannot in general be translated into matrix notation if considered by itself. For it is of the essence of this transformation that every vector $\mathbf{a}_r$ be associated with its mate $\mathbf{a}'_r$ and likewise every $\mathbf{b}_r$ with $\mathbf{b}'_r$; then $\mathbf{a}_r\mathbf{a}'_r$ is symbolically equivalent to A_r and $\mathbf{b}_r\mathbf{b}'_r$ to B_r. If, however, by a simple interchange of a pair of consequents in the polyadic expression, we obtain a pair of factors $(\mathbf{a}_r\mathbf{b}'_r : \mathbf{a}'_s\mathbf{b}_s)$ $(\mathbf{a}_s\mathbf{b}'_s : \mathbf{a}'_r\mathbf{b}_r)$, these may be developed as in (118) and (119) and yield the two factors $(A_rA_s)_S(B_rB_s)_S$. Such factors occurred already in the second term of m_2 and in the second, third, and fourth terms of m_3.

If three consequents change places among themselves, we obtain a product of three factors of the form $(\mathbf{a}_r\mathbf{b}'_r : \mathbf{a}'_s\mathbf{b}_s)$ $(\mathbf{a}_s\mathbf{b}'_s : \mathbf{a}'_t\mathbf{b}_t)$ $(\mathbf{a}_t\mathbf{b}'_t : \mathbf{a}'_r\mathbf{b}_r)$. It is important to notice that *the order of subscripts in the vectors* **a** *is the same as that in the vectors* **b** *but the order of accents is reversed.* It follows that when we develop after the manner of (122) the vectors **a** yield the product of dot products $\mathbf{a}_r\cdot\mathbf{a}'_s\mathbf{a}_s\cdot\mathbf{a}'_t\mathbf{a}_t, \mathbf{a}'_r$, while the vectors **b** yield the product $\mathbf{b}'_r\cdot\mathbf{b}_s\mathbf{b}'_s\cdot\mathbf{b}_t\mathbf{b}'_t, .\mathbf{b}_r$, *where the order of subscripts is the same, but the accents are on the first vector in each dot product, instead of the second.* These factors are equivalent in matrix form to $(A_rA_tA_s)_S$ and $(B_rB_sB_t)_S$, where the *order of sub-*

scripts in the postfactors B is the same as in the polyadic term, but the order of subscripts in the prefactors A is the reverse of the order in the postfactors B. For in the polyadic expression $\mathbf{b}'_s$ follows its mate $\mathbf{b}_s$, and $\mathbf{b}'_t$ follows $\mathbf{b}_t$, and if we rewrite the factor in the form $(\mathbf{b}_r\mathbf{b}'_r\cdot\mathbf{b}_s\mathbf{b}'_s\cdot\mathbf{b}_t\mathbf{b}'_t)_S$ then $\mathbf{b}'_r$ also follows its mate $\mathbf{b}_r$, whence directly the matrix factor $(B_rB_sB_t)_S$. But in the polyadic expression as it stands, each $\mathbf{a}'$, when we follow the same steps, *precedes* its mate on account of the reversal of the order of accents. Thence follows the reversal of order of matrices in the corresponding factor. Transformations of this type already occurred in the last two terms of m_3.

All the italicized statements in the above discussion are true no matter how many consequents have changed places among themselves. We may therefore write a general rule for the formation of m_p.

Rule for forming $p!m_p$ where m_p is the coefficient of θ^p in the Hamilton-Cayley equation for θ.

Let there be p subscripts $i, j, k, \cdots, r, s, t$, each of which may have any value from 1 to h. Choosing a particular set of values for these subscripts, we form a group of $p!$ terms as follows: the leading term is $A_{iS}B_{iS}A_{jS}B_{jS}A_{kS}B_{kS}\cdots A_{rS}B_{rS}A_{sS}B_{sS}A_{tS}B_{tS}$. The other terms are formed from the leading term by first interchanging the postfactors B in all ways, while the prefactors A are at first left fixed in position. If a particular postfactor B is left in position, it yields in the corresponding term a factor B_S and its prefactor yields A_S *precisely as in the leading term.* If a pair of postfactors as B_r and B_q change places, there results in the corresponding term a factor $(A_rA_q)_S(B_qB_r)_S$. If three postfactors, as B_k, B_r, and B_s, change places so that their new order is B_r, B_s, B_k, there results in the corresponding term a factor $(A_kA_sA_r)_S(B_rB_sB_k)_S$ *where the order of prefactors is the reverse of the final order of the postfactors.* In general if any group of postfactors change places among themselves so that their final order is

$$B_kB_r\cdots B_sB_j$$

there results in the corresponding term a factor $(A_jA_s\cdots A_rA_k)_S(B_kB_r\cdots B_sB_j)_S$ where the order of prefactors is the reverse of the final order of postfactors.

The group of $p!$ terms thus obtained is to be summed over the extent of the linear matrix equation.

18. CHARACTER OF THE COEFFICIENTS AS ALGEBRAIC POLYNOMIALS.

It is evident from the form of the leading term that in every term of the expansion of m_p will be found p prefactors and p postfactors, whether alike or different. Hence m_p is homogeneous and of degree $2p$ in the scalar elements of the given matrices which define θ.

Furthermore m_p is non-homogeneous and of degree p (in general) in the scalar elements of any particular matrix. For if the p subscripts which occur in a group of terms are all alike, the p prefactors and the p postfactors are all alike.

Again, each group of terms, corresponding to a particular choice of p subscripts, is homogeneous in the scalar elements of each of the matrices which occur in the group. The degree in any one matrix is determined by the number of times its subscript has been chosen in making up p subscripts. The degree in A_r is the same as that in B_r.

We thus see that each group of terms formed by the above rule is made up of terms essentially *unlike* the terms of all the other groups, each group, in other words, constitutes a distinct polynomial in the scalar elements of certain of the given matrices. Within the group, however, various terms may be like one another, and simplifications may occur, particularly when special values are assigned to some of the matrices; for example, when some of them are allowed to become the idemfactor.

19. The Equation of Extent Unity and Order Two.

As an example, closely allied to that of Art. 10, let us take all given matrices to be of the second order, and, to begin with, let $h = 1$, so that the linear matrix equation reduces to the simple form

$$AxB = C \tag{126}$$

and can be at once solved as $x = A^{-1}CB^{-1}$. Forming the m_p by our rule we have $m_1 = A_SB_S$ which is $(a_{11} + a_{22})\ (b_{11} + b_{22})$. Next $2m_2 = A^2{}_SB^2{}_S - (A^2)_S(B^2)_S$.

The scalar $(A^2)_S$ may be expressed in terms of simpler quantities by using the Hamilton-Cayley equation for the matrix A. Let the coefficients in this equation be A_S and A''_S so that

$$A^2 = A_SA - A''_SI \tag{127}$$

where I is the identical matrix of the second order. Taking the scalar of both sides of the equation we have

$$(A^2)_S = A^2{}_S - 2A''_S \tag{128}$$

and a similar equation for $(B^2)_S$. Substituting in the expression for $2m_2$ we find on simplifying

$$m_2 = A^2_S B''_S - 2A''_S B''_S + A''_S B^2_S \qquad (129)$$

Since the matrices are of the second order, A''_S and B''_S are the determinants of their respective matrices; hence on substituting,

$$\begin{aligned} m_2 = {} & (a_{11} + a_{22})^2 (b_{11}b_{22} - b_{12}b_{21}) - 2(a_{11}a_{22} - a_{12}a_{21}) (b_{11}b_{22} - b_{12}b_{21}) \\ & + (a_{11}a_{22} - a_{12}a_{21}) (b_{11} + b_{22})^2 \\ = {} & (a^2_{11} + a^2_{22}) (b_{11}b_{22} - b_{12}b_{21}) + (b^2_{11} + b^2_{22}) (a_{11}a_{22} - a_{12}a_{21}) \\ & \quad + 2a_{11}a_{22}b_{11}b_{22} - 2a_{12}a_{21}b_{12}b_{21} \qquad (130) \end{aligned}$$

Again

$$6m_3 = A^3_S B^3_S - 3A_S B_S (A^2)_S (B^2)_S + 2(A^3)_S (B^3)_S \qquad (131)$$

by the rule. But $A^3 = A_S A^2 - A''_S A$ hence

$$(A^3)_S = A^3_S - 3A_S A''_S \qquad (132)$$

by (128), and similarly for B. Substituting values in (131) from (128) and (132) to get rid of scalars of powers of matrices we find

$$\begin{aligned} m_3 &= A_S B_S A''_S B''_S \qquad (133) \\ &= (a_{11} + a_{22}) (b_{11} + b_{22}) (a_{11}a_{22} - a_{12}a_{21}) (b_{11}b_{22} - b_{12}b_{21}) \end{aligned}$$

and finally $24m_4 = A^4_S B^4_S - 6A^2_S B^2_S (A^2)_S (B^2)_S + 3(A^2)^2_S (B^2)^2_S$

$$+ 8A_S B_S (A^3)_S (B^3)_S - 6(A^4)_S (B^4)_S \qquad (134)$$

But $A^4 = A_S A^3 - A''_S A^2$ hence

$$(A^4)_S = A^4_S - 4A^2_S A''_S + 2A''^2_S \qquad (135)$$

so that by eliminating the scalars of powers from (134)

$$m_4 = A''^2_S B''^2_S \qquad (140)$$

Now in the present example $\theta^2 C = A^2CB^2$, $\theta^3 C = A^3CB^3$, etc. Hence

$$\begin{aligned} x = \theta^{-1}C = \frac{1}{A''^2_S B''^2_S} [& A_S B_S A''_S B''_S C - (A^2_S B''_S - 2A''_S B''_S \\ & + A''_S B^2_S) ACB + A_S B_S A^2CB^2 - A^3CB^3] \qquad (141) \end{aligned}$$

Simplifying by the aid of the Hamilton-Cayley equations for A and B,

$$\begin{aligned} x &= \frac{1}{A''_S B''_S} [A_S B_S C - B_S AC - A_S BC + ACB] \\ &= \frac{1}{A''_S B''_S} [A_S I - A] C [B_S I - B] = A^{-1}CB^{-1}, \end{aligned}$$

checking with the known solution.

20. The Equation of Extent Two and Order Two.

The following considerations will suggest how the work of solution may be arranged when the equation is of higher extent. We have seen that any coefficient m_p is a sum of groups of terms. Each group is a homogeneous polynomial. Suppose a choice of subscripts in which i occurs a times, j occurs b times, etc. Let the development of terms, under the rule, corresponding to this choice of subscripts, be denoted by $G(i^a j^b \cdots)$. Each choice of subscripts will occur a number of times equal to the coefficient of the corresponding term in the expansion of

$$(i + j + k + \cdots + r + s + t)^p. \tag{142}$$

Thus the entire development may be systematically carried out.

For example, take

$$A_i x B_i + A_j x B_j = C \tag{143}$$

and form m_4. With the above notation we shall have

$$m_4 = G(i^4) + 4G(i^3 j) + 6G(i^2 j^2) + 4G(ij^3) + \mathrm{G}(j^4) \tag{144}$$

By the preceding example we have $\mathrm{G}(i^4) = A''{}_i^2{}_S B''{}_i^2{}_S$ and similarly for $G(j^4)$. By the rule we have

$$\begin{aligned} G(i^3 j) = {} & A^3{}_{iS} A_{jS} B^3{}_{iS} B_{jS} - 3A^2{}_{iS}(A_i A_j)_S B^2{}_{iS}(B_i B_j)_S \\ & - 3A_{iS} A_{jS}(A^2{}_i)_S B_{iS} B_{jS}(B^2{}_i) + 3(A^2{}_i)_S(A_i A_j)_S(B^2{}_i)_S(B_i B_j)_S \\ & + 6A_{iS}(A^2{}_i A_j)_S B_{iS}(B^2{}_i B_j)_S + 2A_{jS}(A^3{}_i)_S B_{jS}(B^3{}_i)_S \\ & - 6\,(A^3{}_i A_j)_S(B^3{}_i B_j)_S \end{aligned} \tag{145}$$

Brevity will be gained in notation, while nothing is lost in explicitness, if, in such expressions, we indicate only the matrices A which are prefactors in the original equation, remembering that the order of subscripts among postfactors, when more than two matrices are multiplied, is the reverse of that for prefactors. With this understanding we may also omit the letter A and the subscript S. Thus (145) may be abbreviated

$$G(i^3 j) = i^3 j - 3i^2(ij) - 3ij(i^2) + 3(i^2)\,(ij) + 6i(i^2 j) + 2j(i^3) - 6(i^3 j) \tag{145a}$$

As a check, if A_j and B_j are replaced by the idemfactor of the second order, $G(i^3 j)$ should reduce to $6A_S B_S A''{}_S B''{}_S$ because m_4 becomes the determinant of $A(\ )B + (\ )$, and the terms of the third degree in A

must be the coefficient m_3 for the example of the last article. In fact we then have, remembering $I_S = 2$,

$$G(i^3j) = 4A^3{}_S B^3{}_S - 3A^3{}_S B^3{}_S - 12A_S(A^2)_S B_S(B^2)_S + 3A_S(A^2)_S B_S(B^2)_S$$
$$+ 6A_S(A^2)_S B_S(B^2)_S + 8(A^3)_S(B^3)_S - 6(A^3)_S(B^3)_S$$

which is the same as (131).

Developments like (145) are the same in form no matter what the order of the matrices involved. They may in general be simplified by the use of the Hamilton-Cayley equations for A and B.

$G(ij^3)$ may be obtained from $G(i^3j)$ by interchange of subscripts. For the middle term of (144) we have

$$G(i^2j^2) = i^2j^2 - i^2(j^2) - j^2(i^2) - 4ij(ij) + (i^2)(j^2) + 2(ij)^2 + 4i(ij^2)$$
$$+ 4j(i^2j) - 2(ijij) - 4(i^2j^2) \qquad (146)$$

which by letting A_j and B_j be I of the second order should become $2A^2{}_S B^2{}_S - 2(A^2)_S(B^2)_S$ as is easily verified.

In simplifying, use is to be made of the identity

$$(A_iA_j)_S = A_{iS}A_{jS} - A^*{}_iA_j \qquad (147)$$

where the last term is the star product of A_i and A_j as already defined. Since this scalar is an invariant of the two matrices it may well be abbreviated $A^*{}_{ij}$. Collecting and reducing results we find

$$m_4 = A''^2{}_{iS}B''^2{}_{iS} + A''{}_{iS}B''{}_{iS}A^*{}_{ij}B^*{}_{ij} + A''{}_{iS}A''{}_{jS}B^{*2}{}_{ij}$$
$$+ B''{}_{iS}B''{}_{jS}A^{*2}{}_{ij} - 2A''{}_{iS}A''{}_{jS}B''{}_{iS}B''{}_{jS} + A''{}_{jS}B''{}_{jS}A^*{}_{ij}B^*{}_{ij}$$
$$+ A''^2{}_{jS}B''^2{}_{jS} \qquad (148)$$

and by similar processes,

$$m_3 = A_{iS}A''{}_{iS}B_{iS}B''{}_{iS} + A_{iS}B_{jS}B''{}_{iS}A^*{}_{ij} + A_{jS}B_{iS}A''{}_{iS}B^*{}_{ij}$$
$$- A_{jS}B_{jS}A''{}_{iS}B''{}_{iS} + A_{jS}B_{jS}A''{}_{jS}B''{}_{jS} + A_{jS}B_{iS}B''{}_{jS}A^*{}_{ijS}$$
$$+ A_{iS}B_{jS}A''{}_{jS}B^*{}_{ij} - A_{iS}B_{iS}A''{}_{jS}B''{}_{jS} \qquad (149)$$

$$m_2 = A^2{}_{iS}B''{}_{iS} - 2A''{}_{iS}B''{}_{iS} + A''{}_{iS}B^2{}_{iS} + A_{iS}A_{jS}B^*{}_{ij} - A^*{}_{ij}B^*{}_{ij}$$
$$+ B_{iS}B_{jS}A^*{}_{ij} + A^2{}_{jS}B''{}_{jS} - 2A''{}_{jS}B''{}_{jS} + A''{}_{jS}B^2{}_{jS} \qquad (150)$$

$$m_1 = A_{iS}B_{iS} + A_{jS}B_{jS} \qquad (151)$$

whence (143) is completely solved.

[1] This symbolic method is due to H. B. Phillips: *Some invariants and co-variants of ternary collineations*," American Journal of Mathematics, **36**, 1914.

[2] Hamilton, *Lectures on Quaternions*, 1853, *Cayley*, *A Memoir on the theory of matrices*, 1858.

[3] E. B. Wilson, *On the theory of double products and strains in hyperspace*. Conn. Acad. Trans. **14**, 1908.

[4] C. L. E. Moore and H. B. Phillips, *The dyadics which occur in a point-space of three dimensions*, Proc. Amer. Acad. of Arts and Sci. **53**, 1918.

[5] For the elementary theory, Bocher's Algebra may be consulted.

[6] See note 3.

[7] For the laws of *p-way determinants* see Amer. Journal of Math. **40**, 1918, by Lepine Hall Rice.

[8] Compare Joly's Appendix to Hamilton's *Elements of Quaternions*.

[9] This result is an extension of Hamilton's invariant property of his coefficients.

[10] *Elements of Quaternions*, 2nd Ed., Vol. I, Art. 348.

[11] In a series of papers over many years. They are all, I think, listed in the bulletins of the Quaternion Association. See, in particular, Wien. Ber. **112**, 1903, pp. 645, 1091, and 1533.

[12] The term "extent" is due to Sylvester. I have elsewhere given a sketch of the present method in its relation to the work of Sylvester. Proc. Nat. Acad. of Sci. **8**, April, 1922.

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IDENTITIES SATISFIED BY ALGEBRAIC POINT FUNCTIONS IN N-SPACE.

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1. The Fundamental Identity.

Let $F(\mathbf{x})$ denote a homogeneous polynomial of degree K in N variables $x_1, x_2, \cdots x_N$. Let the number of terms in this polynomial be n, that is

$$n = (K+1)(K+2)(K+3)\cdots(K+N-1)/1.2.3.\cdots(N-1). \quad (1)$$

Let $F(\mathbf{a}_i)$ be the value of the polynomial when a set of values a_{i1}, $a_{i2}, \cdots a_{iN}$ is assigned to the variables $x_1, x_2, \cdots x_N$. We say that $F(\mathbf{a}_i)$ is the value of the polynomial at the point $\mathbf{a}_i$.

As a temporary notation, let the coefficients of the various terms of the polynomial be $A_1, A_2, \cdots A_n$, with the terms written in some determined order. We may suppose this order to be descending order of x_1, x_2, etc., meaning that the term containing x_1^K is the leading term, followed by all terms containing x_1^{K-1}, then all terms containing x_1^{K-2} and so on; and that in each of these groups we follow descending order of x_2 by a similar rule, and so on. Thus

$$F(\mathbf{x}) = A_1 x_1^K + A_2 x_1^{K-1} x_2 + A_3 x_1^{K-1} x_3 + \cdots + A_n x_N^K. \quad (2)$$

Now let there be a set of n points $\mathbf{a}_1, \mathbf{a}_2, \cdots \mathbf{a}_i \cdots \mathbf{a}_n$. If we expand the value of the polynomial at each of these points in the form (2) we shall have n equations of the form

$$F(\mathbf{a}_i) = A_1 a_{i1}^K + A_2 a_{i1}^{K-1} a_{i2} + A_3 a_{i1}^{K-1} a_{i3} + \cdots + A_n a_{iN}^K, \quad (3)$$

which, together with (2), constitutes a set of $n + 1$ linear homogeneous equations in the $n + 1$ quantities $A_1, A_2, \cdots A_n$ and unity. It is accordingly evident that the determinant

$$\begin{vmatrix} F(\mathbf{x}), & x_1^K, & x_1^{K-1}x_2, & \cdots, & x_N^K \\ F(\mathbf{a}_1), & a_{11}^K, & a_{11}^{K-1}a_{12}, & \cdots, & a_{1N}^K \\ F(\mathbf{a}_2), & a_{21}^K, & a_{21}^{K-1}a_{22}, & \cdots, & a_{2N}^K \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ F(\mathbf{a}_n), & a_{n1}^K, & a_{n1}^{K-1}a_{n2}, & \cdots, & a_{nN}^K \end{vmatrix} \tag{4}$$

must vanish. Developing by the elements of the first column we may write

$$F(\mathbf{x})C_0 + F(\mathbf{a}_1)C_1 + F(\mathbf{a}_2)C_2 + \cdots + F(\mathbf{a}_n)C_n = 0 \tag{5}$$

where C_0, C_1, etc. are the cofactors of the elements into which they are multiplied.

If $\mathbf{x}$ be variable, with $\mathbf{a}_1, \mathbf{a}_2, \cdots \mathbf{a}_n$ constant, the identity (5) will be the expansion of the polynomial $F(\mathbf{x})$ in terms of its values at n fixed points, provided C_0 does not vanish. The coefficients $C_1, C_2, \cdots C_n$ are polynomials of degree K in $\mathbf{x}$. They are also polynomials of degree K in each of the fixed points with exception of the point having the same subscript: C_i is independent of $\mathbf{a}_i$. On the other hand C_0 is a constant in the sense of being independent of $\mathbf{x}$. The C's are all independent of the special choice of the polynomial $F(\mathbf{x})$, that is of $A_1, A_2, \cdots A_n$.

Let us next take $\mathbf{e}_1, \mathbf{e}_2, \cdots \mathbf{e}_N$ a system of linearly independent vectors. Any point $\mathbf{x}$ may be regarded as a point vector,

$$\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_N\mathbf{e}_N \tag{6}$$

and in a similar manner

$$\mathbf{a}_i = a_{i1}\mathbf{e}_1 + a_{i2}\mathbf{e}_2 + \cdots + a_{iN}\mathbf{e}_N. \tag{7}$$

Let us furthermore take a system of N polynomials of degree K, namely $F_1(\mathbf{x}), F_2(\mathbf{x}), \cdots F_N(\mathbf{x})$, and write

$$\mathbf{F}(\mathbf{x}) = \mathbf{e}_1F_1(\mathbf{x}) + \mathbf{e}_2F_2(\mathbf{x}) + \cdots + \mathbf{e}_NF_N(\mathbf{x}). \tag{8}$$

By definition, therefore, $\mathbf{F}(\mathbf{x})$ is a vector function of the point vector $\mathbf{x}$, homogeneous of degree K in the variables $x_1, x_2, \cdots x_N$. Vector functions are distinguished by bold-faced type.

Each of the scalar functions which are thus assigned as components of the vector $\mathbf{F}(\mathbf{x})$ will satisfy an identity of the form (5). We have then a set of N identities

$$\begin{array}{l}
F_1(\mathbf{x})C_0 + F_1(\mathbf{a}_1)C_1 + F_1(\mathbf{a}_2)C_2 + \cdots + F_2(\mathbf{a}_n)C_n = 0 \\
F_2(\mathbf{x})C_0 + F_2(\mathbf{a}_1)C_1 + F_2(\mathbf{a}_2)C_2 + \cdots + F_2(\mathbf{a}_n)C_n = 0 \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
F_N(\mathbf{x})C_0 + F_N(\mathbf{a}_1)C_1 + F_N(\mathbf{a}_2)C_2 + \cdots + F_N(\mathbf{a}_n)C_n = 0
\end{array}$$

wherein we assume that the fixed points $\mathbf{a}_1, \mathbf{a}_2, \cdots \mathbf{a}_n$, while wholly arbitrary, are constant for all the identities of the set. Hence the coefficients $C_0, C_1, C_2, \cdots C_n$ are unaltered as we pass from one identity to another. If, then, we multiply these identities in turn by $\mathbf{e}_1, \mathbf{e}_2, \cdots \mathbf{e}_N$, and add the results, we shall have the vector identity

$$\mathbf{F}(\mathbf{x})C_0 + \mathbf{F}(\mathbf{a}_1)C_1 + \mathbf{F}(\mathbf{a}_2)C_2 + \cdots + \mathbf{F}(\mathbf{a}_n)C_n = 0. \tag{9}$$

It is obvious that the left member of this identity may, if we wish, be written in the form of the determinant (4), the elements of the first column being now vectors.

Just as with the scalar identity (5), we may regard the vector identity (9) in two ways,— either as the statement of the linear relation connecting the values of the vector polynomial $\mathbf{F}(\mathbf{x})$ at $n + 1$ arbitrary points; or as the expansion of this polynomial in terms of its values at n points arbitrary except that C_0 does not vanish. The vector function $\mathbf{F}(\mathbf{x})$ depends on N scalar polynomials, each containing n arbitrary numerical coefficients. Whenever these coefficients need to be separately considered they may be designated according to the matrix

$$\begin{array}{lllll}
A_{11}, & A_{21}, & A_{31}, & \cdots, & A_{n1} \\
A_{12}, & A_{22}, & A_{32}, & \cdots, & A_{n2} \\
A_{13}, & A_{23}, & A_{33}, & \cdots, & A_{n3} \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
A_{1N}, & A_{2N}, & A_{3N}, & \cdots, & A_{nN}
\end{array}$$

Thus the set of numbers A_{ij}, like the set a_{ij}, contains Nn elements.

We may now state the following simple but fundamental theorem:

Theorem I. *The values taken on by an arbitrary vector polynomial at* $n + 1$ *arbitrary points are linearly related. The multipliers* $C_0, C_1, C_2, \cdots C_n$ *are functions of the arbitrary points, but are independent of the coefficients* A_{ij} *of the terms of the polynomial.*

2. GEOMETRIC MEANING OF THE COEFFICIENTS.

To take a specific example, let $K = 1$ and $N = 3$. The vector determinant of the form (4) becomes

$$\begin{vmatrix} \mathbf{F}(\mathbf{x}), & x_1, & x_2, & x_3 \\ \mathbf{F}(\mathbf{a}_1), & a_{11}, & a_{12}, & a_{13} \\ \mathbf{F}(\mathbf{a}_2), & a_{21}, & a_{22}, & a_{23} \\ \mathbf{F}(\mathbf{a}_3), & a_{31}, & a_{32}, & a_{33} \end{vmatrix}$$

The coefficients C_0, C_1, C_2, and C_3 become determinants of the components of three vectors. If we denote these determinants by (123), (x23), (x13), and (x12), the identity (9) becomes

$$\mathbf{F}(\mathbf{x})\,(123) - \mathbf{F}(\mathbf{a}_1)\,(x23) + \mathbf{F}(\mathbf{a}_2)\,(x13) - \mathbf{F}(\mathbf{a}_3)\,(x12) = 0, \quad (10)$$

a relation very familiar to students of vector analysis, $\mathbf{F}(\mathbf{x})$ being now a linear vector function. When $K = 1$ a similar relation holds for any value of N, whence clearly the vanishing of any C means coplanarity of the vectors which enter into it.

Next take $K = 2$ and $N = 3$. The vector determinant of the form (4) becomes

$$\begin{vmatrix} \mathbf{F}(\mathbf{x}), & x_1^2, & x_1x_2, & x_1x_3, & x_2^2, & x_2x_3, & x_3^2 \\ \mathbf{F}(\mathbf{a}_1), & a_{11}^2, & a_{11}a_{12}, & a_{11}a_{13}, & a_{12}^2, & a_{12}a_{13}, & a_{13}^2 \\ \mathbf{F}(\mathbf{a}_2), & a_{21}^2, & a_{21}a_{22}, & a_{21}a_{23}, & a_{22}^2, & a_{22}a_{23}, & a_{23}^2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \mathbf{F}(\mathbf{a}_6), & a_{61}^2, & a_{61}a_{62}, & a_{61}a_{63}, & a_{62}^2, & a_{62}a_{63}, & a_{63}^2 \end{vmatrix} \quad (11)$$

In this case the vanishing of one of the coefficients C means that the six vectors which enter into it lie on a quadric cone, (which may, however, be a degenerate cone). For the equation $C_i = 0$ is homogeneous of the second degree in six vectors, and holds true when any two of them coincide, because two rows of the above determinant become equal. Therefore if, as usual, we regard $\mathbf{x}$ as a variable vector, the equation $C_i = 0$, for values of the subscript other than zero, is the equation of the cone determined by the five fixed vectors which enter into it. Similar reasoning applies for any values of N and K.

3. THE COEFFICIENTS AS ELIMINANTS OF K-ADICS.

The coefficients C may be interpreted in another way, so as to connect them with the theory of matrices and allied operators. We take, as before, a set of n vectors $\mathbf{a}_1, \mathbf{a}_2, \cdots \mathbf{a}_n$. Consider for a moment the case $K = 2$. A symmetrical dyadic may be written as a sum of terms of the form $\mathbf{aa}$ where $\mathbf{a}$ is a vector. Let there be a set of constants $c_1, c_2, \cdots c_n$ not all zero. We define the equation

$$c_1\mathbf{a}_1\mathbf{a}_1 + c_2\mathbf{a}_2\mathbf{a}_2 + \cdots + c_n\mathbf{a}_n\mathbf{a}_n = 0 \tag{12}$$

to be equivalent to the set of n scalar equations obtained by selecting pairs of corresponding components in all possible ways:—

$$c_1a_{1i}a_{ij} + c_2a_{2i}a_{2j} + \cdots + c_na_{ni}a_{nj} = 0 \tag{13}$$

where the subscripts i and j may have any values from 1 to N. These equations are linear and homogeneous in the c's; hence the determinant whose elements are $a_{hi}a_{hj}$ must vanish if (12) holds true. A glance at (11) shows that this determinant is the same as C_0, the minor of $\mathbf{F}(\mathbf{x})$,— by interchanging rows and columns. Therefore the necessary and sufficient condition for the existence of a dyadic equation of the form (12) is that the n vectors which enter in that equation should lie on a quadric cone.

A symmetrical K-ad should properly be written

$$\mathbf{aaa} \cdots \text{to } K \text{ factors} \tag{14}$$

but, when no ambiguity is brought about, we may write it as $\mathbf{a}^K$. We in general define the K-adic equation

$$c_1\mathbf{a}_1{}^K + c_2\mathbf{a}_2{}^K + \cdots + c_n\mathbf{a}_n{}^K = 0 \tag{15}$$

to be equivalent to the system of n equations

$$\sum_h [c_h a_{hi} a_{hj} \cdots \text{to } K \text{ factors}] = 0 \tag{16}$$

where in each equation h runs from 1 to n. Passage from one equation of the system to another is by varying each of the second subscripts from 1 to N. By inspection of (4) it is evident that the eliminant of the set of n equations is the same as C_0. We thus have

Theorem II. *A set of* n *symmetrical K-ads are linearly related if and only if their vector elements lie on a hypercone of order* K.

4. SPECIAL FORMS OF THE GENERAL IDENTITY: METHOD OF STANDARD SETS.

By assigning particular values to the vectors $\mathbf{a}_1, \mathbf{a}_2, \cdots \mathbf{a}_n$ and by selecting particular forms of polynomials $F(\mathbf{a})$, a great number of identities may be obtained from the scalar formula (5) as well as from the vector formula (9).

An elementary method is, evidently, to assign numerical values to the elements a_{ij}, with the sole restriction that C_0 shall not vanish. The coefficients $C_1, C_2, \cdots C_n$ are polynomials of degree K. By (5), an arbitrary polynomial $F(x)$ can be expanded in terms of these n polynomials, the coefficients $F(\mathbf{a}_i)$ being found by direct substitution.

It is evident, therefore, that $C_1, C_2, \cdots C_n$ are linearly independent. It is equally evident that not every linearly independent set of n polynomials homogeneous of degree K in N variables can be taken as these n C's; for the totality of all vectors where two or more C's vanish is comprised by the set of vectors $\mathbf{a}_1, \mathbf{a}_2, \cdots \mathbf{a}_n$. We may embody this distinction in the following definition:

Definition I. *A set of polynomials* $\mathrm{P}_1, \mathrm{P}_2, \cdots P_n$, *homogeneous of degree* K *in* N *variables, such that* $\mathrm{P}_i(\mathbf{a}_j)$ *vanishes when* i *is different from* j *but not when* i *equals* j *for each of the points* $\mathbf{a}_1, \mathbf{a}_2, \cdots \mathbf{a}_n$, *will be called a* **standard set.**

By use of standard sets of polynomials, many of the identities of elementary algebra may be made to appear as special cases of (5). To take the simplest of illustrations, let $N = K = 2$ so that $n = 3$, with variables x, y. Let the matrix a_{ij} be

$$\begin{matrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{matrix}$$

whence the determinant (4) becomes

$$\begin{vmatrix} F(x, y), & x_2, & xy, & y_2 \\ F(1, 0), & 1\,, & 0, & 0 \\ F(1, 1), & 1\,, & 1, & 1 \\ F(0, 1), & 0\,, & 0, & 1 \end{vmatrix}$$

By easy calculation $C_0 = 1, C_1 = xy - x^2, C_2 = -xy, C_3 = xy - y^2$. Choosing $F(x, y) = (x + y)(x - y)$ yields $F(1, 0) = 1$, $F(1, 1) = 0$, $F(0, 1) = -1$. The identity (5) appears as

$$(x+y)(x-y) + (xy - x^2) + 0 + (-1)(xy - y^2) = 0$$

To take a less familiar illustration, let $\mathbf{F}(\mathbf{x})$ be a quadratic vector function in ordinary space, $N=3$, $K=2$. A simple way to build a standard set of polynomials is to choose four vectors $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4$ of which no three are coplanar, and form six products of linear factors $\mathbf{x} \cdot \mathbf{b}_i \mathbf{b}_j \cdot \mathbf{x}$ which are the required polynomials. The six fixed points or vectors $\mathbf{a}_h$ are the six intersections of four planes, and the vector identity (9) becomes, by putting $\mathbf{F}(\mathbf{a}_h) = \mathbf{F}(\mathbf{b}_i \times \mathbf{b}_j) = \mathbf{f}_{pq}$,

$$\mathbf{F}(\mathbf{x}) = \mathbf{x}\cdot(\mathbf{b}_1\mathbf{f}_{23}\mathbf{b}_4 + \mathbf{b}_1\mathbf{f}_{34}\mathbf{b}_2 + \mathbf{b}_1\mathbf{f}_{24}\mathbf{b}_3 + \mathbf{b}_2\mathbf{f}_{14}\mathbf{b}_3 + \mathbf{b}_2\mathbf{f}_{13}\mathbf{b}_4 \\ + \mathbf{b}_3\mathbf{f}_{12}\mathbf{b}_4)\cdot\mathbf{x} \tag{17}$$

where, if $\mathbf{F}(\mathbf{x})$ is an arbitrary quadratic vector function, the six vectors $\mathbf{f}$ are arbitrary.

In both the above illustrations we have $K = 2$, and note that the polynomials of the standard set are formed by selecting pairs of linear polynomial from a group of $K + N - 1$ possibilities. This procedure is applicable in all cases. For the number of ways in which K factors can be selected from a group of $K + N - 1$ factors with no repetitions is the same as the number of ways in which K variables can be selected from N variables allowing repetitions, that is n ways by definition of n. Each linear factor is of the form $\mathbf{b}_i\cdot\mathbf{x}$ where

$$\mathbf{b}_i\cdot\mathbf{x} = b_{i1}x_1 + b_{i2}x_2 + \cdots + b_{iN}x_N \tag{18}$$

and there are $N + K - 1$ vectors $\mathbf{b}_i$ so chosen that no N of them are linearly related. We thus arrive at the following:

Definition II. *A standard set of polynomials which has been formed by taking products of* K *linear polynomials selected from* $N + K - 1$ *such polynomials without repetition will be called a* **factored set**.

The vectors $\mathbf{a}_1, \mathbf{a}_2 \cdots \mathbf{a}_n$ may always be chosen so that $P_i(\mathbf{a}) = 1$. We define the vector $[\mathbf{b}_p\mathbf{b}_q\mathbf{b}_r \cdots$ to $N - 1$ factors$]$ to be the vector whose scalar components are the respective cofactors of the elements of the first row from the determinant

$$\begin{vmatrix} b_{i1}, & b_{i2}, & b_{i3}, & \cdots, & b_{iN} \\ b_{p1}, & b_{p2}, & b_{p3}, & \cdots, & b_{pN} \\ b_{q1}, & b_{q2}, & b_{q3}, & \cdots, & b_{qN} \\ b_{r1}, & b_{r2}, & b_{r3}, & \cdots, & b_{rN} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \multicolumn{5}{l}{\text{to } N \text{ rows}} \end{vmatrix} \tag{19}$$

We then define one of the vectors $\mathbf{a}$, say $\mathbf{a}_1$, as follows; let $\mathbf{m}_1$ denote the vector $[\mathbf{b}_1\mathbf{b}_2\mathbf{b}_3\cdots\mathbf{b}_{N-1}]$ and write

$$\mathbf{a}_1 = \frac{\mathbf{m}_1}{(\mathbf{b}_N\cdot\mathbf{m}_1\mathbf{b}_{N+1}\cdot\mathbf{m}_1\cdots\mathbf{b}_{N+K-1}\cdot\mathbf{m}_1)^{\frac{1}{K}}} \tag{20}$$

The other vectors $\mathbf{a}$ are built up in a similar manner: each $\mathbf{a}$ is a function of all the $\mathbf{b}$'s; the denominator is the Kth root of the product of K determinants of the form $\mathbf{b}\cdot\mathbf{m}$, that is of the form (19); any selection of $N-1$ of the $\mathbf{b}$'s determines one of the n vectors $\mathbf{m}_1$, $\mathbf{m}_2$ $\cdots\mathbf{m}_n$ which forms the numerator of the corresponding $\mathbf{a}$. All the remaining $\mathbf{b}$'s occur explicitly in the denominator.

The polynomials $P_1, P_2, \cdots P_n$ may now be taken of the form

$$P_h = (\mathbf{b}_p\cdot\mathbf{x})\,(\mathbf{b}_q\cdot\mathbf{x})\,(\mathbf{b}_r\cdot\mathbf{x})\cdots \text{to } K \text{ factors.} \tag{21}$$

The $\mathbf{a}$ of corresponding subscript is that $\mathbf{a}$ which contains precisely the same choice of $\mathbf{b}$'s in its denominator; for example, if $\mathbf{a}_1$ be defined by (20) then P_1 must be taken as

$$P_1 = \mathbf{b}_N\cdot\mathbf{x}\mathbf{b}_{N+1}\cdot\mathbf{x}\cdots\mathbf{b}_{N+K-1}\cdot\mathbf{x} \tag{22}$$

By inspection of (20) and (22) it is evident that $P_1(\mathbf{a}_1) = 1$, and similarly for any other distribution of $\mathbf{b}$'s between numerator and, denominator. That is $P_h(\mathbf{a}_h) = 1$ for all values of h, as was to be shown.

It is clear that a set of polynomials defined by (21) will fulfill the conditions for a standard set provided no group of N vectors chosen from the $\mathbf{b}$'s are linearly related. For the $\mathbf{a}$'s may be taken as in (20) and no denominator can vanish. At least one of the $\mathbf{b}$'s which occur in $\mathbf{m}_j$ must occur in P_i when i is different from j. Hence $P_i(\mathbf{a}_j) = 0$.

Definition III. *A factored set of polynomials, together with a set of points* $\mathbf{a}_1, \mathbf{a}_2, \cdots \mathbf{a}_n$ *such that* $P_i(\mathbf{a}_j)$ *vanishes when* i *is different from* j *but equals unity when* i *equals* j, *will be called a* **normal reference system**.

Any polynomial P_i of a normal reference system agrees with the coefficient C_i of (5) except for a constant factor, since P_i and C_i both vanish at every $\mathbf{a}$ except $\mathbf{a}_i$. By putting $\mathbf{a}_i$ for $\mathbf{x}$ in (5) we have

$$F(\mathbf{a}_1)C_0 + F(\mathbf{a}_i)C_i(\mathbf{a}_i) = 0; \tag{23}$$

but F is an arbitrary polynomial, hence $F(\mathbf{a}_i)$ is in general not zero. Therefore we have identically

$$C_0 + C_i(\mathbf{a}_i) = 0 \tag{24}$$

and since $P_i(\mathbf{a}_i) = 1$ it follows that

$$P_i = -\frac{C_i}{C_0} \tag{25}$$

By substituting $C_i = -C_0P_i$ in the vector identity (9) we arrive at the following special case of theorem I,—

Theorem III. *An arbitrary vector function homogeneous of degree* K *in* N *variables may be written as the sum of* n *terms*

$$\mathbf{F}(\mathbf{x}) = \sum_{h}[\mathbf{f}_h(\mathbf{b}_p\cdot\mathbf{x})\,(\mathbf{b}_q\cdot\mathbf{x})\cdots\text{to } K \text{ factors}] \tag{26}$$

where the vectors $\mathbf{f}_1, \mathbf{f}_2, \cdots \mathbf{f}_n$ *are arbitrary; in each term occur* K *linear polynomials selected from* N + K − 1 *linear polynomials; these linear polynomials are under the sole restriction that no* N *of them are linearly related.*

It follows from the form of (26) that, if $\mathbf{a}_h$ be of the form (20),

$$\mathbf{F}(\mathbf{a}_h) = \mathbf{f}_h \tag{27}$$

It is apparent that expansions like (26) will be of advantage in investigations where symmetry of form is to be sought. As a simple illustration suppose $K = 2$. We have $N + K - 1 = N + 1$ and may take as our linear polynomials the N variables $x_1, x_2, \cdots x_n$ together with their sum $x_i + x_2 + \cdots + x_N$. We see at once that any quadratic vector function can be written in the form

$$\Sigma\mathbf{f}_{ij}x_ix_j + \varphi(\mathbf{x})\Sigma x_h;\ [i \text{ different from } j;\ h = 1, 2, \cdots N] \tag{28}$$

where $\varphi(\mathbf{x})$ is a linear vector function in N-space.

5. Special Forms of the General Identity: Method of Factoring.

I propose next to consider a large class of identities which we may obtain from (5) and from (9) if, instead of making some special choice of the vectors $\mathbf{a}_1, \mathbf{a}_2, \cdots \mathbf{a}_n$, we take particular polynomials $F(\mathbf{x})$ and $\mathbf{F}(\mathbf{x})$.

As a first example, let $E(\mathbf{x})$ be a polynomial of degree $K - 1$ and take $\mathbf{F}(\mathbf{x}) = \mathbf{x}E(\mathbf{x})$; that is, the vector function $\mathbf{F}(\mathbf{x})$ is chosen to be the point vector $\mathbf{x}$ multiplied by an arbitrary scalar polynomial. The identity (9) becomes

$$\mathbf{x}E(\mathbf{x})C_0 + \mathbf{a}_1E(\mathbf{a}_1)C_1 + \mathbf{a}_2E(\mathbf{a}_2)C_2 + \cdots + \mathbf{a}_nE(\mathbf{a}_n)C_n = 0 \tag{29}$$

We may now choose the polynomial $E(\mathbf{x})$ so that it shall vanish at a number of points, say at m points. Let these points be $\mathbf{a}_{n-m+1}$, $\mathbf{a}_{n-m+2}$, etc. up to $\mathbf{a}_n$. In other words let $E(\mathbf{x})$ be a function of the same type as the C's but associated with the case $K - 1$. This particular form of $E(\mathbf{x})$ may be distinguished as $E_m(\mathbf{x})$. The last m terms of (29) will now vanish and we have

$$\mathbf{x}E_m(\mathbf{x})C_0 + \mathbf{a}_1E_m(\mathbf{a}_1)C_1 + \cdots + \mathbf{a}_{n-m}E_m(\mathbf{a}_{n-m})C_{n-m} = 0 \qquad (30)$$

If the number of terms in the general polynomial of degree K in N variables be denoted by $n(N, K)$ we have the identity

$$n(N, K) - n(N, K - 1) = n(N - 1, K) \qquad (31)$$

while $m = n(N, K - 1) - 1$. The number of terms in (29) is $n(N, K) + 1$. Hence the number of terms in (30) is $n(N - 1, K) + 2$. It is clear that we may obtain from (29) as many identities of the form (30) as there are ways of selecting m points from $n + 1$ points.

Again, we may take $F(\mathbf{x}) = L(\mathbf{x})E(\mathbf{x})$ where $L(\mathbf{x})$ is a linear polynomial. Repeating the same steps, with the same meaning of $E_m(\mathbf{x})$, we shall obtain, instead of the vector identity (30), a scalar identity

$$L(\mathbf{x})E_m(\mathbf{x})C_0 + L(\mathbf{a}_1)E_m(\mathbf{a}_1)C_1 + \cdots + L(\mathbf{a}_{n-m})E_m(\mathbf{a}_{n-m})C_{n-m} = 0 \qquad (32)$$

We may now choose $L(\mathbf{x})$ so as to vanish at the $N - 1$ points whose subscripts are $n - m - N + 2$, $n - m - N + 3$, etc. up to $n - m$; that is, $L(\mathbf{x})$ may be taken to be the determinant of the components of these $N - 1$ point vectors and the variable vector $\mathbf{x}$. The last $N - 1$ terms of (32) will now vanish, leaving $n(N - 1, K) - N + 3$ terms. We may obtain as many identities of this last form as there are ways of selecting $N - 1$ points from $n(N - 1, K) + 2$ points, multiplied by the number of ways (above noted) in which (30) could be formed.

Instead of factoring $F(\mathbf{x})$ into a linear polynomial and one of degree $K - 1$ we can evidently obtain other identities in a similar manner by factoring $F(\mathbf{x})$ into any pair of polynomials, or into any number of polynomials such that the sum of their degrees is equal to K. Clearly then there exist a vast number of identities connecting the coefficients C in (5) or (9) with the anologous coefficients in expansions associated with smaller values of K.

In a former paper [1] I have made a somewhat detailed study of the case $K = 2$, $N = 3$. The "Aconic Function" of Hamilton is a function of six vectors, in the form of a scalar product, which vanishes when these six vectors lie on a quadric cone; equivalent, therefore, to C_0 for the case in question. In the paper referred to, the properties of the seven C's for this case were derived from Hamilton's function, instead of from the determinant (4). It was shown that in this case any three C's are connected by a relation of the form

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[1] An Identical Relation Connecting Seven Vectors. Proc. Royal Soc. Edin. vol. XL, Part II(No. 14), pp. 129–139.

$$L_pC_p + L_qC_q + L_rC_r = 0 \tag{33}$$

where the L's are linear polynomials. It is easy to see from (30) that a similar relation holds for all values of N and K. For, writing for the sake of brevity

$$n - m - N + 1 = n(N-1, K) - N + 2 = q;\ m = n(N, K-1) - 1 \tag{34_1}$$

and with a notation like that of the preceding article,

$$\mathbf{b} = [\mathbf{x}\mathbf{a}_{q+2}\mathbf{a}_{q+3}\cdots\mathbf{a}_{n-m}] \tag{34_2}$$

we multiply $\mathbf{b}$ into (30), taking the dot product. The first term vanishes because $\mathbf{b}\cdot\mathbf{x} = 0$. The last $N - 2$ terms also vanish. The products $\mathbf{b}\cdot\mathbf{a}_1$ etc. up to $\mathbf{b}\cdot\mathbf{a}_{q+1}$ are obviously linear polynomials in $\mathbf{x}$, while $E_m(\mathbf{a}_1)$ etc. are constants. We now have

$$\mathbf{b}\cdot\mathbf{a}_1E_m(\mathbf{a}_1)C_1 + \cdots + \mathbf{b}\cdot\mathbf{a}_{q+1}E_m(\mathbf{a}_{q+1})C_{q+1} = 0 \tag{35}$$

A standard set of polynomials, as already pointed out, becomes a set of C's when each is multiplied by a proper constant. We have therefore by (35) the following theorem:

Theorem IV. *Any $q + 1$ polynomials selected from a standard set are connected by relations of the form $\Sigma \mathrm{L}_i\mathrm{P}_i = 0$, where $\mathrm{L}_1\cdots\mathrm{L}_{q+1}$ are linear polynomials of the form $\mathbf{b}\cdot\mathbf{a}_i$.*

If $S[p, q]$ be the number of ways in which p points may be selected from q points, (i.e. S is a binomial coefficient), the number of such relations connecting the polynomials of a given standard set is $S[q + 1, n]S[N - 2, n - q - 1]$.

The factor last written may be denoted by Q, that is

$$Q = S[N-2, n-q-1] \tag{36}$$

This is the number of relations of the form $\Sigma L_iP_i = 0$ which may be written down connecting a *given group* of $q + 1$ polynomials selected from the n polynomials of a standard set. (We assume $N > 2, K > 1$). If $N = 3$, $K = 2$, we have $n = 6$, $q = 2$, $Q = 3$. If either N or K be larger, we have $Q > q + 1$.

We may now prove that, regarding a relations of form (35) as equations satisfied by a particular selection of $q + 1$ of the C's, not more than q of these equations can be independent. For consider the $m + N - 2$ points $\mathbf{a}_{q+2}, \mathbf{a}_{q+3}, \cdots \mathbf{a}_n$. These are the $\mathbf{a}$'s which correspond to the terms absent from (35). Let $E_1, E_2, \cdots E_{q+1}$ be polynomials of degree $K - 1$ which vanish, respectively, at $q + 1$ different selected groups of m points chosen from the above $m + N - 2$ points. Let $\mathbf{b}_1, \mathbf{b}_2, \cdots \mathbf{b}_{q+1}$ denote those linear vector functions of $\mathbf{x}$ of form (34) into which, in each case, enter the $\mathbf{a}$'s not occurring in the corresponding E. Corresponding to each choice of E and corresponding $\mathbf{b}$ we may write an identity of the form (35). Thus

$$\begin{array}{ll} \mathbf{b}_1 \cdot \mathbf{a}_1 E_1(\mathbf{a}_1) C_1 + \cdots + \mathbf{b}_1 \cdot \mathbf{a}_{q+1} E_1(\mathbf{a}_{q+1}) C_{q+1} = 0 & \\ \mathbf{b}_2 \cdot \mathbf{a}_1 E_2(\mathbf{a}_1) C_1 + \cdots + \mathbf{b}_2 \cdot \mathbf{a}_{q+1} E_2(\mathbf{a}_{q+1}) C_{q+1} = 0 & (37) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ \mathbf{b}_{q+1} \cdot \mathbf{a}_1 E_{q+1}(\mathbf{a}_1) C_1 + \cdots + \mathbf{b}_{q+1} \cdot \mathbf{a}_{q+1} E_{q+1}(\mathbf{a}_{q+1}) C_{q+1} = 0 & \end{array}$$

We may regard these as $q + 1$ linear equations satisfied by the $q + 1$ C's which enter. We know that these C's are actual polynomials of degree K in $\mathbf{x}$. Hence the equations cannot be independent, as was to be demonstrated.

On the other hand we may in general select q such equations which shall be independent. For consider the coefficients of C_j in the successive equations: they are of the form $\mathbf{b}_i \cdot \mathbf{a}_j E_i(\mathbf{a}_j)$, and may be thought of as polynomials of degree K in $\mathbf{a}_j$. They all vanish if $\mathbf{a}_j$ be made to coincide with any one of the points $\mathbf{a}_{q+2} \cdots \mathbf{a}_n$ or $\mathbf{x}$, by hypothesis, that is at $m + N - 1$ points. In general n polynomials may be linearly independent. Hence of these polynomials $n - (m + N - 1)$ may be linearly independent. But this number is q, as was to be shown. A specific rule for selecting the q equations will be given below.

Corollary to Theorem IV. The following corollary to theorem IV will be essential in the applications made below. We suppose given a standard set of polynomials $P_1(\mathbf{x}), \cdots P_n(\mathbf{x})$ based on a set of vectors $\mathbf{a}_1, \cdots \mathbf{a}_n$ as already defined. We now adjoin $N - 3$ other vectors which need not be distinct from the others, and which we may call $\mathbf{a}_{n+1} \cdots$ etc.; and write as in (34_2)

$$\mathbf{b}_1 = [\mathbf{x} \mathbf{a}_h \mathbf{a}_j \cdots] \qquad (37_A)$$

where the $N - 2$ vectors $\mathbf{a}_h, \mathbf{a}_j, \cdots$ which occur in $\mathbf{b}_i$ are precisely those $\mathbf{a}$'s (out of the total number of $n + N - 3$), which do not occur in P_i. It is evident that $\mathbf{b}_i \cdot \mathbf{y}$ will be a linear polynomial in $\mathbf{x}$. Write

$$L_i(\mathbf{x}) = \mathbf{b}_i \cdot \mathbf{y} \qquad (37_B)$$

where $\mathbf{y}$ is an arbitrary vector. The corollary may now be stated:

Of the polynomials of a standard set, not fewer than $n - m - t$ can be connected by a relation of the form

$$\Sigma c_i L_i(\mathbf{x}) P_i(\mathbf{x}) = 0 \tag{37_C}$$

unless the linear polynomials $L_i(\mathbf{x})$ *all vanish at more than* t *of the points* $\mathbf{a}_1, \cdots \mathbf{a}_n$; *it is understood that* $L_1(\mathbf{x})$ *is of the form* (37_B), *and that* c_i *is independent of* $\mathbf{x}$.

Proof. Suppose a relation of the form

$$c_1 L_1(\mathbf{x}) P_1(\mathbf{x}) + \cdots + c_w L_w(\mathbf{x}) P_w(\mathbf{x}) = 0 \tag{37_D}$$

where $w = n - m - t - 1$ and where the L's all vanish at the t points $\mathbf{a}_{w+1}, \cdots, \mathbf{a}_{w+t}$, but at no others of subscript less than $n + 1$.

The polynomial $L_i(\mathbf{x})$ is the determinant of the coefficients of the N vectors $\mathbf{x}$, $\mathbf{y}$, $\mathbf{a}_i$, and the $N - 3$ adjoined vectors, any or all of which might coincide with an equal number of the original set $\mathbf{a}_1 \cdots \mathbf{a}_n$, according to the value of t. Thus $L_i(\mathbf{x})$ is a linear function of $\mathbf{a}_i$, and $L_j(\mathbf{x})$ is the same linear function of $\mathbf{a}_j$ except perhaps in sign. If we write $L_i(\mathbf{x}) = L(\mathbf{a}_i)$ we may write (37_D) in the form

$$c_1 L(\mathbf{a}_1) P_1(\mathbf{x}) + \cdots + c_w L(\mathbf{a}_w) P_w(\mathbf{x}) = 0 \tag{37_E}$$

It suffices to show that $L(\mathbf{a}_{n-m}) = 0$ contrary to hypothesis.

Now c_i must be a polynomial of degree $K - 1$ in $\mathbf{a}_i$. For all the terms of the supposed identity (37_E) are of degree K in $\mathbf{a}_i$ with exception of the ith term; is independent of $\mathbf{a}_i$; and $L(\mathbf{a}_i)$ is linear in $\mathbf{a}_i$.

Furthermore c_i is of degree $K - 1$ in each of the $\mathbf{a}$'s from $\mathbf{a}_{n-m+1}$ to $\mathbf{a}_n$ inclusive. For (37_E), being an identity, will subsist if we make $\mathbf{a}_i$ coincide with $\mathbf{a}_{n-m+j}$; P_i will be unaltered since it is independent of $\mathbf{a}_i$; all other P's vanish; $L(\mathbf{a}_{n-m+j})$ by hypothesis does not vanish; hence c_i must vanish when $\mathbf{a}_i$ coincides with $\mathbf{a}_{n-m+j}$; that is, c_i is a polynomial of degree $K - 1$ in $\mathbf{a}_i$ vanishing at the $m = n(N, K - 1) - 1$ points $\mathbf{a}_{n-m+1} \cdots \mathbf{a}_n$, and is therefore a polynomial of degree $K - 1$ in each of these points. We may accordingly identify the c's of (37_E) with the polynomials $E_m(\mathbf{a}_i)$ of (32) except, in each case, for a constant multiplier g_i.

If, finally, we make $\mathbf{a}_i$ coincide with $\mathbf{a}_{n-m}$, c_i will not vanish, for a polynomial of degree $K - 1$ cannot be made to vanish at more than m arbitrary points; P_i is independent of $\mathbf{a}_i$; and so $L(\mathbf{a}_{n-m})$ must vanish, contrary to hypothesis.

Hence the identity (37_D) is impossible unless the L's all vanish at more than t of the points $\mathbf{a}_1 \cdots \mathbf{a}_n$, as was to be proved.

6. APPLICATION TO THE PROBLEM OF EXPRESSING POLYNOMIALS AS DETERMINANTS WHOSE ELEMENTS ARE LINEAR POLYNOMIALS.

If we take $M = 3$ we have $q = n(2, K) - 1 = K$. We may then regard K equations of type (37) as so many linear equations in the $K + 1$ unknowns $C_1, \cdots C_{K+1}$. The K-row determinants from the matrix of the coefficients are polynomials of degree K; their elements are linear in $\mathbf{x}$; and, aside from a constant factor, they must be equal to the respective C's which enter. We therefore have the means of setting up polynomials of degree K in 3 variables in the form of determinants with linear elements and vanishing at $n - 1$ given points.

By use of the theory of matrices, Dickson has recently given a highly elegant proof that all homogeneous polynomials in two and in three variables, as well as quadrics and sufficiently general cubics in four variables, can be expressed as determinants with linear elements; and that no other general polynomials can be so expressed.[2] He has shown how such a transformation can be accomplished for an arbitrary plane curve by employing no irrationalities except the roots of a single equation of degree K. The present investigation is equally general if we have no regard for rationality: a polynomial of degree K through $n - 1$ arbitrary points is an arbitrary polynomial. With regard to rationality, however, the method of the present article is less general, for Dickson does not assume any points on the curve to be known. But an explicit formulation of the polynomial as a function of the n points $\mathbf{a}_1, \cdots \mathbf{a}_{n-1}$ and $\mathbf{x}$ is not without utility.

To take a concrete case, let $N = 3$ and $K = 3$. Let it be required to express as a determinant with linear elements the cubic through the nine points $\mathbf{a}_1 \cdots \mathbf{a}_9$. Let the six points $\mathbf{a}_1 \cdots \mathbf{a}_6$ be regarded as $n(3, 2)$ points corresponding to the case $N = 3$, $K = 2$. Let $E_1 \cdots E_6$ be the respective quadrics which vanish at five of these six points, the omitted point having corresponding subscript. Let (xij) and (kij) denote the determinants of the components of the vectors $\mathbf{x}$, $\mathbf{a}_i$, $\mathbf{a}_j$ and $\mathbf{a}_k$, $\mathbf{a}_i$, $\mathbf{a}_j$ respectively. We may chose $\mathbf{b}_1 = [\mathbf{x}\mathbf{a}_i]$; so that $\mathbf{b}_i \cdot \mathbf{a}_j = (xij)$. We may write $\mathbf{a}_{10}$ for an arbitrary tenth point. With the notation already used C_{10} will be our required cubic. We may form six equations of the form (37) each of which involves the cubics $C_7 \cdots C_{10}$. Three of these are sufficient and may be taken thus:

$$
\begin{aligned}
(x17)E_1(\mathbf{a}_7)C_7 + (x18)E_1(\mathbf{a}_8)C_8 + (x19)E_1(\mathbf{a}_9)C_9 + (x1_{10})E_1(\mathbf{a}_{10})C_{10} &= 0\\
(x27)E_2(\mathbf{a}_7)C_7 + (x28)E_2(\mathbf{a}_8)C_8 + (x29)E_2(\mathbf{a}_9)C_9 + (x2_{10})E_2(\mathbf{a}_{10})C_{10} &= 0\\
(x37)E_3(\mathbf{a}_7)C_7 + (x38)E_3(\mathbf{a}_8)C_8 + (x39)E_3(\mathbf{a}_9)C_9 + (x3_{10})E_3(\mathbf{a}_{10})C_{10} &= 0
\end{aligned}
$$

---

[2] Trans. Amer. Math. Soc. Vol. 22, No. 2 (April 1921), pp. 167–179.

Hence we may take as the required cubic the determinant

$$\begin{vmatrix} (x17)E_1(\mathbf{a}_7), & (x18)E_1(\mathbf{a}_8), & (x19)E_1(\mathbf{a}_9) \\ (x27)E_2(\mathbf{a}_7), & (x28)E_2(\mathbf{a}_8), & (x29)E_2(\mathbf{a}_9) \\ (x37)E_3(\mathbf{a}_7), & (x38)E_3(\mathbf{a}_8), & (x39)E_3(\mathbf{a}_9) \end{vmatrix} \qquad (38)$$

That this determinant vanishes when $\mathbf{x}$ coincides with $\mathbf{a}_1$ is evident, for the elements of the first row all vanish; similarly the second and third rows vanish when $\mathbf{x}$ coincides with $\mathbf{a}_2$ and $\mathbf{a}_3$ respectively. The columns vanish when $\mathbf{x}$ coincides with $\mathbf{a}_7$, $\mathbf{a}_8$, $\mathbf{a}_9$. Since as has been shown the determinant must also vanish when $\mathbf{x}$ coincides with $\mathbf{a}_4$, $\mathbf{a}_5$, or $\mathbf{a}_6$, we are presented with three new identities of the form

$$\begin{vmatrix} (417)E_1(\mathbf{a}_7), & (418)E_1(\mathbf{a}_8), & (419)E_1(\mathbf{a}_9) \\ (427)E_2(\mathbf{a}_7), & (428)E_2(\mathbf{a}_8), & (429)E_2(\mathbf{a}_9) \\ (437)E_3(\mathbf{a}_7), & (438)E_3(\mathbf{a}_8), & (439)E_3(\mathbf{a}_9) \end{vmatrix} = 0 \qquad (39)$$

the other two having 5 and 6 in place of 4. In fact if we multiply the elements of the first row of (39) by the three-row determinant (561), those of the second row by (562), of the third by (563), and add, the sum of the elements of each column is zero; for

$$(561)\,(41x)E_1(\mathbf{x}) + (562)\,(42x)E_2(\mathbf{x}) + (563)\,(43x)E_3(\mathbf{x}) = 0 \qquad (40)$$

is an identity of the type (33), a special case of (35). It holds therefore when $\mathbf{x}$ is replaced by $\mathbf{a}_7$, $\mathbf{a}_8$, or $\mathbf{a}_9$.

Determinants of similar form to (38) may evidently be written down at once for any value of K when $N = 3$. The E's will in all cases denote polynomials of degree $K - 1$; vanishing at $n(N, K - 1) - 1$ points.

When N is greater than 3 we have always $q > K$; hence if we select from the equations of form (37) a set of q equations which are independent we may write the C's proportional to a set of determinants having linear elements; *these determinants must accordingly be reducible polynomials and must possess a common factor of degree* q − K, *the other factor being one of the C's.*

Not every set of q equations of the type of (37) will be independent. For example let $N = 4$, $K = 2$, so that $n = 10$, $m = 3$, $q = 4$, and $Q = S[2, 5] = 10$. If we choose our four vectors $\mathbf{b}_1 \cdots \mathbf{b}_4$ to be $[\mathbf{x}\mathbf{a}_i\mathbf{a}_5]$ where $i = 1, 2, 3, 4$, the coefficients of C_j in our four equations will be $(xi5j)\ (pqrj)$ where p, q, and r are different from i and from each other and are less than five; while j has any value from six to ten inclusive. The four polynomials in $\mathbf{a}_j$ are linearly related; for we may write as a special case of (29)

$$\mathbf{a}_j(1234) - \mathbf{a}_1(j234) + \mathbf{a}_2(j134) - \mathbf{a}_3(j124) + \mathbf{a}_4(j123) = 0 \quad (41)$$

and by multiplying by $[\mathbf{x}\mathbf{a}_5\mathbf{a}_j]$

$$(x5j1)\,(j234) - (x5j2)\,(j134) + (x5j3)\,(j124) - (x5j4)\,(j123) = 0 \quad (42)$$

which, being an identity, holds when j runs from 6 to 10. Hence when $\mathbf{a}_5$ is common to all four $\mathbf{b}$'s the four equations are linearly related.

When the four $\mathbf{b}$'s do not contain an $\mathbf{a}$ in common the four equations are independent. For suppose the $\mathbf{b}$'s to be $[\mathbf{x}\mathbf{a}_1\mathbf{a}_5]$, $[\mathbf{x}\mathbf{a}_2\mathbf{a}_5]$, $[\mathbf{x}\mathbf{a}_3\mathbf{a}_5]$, and $[\mathbf{x}\mathbf{a}_1\mathbf{a}_2]$. The four corresponding equations are $\Sigma(x15j)(234j)C_j = 0$ where j runs from 6 to 10, and three other equations of similar form. The determinant of the coefficients of the first four columns is

$$\begin{vmatrix} (x156)\,(2346), & (x157)\,(2347), & (x158)\,(2348), & (x159)\,(2349) \\ (x256)\,(1346), & (x257)\,(1347), & (x258)\,(1348), & (x259)\,(1349) \\ (x356)\,(1246), & (x357)\,(1247), & (x358)\,(1248), & (x359)\,(1249) \\ (x126)\,(3456), & (x127)\,(3457), & (x128)\,(3458), & (x129)\,(3459) \end{vmatrix} \quad (43)$$

In order that this determinant might vanish identically it would be necessary and sufficient that four numbers c_1, c_2, c_3, c_4 exist, independent of $\mathbf{a}_j$ and not all zero, such that

$$c_1(x15j)\,(234j) + c_2(x25j)\,(134j) + c_3(x35j)\,(124j) + c_4(x12j)\,(345j) = 0 \quad (44)$$

for the vectors $\mathbf{a}_6$, $\mathbf{a}_7$, $\mathbf{a}_8$, $\mathbf{a}_9$, represented by $\mathbf{a}_j$, are arbitrary. If we let $\mathbf{a}_j = \mathbf{x} + \mathbf{a}_5$ this equation reduced to its last term, namely

$$c_4(x125)\,(345x) = 0$$

whence $c_4 = 0$, for the vectors $\mathbf{x}$, $\mathbf{a}_1 \cdots \mathbf{a}_5$ may have any values whatever. By letting $\mathbf{a}_j$ be $\mathbf{a}_1 + \mathbf{a}_4$, $\mathbf{a}_2 + \mathbf{a}_4$, and $\mathbf{a}_3 + \mathbf{a}_4$ we see that c_1, c_2, and c_3 must all vanish, contrary to hypothesis. Hence (43) does not vanish identically. We may proceed similarly whenever the four $\mathbf{b}$'s do not have an $\mathbf{a}$ in common.

This determinant must accordingly contain as a factor C_{10}, a quadric through the nine points $\mathbf{a}_1, \cdots \mathbf{a}_9$. That it vanishes at all these points can be verified by inspection: if $\mathbf{x} = \mathbf{a}_4$ the last two rows are equal numerically and opposite in sign; if $\mathbf{x}$ equals any other $\mathbf{a}$ all the elements of a row or of a column vanish.

This determinant is of the fourth degree in $\mathbf{x}$ and in $\mathbf{a}_1 \cdots \mathbf{a}_5$. It is of the second degree in $\mathbf{a}_6 \cdots \mathbf{a}_9$. Since the factor C_{10} is a quadric in all ten vectors the other factor must be a quadric in $\mathbf{x}$ and in $\mathbf{a}_1 \cdots \mathbf{a}_5$ and be independent of $\mathbf{a}_6 \cdots \mathbf{a}_9$. If we wish to find this latter factor it is allowable to assign to $\mathbf{a}_6 \cdots \mathbf{a}_9$ any values we please. If we take

$$\mathbf{a}_6 = \mathbf{a}_3 + \mathbf{a}_4, \quad \mathbf{a}_7 = \mathbf{a}_2 + \mathbf{a}_4, \quad \text{and } \mathbf{a}_8 = \mathbf{a}_2 + \mathbf{a}_3 \tag{45}$$

the determinant (43) becomes

$$\begin{vmatrix} 0 & 0 & 0 & (x159)\,(2349) \\ 0 & (x254)\,(1342), & (x253)\,(1342), & (x259)\,(1349) \\ (x354)\,(1243), & 0 & (x352)\,(1243), & (x359)\,(1249) \\ 0 & (x124)\,(3452), & (x123)\,(3452), & (x129)\,(3459) \end{vmatrix} \tag{46}$$

which is equal to the product of $(x354)\,(1243)\,(x159)\,(2349)$ times the determinant

$$\begin{vmatrix} (x254)\,(1342), & (x253)\,(1342) \\ (x124)\,(3452), & (x123)\,(3452) \end{vmatrix} .$$

which in turn is

$$(1342)\,(3452)\,[(x254)\,(x123) - (x124)\,(x253)]; \tag{48}$$

but from (41), by letting $\mathbf{a}_j = \mathbf{x}$ and multiplying by $[\mathbf{x}\mathbf{a}_2\mathbf{a}_5]$

$$-\,(x251)\,(x234) - (x253)\,(x124) + (x254)\,(x123) = 0 \tag{49}$$

that is, the quantity in brackets in (48) is equal to $(x251)\,(x234)$. Collecting results, we see that the determinant (43), by virtue of the substitutions (45), become the product of eight determinants:

$$(x354)\,(x251)\,(x234)\,(x159)\,(1243)\,(1342)\,(3452)\,(2349) \tag{50}$$

It is evident by inspection of this product that the adventitious factor of the determinant (43) must be

$$(x354)\,(x251)\,(1234) \tag{51}$$

for in no other way can we pick from (50) a factor quadratic in $\mathbf{a}_1 \cdots \mathbf{a}_5$ and $\mathbf{x}$. The algebraic sign is, however, undetermined.

Determinants similar in form to (43) can evidently be written down for any values of K and N. The second factor of each element will in every case be a polynomial of degree $K - 1$ in the variables which enter into it. The first factor is linear in $\mathbf{x}$ and in $N - 1$ other points. We have therefore the theorem:

Theorem V. *Given a polynomial homogeneous of degree* K *in* N *variables, and* n − 1 *points at which this polynomial vanishes: it is in general possible to write down a determinant of order* q *whose elements* *are linear in* **x** *and are rational functions of these points and which shall contain the polynomial as a factor.*

One further illustration may be given of selecting the **b**'s so that the determinant shall not vanish identically, and of factoring the determinant. Take $N = 5$, $K = 2$, and let the seven **b**'s be $[x145]$, $[x356]$, $[x346]$, $[x567]$, $[x467]$, $[x457]$, and $[x456]$. The elements of the first row of our determinant will be $(x145j)\,(2367j)$ where $j = 8, 9, \cdots, 14$; and similarly for the other rows. This determinant does not vanish identically. For if we make the substitutions $\mathbf{a}_8 = \mathbf{a}_1 + \mathbf{a}_2$, $\mathbf{a}_9 = \mathbf{a}_2 + \mathbf{a}_3$, $\mathbf{a}_{10} = \mathbf{a}_1 + \mathbf{a}_3$, $\mathbf{a}_{11} = \mathbf{a}_1 + \mathbf{a}_7$, $\mathbf{a}_{12} = \mathbf{a}_2 + \mathbf{a}_7$, $\mathbf{a}_{13} = \mathbf{a}_3 + \mathbf{a}_7$, and $\mathbf{a}_{14} = \mathbf{y}$, an arbitrary vector, the determinant becomes[3]

$$\begin{vmatrix}
\begin{array}{c}(x1452)\\(12367);\end{array} & 0\ ; & P_1\ ; & P_2\ ; & P_3\ ; & P_4\ ; & P_5 \\
0\ ; & \begin{array}{c}(x3562)\\(12473);\end{array} & \begin{array}{c}(x3561)\\(12473);\end{array} & 0\ ; & 0\ ; & P_6\ ; & P_7 \\
0\ ; & \begin{array}{c}(x3462)\\(12573);\end{array} & \begin{array}{c}(x3461)\\(12573);\end{array} & 0\ ; & 0\ ; & P_8\ ; & P_9 \\
0\ ; & 0\ ; & 0\ ; & \begin{array}{c}(x5671)\\(12347);\end{array} & \begin{array}{c}(x5672)\\(12347);\end{array} & \begin{array}{c}(x5673)\\(12347);\end{array} & P_{10} \\
0\ ; & 0\ ; & 0\ ; & \begin{array}{c}(x4671)\\(12357);\end{array} & \begin{array}{c}(x4672)\\(12357);\end{array} & \begin{array}{c}(x4673)\\(12357);\end{array} & P_{11} \\
0\ ; & 0\ ; & 0\ ; & \begin{array}{c}(x4571)\\(12367);\end{array} & \begin{array}{c}(x4572)\\(12367);\end{array} & \begin{array}{c}(x4573)\\(12367);\end{array} & P_{12} \\
0\ ; & 0\ ; & 0\ ; & 0\ ; & 0\ ; & 0\ ; & P_0
\end{vmatrix}$$

where $P_1, \cdots P_{12}$ denote elements which are of no consequence, but $P_0 = (x456y)\,(1237y)$. It is evident that this determinant factors into a number of linear determinants together with the two-row determinant

$$\begin{vmatrix} (x3562); & (x3561) \\ (x3462); & (x3461) \end{vmatrix}$$

and the three-row determinant

$$\begin{vmatrix} (x5671); & (x5672); & (x5673) \\ (x4671); & (x4672); & (x4673) \\ (x4571); & (x4572); & (x4573) \end{vmatrix} \tag{53}$$

---

[3] For compactness in printing this determinant, the two factors of each element are set one above the other. Thus the element in the first row and first column is

$$(x1452)\ (12367)$$

and similarly for the other factored elements.

The two-row determinant does not vanish identically since it does not vanish when $\mathbf{x} = \mathbf{a}_2 + \mathbf{a}_4$. Nor does the three-row determinant vanish identically, for if $\mathbf{x} = \mathbf{a}_4 + \mathbf{a}_1$ it becomes (45671) times the two-row determinant

$$\begin{vmatrix} (14672); & (14673) \\ (14572); & (14573) \end{vmatrix}$$

which, in turn, does not vanish if we let $\mathbf{a}_1 = \mathbf{a}_2 + \mathbf{a}_5$. Hence our original determinant does not vanish identically.

To complete the factorization, consider the minor of the leading element in (53), namely

$$(x4672)\,(x4573) - (x4572)\,(x4673); \tag{54}$$

and let $\mathbf{x}$ be expanded in terms of the five vectors $\mathbf{a}_2$, $\mathbf{a}_3$, $\mathbf{a}_4$, $\mathbf{a}_5$, and $\mathbf{a}_i$,—

$$\mathbf{x}(23457) - \mathbf{a}_2(x3457) + \mathbf{a}_3(x2457) - \mathbf{a}_4(x2357) + \mathbf{a}_5(x2347) - \mathbf{a}_7(x2345) = 0 \tag{55}$$

and multiplying by $[x467]$

$$-(x4672)\,(x3457) + (x4673)\,(x2457) + (x4675)\,(x2347) = 0 \tag{56}$$

whence it is evident that (54) contains the factor $(x4567)$. In a similar manner it may be shown that the other minors of the elements of the first row in (53) contain the same factor. Thus the determinant contains this factor. In the same way we may show that the minors of any other row contain a common factor, hence the determinant is the product of three linear factors. Again, the same process shows that (52) is a product of two linear factors. The factorization is therefore complete, and the adventitious factor of our original determinant is here, as in the former case, a mere product of determinants linear in the points which enter into them.

7. Rule for Constructing these Determinants.

It remains to indicate how, in general, the $\mathbf{b}$'s may be chosen so that the determinant contemplated in theorem V shall not vanish identically. We have seen that the order q of this determinant is $n[(N-1), K] - N + 2$. Let q' be the number analogous to q but associated with polynomials of degree less by one, that is

$$q' = n[N-1, K-1] - N + 2 \tag{57}$$

The elements of the determinant are of the form $\mathbf{b}\cdot\mathbf{a}_j E(\mathbf{a}_j)$. A necessary condition that the determinant shall not vanish identically is that not more than $q' + N - 2$ of the E's shall be polynomials of the same standard set. For when several E's belong to the same standard set they are functions of a set of $m + 1$ of the **a**'s not including $\mathbf{a}_j$. Since any element of the determinant contains every one of a set of $m + N - 2$ of the **a**'s (exclusive of $\mathbf{a}_j$), and also contains **x**, it follows that the corresponding **b**'s will possess $N - 3$ of the **a**'s in common. That is, they have in common $N - 2$ arbitrary vectors not belonging to the reference system of the E's which occur in the same elements. Now by theorem IV, if $q' + 1$ of the E's belong to a standard set they satisfy an identity $\Sigma L(\mathbf{a}_j)E(\mathbf{a}_j) = 0$, but the L's are functions of those **a**'s only which occur in the E's. By writing for the E's an identity of the form (32) it is evident that even when the L's contain $N - 2$ arbitrary vectors we shall have $q' + 1 + (N - 2)$ of the E's (which now correspond to the C's of (32)), connected by a relation $\Sigma LE = 0$. Hence only $q' + N - 2$ of the E's can belong to the same standard set, as was to be shown.

We are now in a position to prove that the **b**'s may in general be selected by the following rule:

Rule for choice of b's. Select any $m + N - 2$ from the given set of $n - 1$ vectors $\mathbf{a}_1 \cdots \mathbf{a}_{n-1}$. Choose $q' + N - 2$ polynomials $E_1, E_2, \cdots$ out of the standard set of degree $K - 1$ based on $m + 1$ of these vectors, which we may number from 1 to $m + 1$.

Make a second selection of $m + 1$ **a**'s by omitting $\mathbf{a}_1$ from the first selection and adjoining $\mathbf{a}_{m+2}$, choose $q' + N - 3$ polynomials of the new standard set based on this second selection of **a**'s.

If we have not yet q polynomials, make a third selection of $m + 1$ **a**'s by omitting $\mathbf{a}_{m+2}$ from the second selection and adjoining $\mathbf{a}_{m+3}$; choose $q' + N - 3$ polynomials from a third standard set based on this third selection of **a**'s.

If we have not yet q polynomials, proceed in a similar manner to form new groups until q polynomials have been obtained: every selection of **a**'s consists of $\mathbf{a}_2, \mathbf{a}_3, \cdots \mathbf{a}_{m+1}$ together with one other. Each of the **b**'s contains every **a** not in its own 'E' but included in the original choice of $m + N - 2$ of the **a**'s.

We have to prove that the determinant whose elements are $\mathbf{b} \cdot \mathbf{a}_j E(\mathbf{a}_j)$ does not vanish identically, and that the rule is always possible.

We note that, by the rule, $\mathbf{a}_1$ occurs only in the E's numbered from 2 to $q' + N - 2$ inclusive. Hence $\mathbf{a}_1$ occurs in all the **b**'s except those so numbered. If the determinant vanishes identically a set of numbers $c_1 \cdots c_q$ exists such that $\Sigma c\mathbf{b} \cdot \mathbf{a}_j E(\mathbf{a}_j) = 0$, the c's being independent of $\mathbf{a}_j$ and not all zero. Now let $\mathbf{a}_j = \mathbf{x} + g\mathbf{a}_1$ where g is an arbitrary scalar. All terms vanish except those from 2 to $q'+N-2$. The equation takes the form $\Sigma c_i \mathbf{b}_i \cdot \mathbf{a}_1 E_1(\mathbf{x} + g\mathbf{a}_1) = 0$ where i runs from 2 to $q' + N - 2$.

Consider the quantity $E_i\,(\mathbf{x} + g\mathbf{a}_1)$. It is a polynomial in g whose absolute term is $E_i(\mathbf{x})$. Since the above equation is to hold for all values of $\mathbf{a}_j$ it must hold for all values of g. Hence the sum of the absolute terms must vanish, that is $\Sigma c_i \mathbf{b}_i \cdot \mathbf{a}_i E_i(\mathbf{x}) = 0$ where i runs through the $q' + N - 3$ values above noted. But this is impossible unless the c's which have these subscripts are all zero: for the equation now has the form $\Sigma L(\mathbf{x})E(\mathbf{x}) = 0$, the only $\mathbf{a}$ common to L and E is $\mathbf{a}_1$, whence not fewer than $q' + N - 2$ E's satisfy such a relation, by the corollary to theorem IV.

Similarly we note that $\mathbf{a}_{m+2}$ occurs only in the second group of E's, and prove in the same way that all c's corresponding to this group must be zero, and so on, till finally only c_1 remains. Hence c_1 is also zero. This is contrary to hypothesis that the c's be not zero. Hence the determinant does not vanish identically.

To show, finally, that the rule is always possible, we note, first, that the rule observes the plan of grouping the E's so that no more than $q' + N - 2$ belong to a standard set, E_1 being common to all the groups. The first group requires $m + 1$ of the $\mathbf{a}$'s. Each new group requires one additional $\mathbf{a}$. We have $m + N - 2$ $\mathbf{a}$'s available for selection. It therefore suffices to show that the number of groups is not greater than $N - 2$. We have, in other words to prove the inequality

$$q \stackrel{=}{<} (N - 2)\,(q' + N - 3) + 1 \tag{58}$$

Putting for q and q' their values in terms of N and K this becomes

$$\frac{(K+1)\,(K+2)\cdots(K+N-2)}{(N-2)!} - N + 2 \stackrel{=}{<}$$
$$\frac{(N-2)K(K+1)\,(K+2)\cdots(K+N-3)}{(N-2)!} - (N-2)$$

which simplifies to $0 \stackrel{=}{<} (N - 2)K(K + N - 3)$. This holds when N is 3 or more, and binary forms were above explicitly excluded from consideration. The method is thus completely established.

In agreement with these results we note that when $N = 3$ we have but one group, as exemplified by (38). When $N = 4$ we shall always have two groups, but they are not both complete. If the first group be always completed, a smaller and smaller proportion of the E's of the second group is needed as K increases. When $N = 5$ both groups are complete only in the case $K = 2$. When $N = 6$ we shall need part of a third group only when $K = 2$ or 3. These and many other facts can be seen at once by a table of the numbers used in the discussion:

Case $N = 3$:

$K =$	1	2	3	4	5	6	7	
$n =$	3	6	10	15	21	28	36	
$m =$	0	2	5	9	14	20	27	$= n[N,K-1]-1$
$m + N - 2 =$	1	3	6	10	15	21	28	
$q =$	1	2	3	4	5	6	7	$= n - m - N + 1$
$q' =$		1	2	3	4	5	6	

and $q' + N - 2 = q$ so that one group suffices in 3-space.

Case $N = 4$:

$K =$	1	2	3	4	5	6	7
$n =$	4	10	20	35	56	84	120
$m =$	0	3	9	19	34	55	83
$m + N - 2 =$	2	5	11	21	36	57	85
$q =$	1	4	8	13	19	26	34
$q' =$		1	4	8	13	19	26
$q' + N - 2 =$		3	6	10	15	21	28

so that $q' + N - 2 > q/2$ for any K.

Case $N = 5$:

$K =$	1	2	3	4	5	6	7
$n =$	5	15	35	70	126	210	330
$m =$	0	4	14	34	69	125	209
$m + N - 2 =$	3	7	17	37	72	128	212
$q =$	1	7	17	32	53	81	117
$q' =$		1	7	17	32	53	81
$q' + N - 2 =$		4	10	20	35	56	84

whence again $q' + N - 2 > q/2$ for any K.

Case $N = 6$:

$K =$	1	2	3	4	5
$n =$	6	21	56	126	252
$m =$	0	5	20	55	125
$m + N - 2 =$	4	9	24	59	129
$q =$	1	11	31	66	122
$q' =$		1	11	31	66
$q' + N - 2 =$		5	15	35	70

so that $q' + N - 2 > q/2$ except when $K = 2$ or 3.

It is interesting to note that $q - q' = n[N - 2, K]$, serving as a check on the correctness of the table.

8. Special Cases of the General Identity: Method of Reduplication.

A third method of obtaining identities from the fundamental identities (5) or (9) depends on the use of a function of two variable points. I therefore call it the method of reduplication. An example of the use of this method for the case $N = 3$, $K = 2$ I have elsewhere given.[4] Let $F(\mathbf{x}, \mathbf{y})$ be a function (vector or scalar) of two points $\mathbf{x}$ and $\mathbf{y}$. Applying (5) or (9) we have

$$F(\mathbf{x}, \mathbf{y})C_0 + F(\mathbf{a}_1, \mathbf{y})C_1 + F(\mathbf{a}_2, \mathbf{y})C_2 + \cdots + F(\mathbf{a}_n, \mathbf{y})C_n = 0 \quad (59_0)$$

and by giving to $\mathbf{y}$ in succession the values $\mathbf{a}_1, \mathbf{a}_2, \cdots \mathbf{a}_n$ we have the n further equations

$$F(\mathbf{x}, \mathbf{a}_1)C_0 + F(\mathbf{a}_1, \mathbf{a}_1)C_1 + F(\mathbf{a}_2, \mathbf{a}_1)C_2 + \cdots + F(\mathbf{a}_n, \mathbf{a}_1)C_n = 0 \quad (59_1)$$

$$F(\mathbf{x}, \mathbf{a}_2)C_0 + F(\mathbf{a}_1, \mathbf{a}_2)C_1 + F(\mathbf{a}_2, \mathbf{a}_2)C_2 + \cdots + F(\mathbf{a}_n, \mathbf{a}_2)C_n = 0 \quad (59_2)$$

. . . .

. . . .

$$F(\mathbf{x}, \mathbf{a}_n)C_0 + F(\mathbf{a}_1, \mathbf{a}_n)C_1 + F(\mathbf{a}_2, \mathbf{a}_n)C_2 + \cdots + F(\mathbf{a}_n, \mathbf{a}_n)C_n = 0 \quad (59_n)$$

We now let $\mathbf{x} = \mathbf{y}$ in (59_0), then multiply these $n + 1$ equations by $C_0, C_1, C_2, C_3, \cdots C_n$, respectively, and add. For simplicity of notation we write $\mathbf{a}_0 = \mathbf{x}$. The result is

$$\Sigma F(\mathbf{a}_i, \mathbf{a}_j)C_iC_j = 0 \quad (60)$$

where both i and j run from 0 to n inclusive. From this identity, by giving to the function F various forms, scalar or vector, a great number of relations connecting the C's may be obtained. The method may also be applied, by triplication, to functions of three points, and so on.

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[4] The Axes of a Quadratic Vector, Proc. Amer. Acad. Arts and Sci., vol. 56, No. 9. June 1921, p. 333.

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THE MINIMUM AUDIBLE INTENSITY OF SOUND.

By Clifford M. Swan.

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The absolute energy value at the threshold of audibility has engaged the attention of experimenters for a number of years. It is of interest in the domain of experimental psychology, and of late has figured in researches in telephone engineering. It is also of importance in the work of architectural acoustics, since the laws governing reverberation as laid down by W. C. Sabine[1] make use of the minimum audible sound as the unit of intensity. Previous researches having shown wide discrepancies and uncertainties in results, the investigation here reported was undertaken several years ago to establish more reliable values if possible. The author is indebted to the late Professor Sabine for his assistance in making check observations in all of the determinations.

Previous Work.

In reviewing previous work in this field, we find that the first quantitative determination was made by Toepler & Boltzmann[2] in 1870. The variation in density at the node at the end of a closed organ pipe, blown at constant pressure, was measured by an optical interference method, with a stroboscopic device to slow down the rapidly shifting interference pattern and render it visible to the eye. The amplitude of vibration at any distance outside the pipe was deduced theoretically in terms of the density variation within, and the distance measured to the point at which the sound became inaudible. No attempt was made to reach an exact result, the experiment being supplementary to another piece of work, and made only to determine the order of magnitude of the quantity sought.

In 1877, Rayleigh[3] attempted to solve the problem from the theoretical amount of energy consumed and the distance at which the sound became inaudible when a whistle of high pitch was blown at constant pressure.

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[1] Contributions, Jeff. Phys. Lab., Harvard Univ., 1906, Vol. 4, pp. 1–74. Also, Collected Papers on Acoustics, W. C. Sabine, Harvard Univ. Press, 1922.

[2] Pogg. Ann., 1870, Vol. 141, p. 349.

[3] Proc. Roy. Soc., 1877, Vol. 26, p. 248.

Wead [4] followed with a series of experiments on tuning forks. An observer was stationed at a fixed distance, and the vibrations of the fork were allowed to die away, the amplitude being noted at the instant the observer ceased to hear the sound. The rate of emission of energy at the moment was calculated and thence the minimum audible energy. A somewhat similar method was employed by Rayleigh [5] a few years later.

Investigations in more recent years were made with telephone diaphragms. The experiments of Rayleigh,[6] Franke,[7] Shaw,[8] and Abraham,[9] along this line are fragmentary and do not call for extended consideration. Zwaardemaker and Quix [10] made an extensive study of the problem between pitches 32 and 12228, using tuning forks for the lower register and organ pipes for the upper. The methods employed give only the relative sensitiveness, and are open to serious criticisms which have been ably discussed by Wien.[11]

The results of all of these investigations are inconsistent, a fact which is not surprising when the sources of error are considered. The experiments in the first group were carried on in the daytime, out-of-doors, and all of the observers speak of the annoyance caused by wind, birds and other extraneous sources of sound. As anyone will appreciate who has tried such experiments, the slightest disturbing sound is sufficient to preclude accuracy in measurements, the sound being so faint near the threshold value that it is entirely obliterated unless absolute quiet prevails. Further criticism may be made of the use of the inverse square law in the computations. It is not conceivable that this law should hold, even in an open field, as the absorbing power of the surface of the ground alone is sufficient to render it invalid. Beyond these obvious difficulties, which amply account for inconsistencies in the results, note should be made of the fact that, contrary to assumption, all of the energy consumed by a sounding body does not reappear as sound, thus furnishing another source of error in many of the cases mentioned.

Some interesting work has been done by Wien. The early investigation of that experimenter[12] consisted in the measurement of the pressure variations in a sound wave at the limit of audibility by means of a sensitive resonator. This method was finally discarded on account of the large errors of observation inherent in it, and another line of attack adopted.[13] The relative sensitiveness of the ear for tones of different pitch ranging from 50 to 16,000 vibrations was determined by two methods. One of these involved a direct measurement of the relative minimum intensity by means of the current necessary to produce a sound in a telephone diaphragm pressed against the ear and acting like a piston in the outer chamber of the ear. The other method consisted in the calculation of the relative intensity from the current necessary to produce audibility at a distance 30 centimeters from the telephone. In the latter case the diaphragm was placed in an aperture in the extended plane surface, so that it was theoretically possible to calculate the energy at any point in the neighborhood for any amplitude of the diaphragm. By the second method the absolute as well as the relative values of the minimum audible sound for different pitches were determined. The results of the two methods give curves for the *relative* sensitiveness with differing pitch which are in close agreement. The *absolute* values are, however, much smaller than those obtained by any other experimenter. Rayleigh [14] doubted the validity of the Wien sensitiveness curve, owing to certain experiments of his own, but in general Wien's results were, for many years, accepted as correct.

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[4] Sill. Jour., 3rd Series, 1883, Vol. 26, p. 177; and 1891, Vol. 41, p. 232.

[5] Phil. Mag., 1894, Vol. 38, p. 365.

[6] Phil. Mag., 1894, Vol. 38, p. 285.

[7] Elektrotech. Zeit., 1890, Vol. 11, p. 288; and Jour. Inst. Elec. Engrs., Vol. 19, p. 754.

[8] Proc. Roy. Soc., Series A, 1905, Vol. 76, p. 360.

[9] Comptes Rendus, 1907, Vol. 144, p. 1099.

[10] Arch. f. Anat. und Physiol., Suppl., 1902, p. 367.

[11] Pflueger's Arch. f. Gesammte Physiol., 1903, Vol. 97, p. 48.

In 1921, Kranz [15] employed a thermophone pressed against the ear as a calculable source of sound, and by gradually reducing the current until he could no longer detect the opening or closing of the circuit, he determined the minimum audible energy. His results have been published for only one pitch.

Lane [16] has conducted an investigation over the upper range of pitch, above 2000, using a Hewlett tone generator as a source. His experiments were carried on at night, out-of-doors, and the disturbance due to the noise of insects is mentioned. His results are therefore open to the same criticism as the work of the early experimenters already noted. He finds the minimum audible intensity to remain practically constant between pitches 2000 to 14000.

More recently, Fletcher and Wegel [17] have completed an extensive research using a special air damped telephone receiver pressed to the ear, with an alternating current from a vacuum tube oscillator, reduced by an attenuator until the sound was just audible. Their results cover a range of pitch from 60 to 4000 cycles.

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[12] Wied. Ann., 1889, Vol. 36, p. 834.
[13] Pflueger's Arch. f. Gesammte Physiol., 1903, Vol. 97, p. 1.
[14] Phil. Mag., 1907, Vol. 14, p. 596.
[15] Physical Review, Second Series, March, 1921, Vol. 17, p. 384.
[16] Phys. Rev., Second Series, May, 1922, Vol. 19, p. 492.
[17] Phys. Rev., Second Series, June, 1922, Vol. 19, p. 553.

Table 1 gives a summary of the results of the principal investigations already mentioned, and shows their manifest discrepancies. The figures give the energy in ergs per square centimeter per second.

Aside from the criticisms which each investigator has made of the researches which have preceded his own, the criticism offered from the standpoint of this paper is on the one hand the impossibility of getting conditions for testing minimum audibility out-of-doors, and on the other, the neglect in indoor experiments of the intensifying effect of reverberation and the complication arising from interference. In this research a new method has been adopted which eliminates the one and takes account of the other.

METHOD.

Some work in architectural acoustics by Sabine[18] suggested a new method of approaching the problem. Sabine shows that the time of decay of sound in a closed room after the source has ceased is a function of the initial intensity in the room and of the minimum audible intensity. Hence if the first two of these quantities can be measured experimentally, the last can be calculated. In order to compute the initial intensity in a room, however, it is necessary not only to know certain constants of the room, but also the rate of emission of energy from the sounding source. That form of sound emission which lends itself most readily to theoretical computation is a circular diaphragm, free at the edges, vibrating in an infinite plane wall. In the present investigation, a near approximation to such a source was employed by a circular diaphragm attached to one prong of a tuning fork vibrating in an aperture in a door flush with one wall of a room. By means of electrical connections to be described presently, the sounding source could be damped almost instantly and the time of decay of the residual sound in the room measured by a chronometer.

THEORY.

The rate of emission of energy from a circular diaphragm, free at the edges, vibrating in an infinite plane wall, may be calculated as follows:

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[18] loc. cit.

TABLE 1.

Pitch	Toepler & Boltzmann 1870	Rayleigh 1877	Wead 1883	Wien 1889	Rayleigh 1894	Wien 1902	Kranz 1921	Lane 1922	Fletcher & Wegel (approx.) 1922
60									5.5×10^{-4}
128			8×10^{-5}						3×10^{-5}
181	8.3×10^{-5}								2.3×10^{-6}
200						5.8×10^{-9}			
220				1×10^{-8}					
256			3×10^{-6}		9.3×10^{-7}				1.2×10^{-6}
384			2.6×10^{-6}		5.5×10^{-7}				
400						11×10^{-11}			
440				7.3×10^{-9}					
512			11×10^{-6}		5.5×10^{-7}				7.8×10^{-8}
600						6.6×10^{-11}			
768			16×10^{-6}						
1024			7.1×10^{-6}						2×10^{-8}
1050						3.5×10^{-12}			
2048							2.2×10^{-8}	7×10^{-8}	1.5×10^{-8}
2730		4×10^{-5}						"	"
3500								"	"

The velocity potential at a distance r from a given point in a medium vibrating with simple harmonic motion may be expressed as

$$P \cos (2\pi n t + \epsilon)$$

where n is the pitch of the sound, and P and ϵ are independent of the time t, but are dependent on r. This may be represented by the real part of the imaginary expression

$$\varphi = P e^{i(kvt+\epsilon)}$$

where

$$k = \frac{2\pi n}{v} = \frac{2\pi}{\lambda}$$

v being the velocity of sound and λ the wave length.

The equation of motion has the general form

$$\frac{\partial^2 \varphi}{\partial t^2} - v^2 \nabla^2 \varphi = \Phi$$

but

$$\frac{\partial^2 \varphi}{\partial t^2} = - k^2 v^2 P e^{i(kvt+\epsilon)}$$

$$= - k^2 v^2 \varphi$$

hence

$$\nabla^2 \varphi + k^2 \varphi + v^2 \Phi = 0$$

The solution of this equation is [19]

$$\varphi = \frac{1}{4\pi v^2} \int\!\!\int\!\!\int \frac{e^{-ikr}}{r} \Phi \, d\tau$$

When the force has the value Φ_1 and is distributed only over a surface S in the medium,

$$\varphi = \frac{1}{4\pi v^2} \int\!\!\int \frac{e^{-ikr}}{r} \Phi_1 \, dS$$

If the surface is plane, and if $\frac{d\varphi}{dn}$ is the normal velocity of the medium at the surface of the plane, then

$$\varphi = - \frac{1}{2\pi} \int\!\!\int \frac{d\varphi}{dn} \frac{e^{-ikr}}{r} dS$$

---

[19] Rayleigh, Theory of Sound, Art. 277, (3).

If the force is distributed over a solid plane of infinite extent, double the energy is emitted, and

$$\varphi = -\frac{1}{\pi}\int\int \frac{d\varphi}{dn}\frac{e^{-ikr}}{r}\,dS$$

If we consider the reaction of the air upon the vibrating plane, the rate of emission of energy will be

$$\frac{dW}{dt} = \int\int \frac{d\varphi}{dn}\cdot\delta p\cdot dS$$

where δp is the pressure variation. If σ is the density of the air,

$$\delta p = -\sigma\frac{d\varphi}{dt}$$

$$\frac{dW}{dt} = -\sigma\int\int \frac{d\varphi}{dn}\frac{d\varphi}{dt}\,dS$$

$$= -ikv\sigma\frac{d\varphi}{dn}\int\int \varphi\,dS$$

if $\frac{d\varphi}{dn}$ is constant over the plane.

Then

$$\frac{dW}{dt} = \frac{ikv\sigma}{\pi}\left[\frac{d\varphi}{dn}\right]^2 \Sigma\,\Sigma\,\frac{e^{-ikr}}{r}\,dS\cdot dS'$$

This is to be solved for the case where a circular rigid portion of the plane is vibrating with simple harmonic motion, while the remainder of the plane is fixed.

The evaluation of this expression for the case in question has been shown by Rayleigh[20] to be

$$\frac{dW}{dt} = \frac{ikv\sigma}{\pi}\left[\frac{d\varphi}{dn}\right]^2 \left\{\frac{\pi^2}{2k^3}K_1(2ka) - \frac{i\pi^2a^2}{k}\left[1 - \frac{J_1(2ka)}{ka}\right]\right\}$$

where a is the radius of the disc, and J_1 and K_1 are Bessel's functions of the first and second kinds respectively.

If x is the displacement of the disc at time t, and α is its amplitude of vibration,

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[20] Theory of Sound, Vol. II, Art. 302.

$$x = \alpha e^{i(kvt+\eta)}$$

$$\frac{d\varphi}{dn} = \frac{dx}{dt} = ikv\alpha e^{i(kvt+\eta)}$$

$$\frac{d^2x}{dt^2} = -k^2v^2\alpha e^{i(kvt+\eta)}$$

Then

$$\frac{dW}{dt} = \frac{\pi\sigma}{2k^3}\cdot\frac{dx}{dt}\cdot\frac{d^2x}{dt^2}K_1(2ka) + \pi a^2 v\sigma\left[\frac{dx}{dt}\right]^2\left[1 - \frac{J_1(2ka)}{ka}\right]$$

Taking the real part of $\frac{dx}{dt}$ and $\frac{d^2x}{dt^2}$, we have

$$\frac{dW}{dt} = \frac{\pi v^3\sigma\alpha}{2}\sin(kvt+\eta)\cos(kvt+\eta)\,K_1(2ka)$$
$$+ \pi a^2k^2v^3\sigma\alpha^2\sin(kvt+\eta)\left[1 - \frac{J_1(2ka)}{ka}\right]$$

The first term is entirely periodic and hence disappears after the fork has made a few vibrations and gained its full amplitude. The mean value of $\sin^2(kvt+\eta)$ is $\frac{1}{2}$. Hence, in the steady condition,

$$W = \frac{\pi a^2k^2v^3\sigma\alpha^2 t}{2}\left[1 - \frac{J_1(2ka)}{ka}\right]$$

Hence the average rate of emission of energy is

$$E = \frac{W}{t} = 2\pi^3a^2v\sigma n^2\alpha^2\left\{1 - \frac{vJ_1\left(\frac{4\pi na}{v}\right)}{2\pi na}\right\}$$

Sabine [21] has shown that in a closed room of volume V,

$$E = VA\,i'\,e^{At}$$

where i' is the minimum audible energy per cubic unit, and

$$A = \frac{\log_e\frac{I_2}{I_1}}{t_2 - t_1} = \frac{\log_e\left[\frac{\alpha_2}{\alpha_1}\right]^2}{t_2 - t_1}$$

---

[21] loc. cit.

Combining these expressions, we have

$$i' = \frac{2\pi^3 a^2 v\sigma n^2 \alpha^2}{VAe^{At}} \left\{ 1 - \frac{vJ_1\left(\frac{4\pi na}{v}\right)}{2\pi na} \right\}$$

In order to find the energy per square centimeter per second, it is necessary only to multiply the value of i' by the velocity of sound v, thus giving the expression

$$E' = i'v = \frac{2\pi^3 a^2 v^2 \sigma n^2 \alpha^2}{VAe^{At}} \left\{ 1 - \frac{vJ_1\left(\frac{4\pi na}{v}\right)}{2\pi na} \right\}$$

from which the results of this investigation were computed. In order to determine A, it is evidently necessary to measure the time of decay of the sound for at least two different amplitudes of the fork.

APPARATUS.

The experiments to be described were made in the Constant Temperature Room of the Jefferson Physical Laboratory of Harvard University. This room has a volume of 76 cubic meters and is constructed with heavy double brick walls situated in the sub-basement of the tower of the Laboratory and free from structural connection with the surrounding building. It is situated at some distance from the street and its attendant noises. All observations were made at night under conditions of absolute quiet.

The source of sound consisted of a circular aluminum diaphragm, free at the edges, vibrating in a circular aperture in the door of the room, the diaphragm being attached to one prong of a tuning fork F outside of the door.

Symmetrical conditions in respect to weight of the fork were secured by attaching an exactly similar diaphragm to the other tyne, idle of course as far as producing sound in the room was concerned, since it was located outside of the room. Symmetrical damping conditions were secured by surrounding the second diaphragm with a guard-ring. The air damping of the diaphragms was reduced by tuning to resonance the space between them. This was accomplished by means of a plunger resonator. This device also served the even more important purpose of causing the air in the very small annulus between the guard-rings and the diaphragms to follow the motion of the latter. The amplitude of vibration of the tyne of the fork was measured by observing through a microscope provided with a micrometer eyepiece the edge of a razor blade attached to the tyne.

The fork *F* was driven electromagnetically by a second fork *D* situated in a distant room to remove it from the range of hearing. On account of unequal changes of temperature in the neighborhood of the two forks, it was necessary to tune the driving fork frequently to keep it in unison with the natural period of the sounding fork. The driving fork was operated without intermission, the sounding fork only at the pleasure of the observer in the Constant Temperature Room. The latter fork was controlled by a key *K* in the hands of the observer which closed a circuit through a relay *R*, causing the driving current

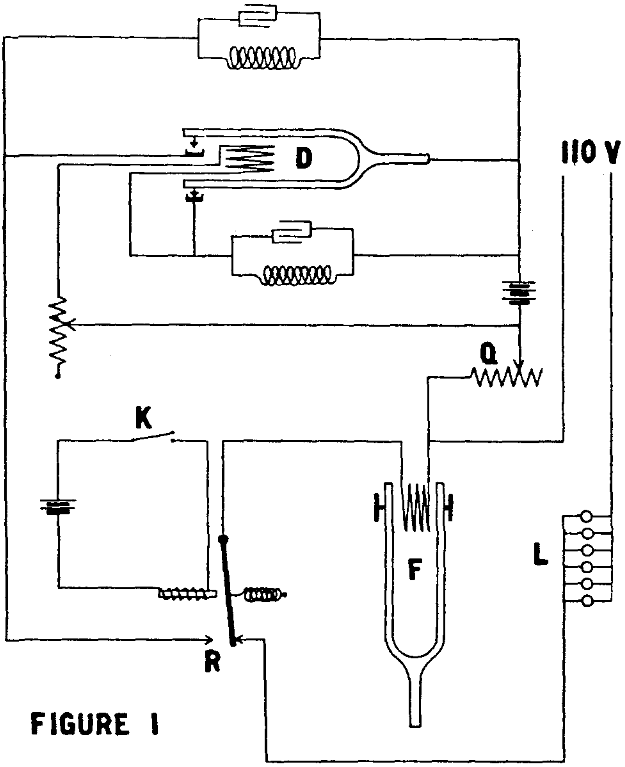

FIGURE 1

to flow through the coils of the fork *F* and set the latter in vibration. The amplitude of this vibration was governed by a rheostat *Q* in the driving circuit.

During the course of an experiment, the observer sat within a box placed against the wall of the room, so constructed that only his head was exposed. The aggregate of absorbing material in the room was thus diminished, the residual sound prolonged, and the accuracy of the observations increased. On the other hand, it created a certain difficulty. In an enclosed room the sound establishes an interference system which must be reckoned with in any acoustical investigation. On account of this interference pattern, the observer, just as the sound ceases to be audible, may be either at a point of maximum or of minimum disturbance, or between the two. It is therefore desirable to take a number of observations at different points relative to the interference system in the room, which is obviously impossible if the observer is to be confined within the box. The difficulty was overcome, however, in a manner much more effective than by moving the observer. The interference system itself was made to shift by a pair of large reflectors reaching more than half way across the room, suspended from a horizontal bar, and revolved by clock work about a vertical axis. This served also to eliminate the effect of the reaction of the sound in the room on the source.

The time of decay was measured by a stop-watch so arranged that it began to record at the instant the source of sound was stopped by the key *K*, and continued to record until stopped by the observer when the residual sound in the room reached the limit of audibility.

Much difficulty was experienced, at first, owing to the continued vibration of the sounding-fork after the current had been broken. An effective method of damping was secured by causing the relay *R* to close a 110 volt direct current through the coil of the fork *F* when the driving current was broken. This pulled the prongs of the fork violently together and stopped the vibration instantly. The amount of current was so great, however, as to burn out the coil of the fork in a very short time. The ideal to be sought, evidently, was some means of permitting a momentary flow of a current sufficiently great to damp the fork instantly, followed by an immediate reduction in the current-strength to an amount which could be safely maintained through the coil. Not only this, but the current must be reduced gradually in strength in order to allow the tynes of the fork to resume their normal position slowly and not produce further sound, as would be the case if they were allowed to snap back into place. A way of solving these difficulties was found by introducing into the 110 volt circuit several 60 watt Mazda lamps, L, connected in multiple. The lamps being cold when the driving current of the fork is broken by the relay, they permit a momentary large damping current to flow, but their rapid increase of resistance as they heat causes exactly the amount and nature of reduction of current-strength which is demanded. The practical operation of this simple but effective device leaves nothing to be desired, and is evidently applicable to many laboratory problems. It was found necessary to use a current from a storage battery as the small periodic fluctuations in the current from a generator were sufficient to produce an audible vibration of the fork. To prevent arcing at the relay on breaking the 110 volt circuit, the gap was immersed in oil.

Observations of the time of decay of the sound were made with four pitches approximately one octave apart, this being the maximum range over which the tuning fork method was found to be practicable. Each series of observations was repeated on several successive nights and checked by two different observers. In all of the experiments an assistant was required to note the amplitude of the sounding fork and to keep it constant. This required incessant watching and adjustment by means of the rheostat on account of small fluctuations in the driving current. With every pitch several different amplitudes of vibration were used for determining the rate of decay in the room, and the mean time of decay corresponding to these amplitudes was obtained from fifty to one hundred chronometer observations in every case. The results are shown in Table 2, where

a = radius of diaphragm
α = amplitude of vibration
n = pitch
t = time of decay of residual sound
A = rate of decay
E' = energy in ergs per sq. cm. per second.

TABLE 2

Observer M				Observer N			
α	t	A	E'	α	t	A	E'
$n = 121$ $a = 2.9$ cm.							
0.040 cm. 0.090 0.182	2.87 3.37 3.86	3.1	5.2×10^{-5}				
0.060 0.180	3.54 4.31	2.9	3.2×10^{-5}	0.060 0.176	3.73 4.59	2.5	8.0×10^{-5}
0.060 0.180	3.31 4.10	2.8	8.1×10^{-5}	0.060 0.180	3.58 4.37	2.8	3.9×10^{-5}
$n = 246$ $a = 2.1$ cm.							
0.004 0.017 0.077	3.09 3.81 4.52	4.2	1.9×10^{-8}	0.003 0.017 0.075	3.10 3.98 4.62	4.3	0.58×10^{-8}
0.005 0.020 0.077	3.10 3.81 4.51	3.9	6.6×10^{-8}	0.005 0.020 0.080	3.14 3.82 4.53	4.0	3.9×10^{-8}
0.005 0.020 0.080	3.14 3.70 4.36	4.6	0.61×10^{-8}				
$n = 493$ $a = 1.1$ cm.							
0.005 0.015 0.045	2.94 3.35 3.86	4.8	4.8×10^{-9}	0.0025 0.015 0.045	2.72 3.40 3.83	5.2	1.0×10^{-9}
0.005 0.015 0.045	2.92 3.36 3.81	4.9	3.4×10^{-9}	0.005 0.015 0.045	2.98 3.39 3.94	4.6	6.9×10^{-9}
0.005 0.015 0.045	3.12 3.58 4.09	4.5	4.8×10^{-9}	0.005 0.015 0.045	3.20 3.68 4.09	5.0	0.7×10^{-9}
$n = 1021$ $a = 0.93$ cm.							
0.0048 0.012	2.97 3.30	5.6	11×10^{-10}	0.0048 0.0144	2.82 3.27	5.8	8.5×10^{-10}
0.0048 0.012	2.98 3.27	6.3	0.97×10^{-10}	0.0048 0.012	2.82 3.14	5.7	15×10^{-10}

RESULTS.

The results are summarized in Table 3. Following Wien, the logarithm of the reciprocal of the minimum energy is used as an index of the sensitivity. In Figure 2, these sensitivity values are plotted against the pitch. Curve 1 shows the result of the present work. Curve 2 shows Wien's final work, and Curve 3 shows that of Fletcher and Wegel. The scattering results of other investigators are also indicated.

TABLE 3.

Pitch	E' Observer M	E' Observer N	E' Weighted mean	$\frac{1}{E'} = S$	$\log_{10} S$
121	5.5×10^{-5}	5.9×10^{-5}	5.7×10^{-5}	1.8×10^{4}	4.3
246	3.0×10^{-8}	2.2×10^{-8}	2.7×10^{-8}	3.7×10^{7}	7.6
493	4.3×10^{-9}	2.9×10^{-9}	3.6×10^{-9}	2.8×10^{8}	8.4
1021	6.0×10^{-10}	12×10^{-10}	9.0×10^{-10}	1.1×10^{9}	9.0

It will be noted that the energy values of the present investigation are somewhat smaller than those obtained by Rayleigh and lie very near the earlier results of Wien, but are considerably larger than Wien's later determinations. This is not surprising as we have already seen that the work of early experimenters was carried on under conditions lacking in quiet which would give too large results, while the later work of Wien is open to criticisms quite as serious operating in the opposite direction.

It is a surprising fact that, in all previous investigations either by physicists or psychologists, the assumption has been made that the sound energy is propagated in spherical or hemispherical waves, obeying the law of inverse squares. Such an assumption is unjustifiable even in experiments conducted out-of-doors, owing to the porosity of the surface of the ground which absorbs the energy and alters its distribution. In a closed room, the sources of error are multiplied on account of reflection from the walls, and here lies the principal criticism to be applied to the later work of Wien.

No matter how small the amount of energy emitted by a source in a closed room, the wave will be reflected from surface to surface many times before it is dissipated by absorption and transmission. There are three effects of this multiple reflection; first, it increases to a large degree the average intensity of sound within the room, second, it sets

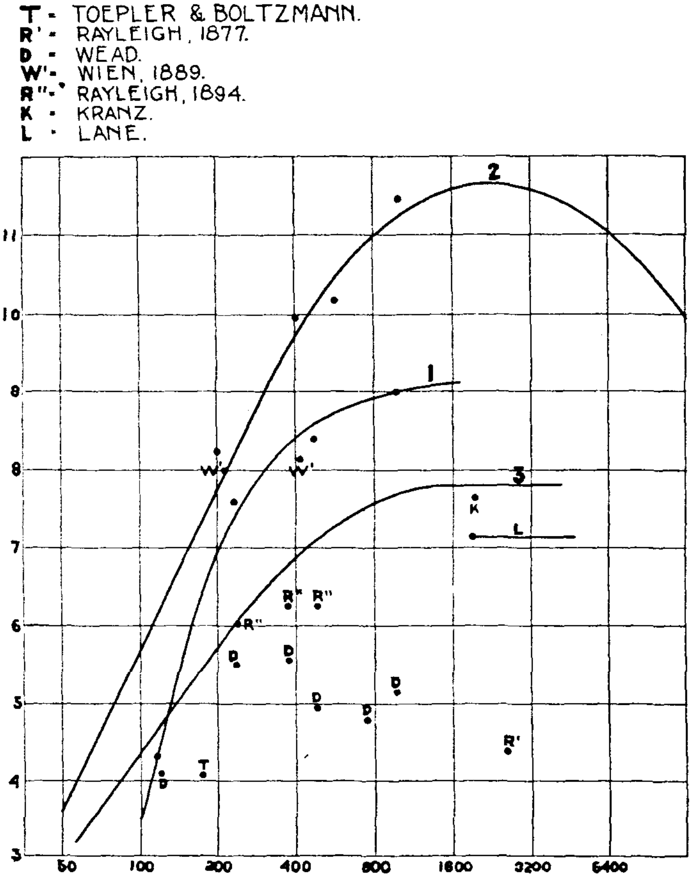

FIGURE 2

up a stationary wave system with definite regions of maximum and minimum intensity, and third, it reacts on the source to change the amount of energy emitted.

In a room such as the lecture hall used by Wien for his experiments, the actual average intensity of sound throughout the room under normal conditions may be roughly estimated to be at least twenty-five times the average intensity which would exist with totally absorbent walls. Hence, as the current in Wien's telephone was reduced until the sound was just audible at a distance of thirty centimeters, only a fraction reaching the ear came in a direct path from the telephone, the balance being contributed by the reverberation in the room. In other words, at the limit of audibility the current in the telephone was less than would have been required for minimum audibility in a non-reverberant space, thus giving an apparent sensitiveness greater than actually was the case.

Furthermore, with a fixed relation between the source and the ear, as in Wien's experiments, the interference system in the room is not to be disregarded, as the ear cannot conceivably be at a point of average disturbance for all pitches with a constant distance between ear and telephone, and is not likely to be at such a point for even a single pitch.

Add to this the reaction of the room upon the source, making it impossible to determine the amount of energy emitted by the telephone diaphragm or even to compare sources of different pitch, and the wonder is, not that Wien's results are too small, but that there is any possible coincidence that should have caused them to fall on anything approaching a regular curve.

The claim has been made that the wide discrepancies in previous investigations are due to inadequate apparatus and to differences in hearing in the normal ear. These criticisms are not altogether justified, however, and the true causes of the variations are undoubtedly due to the reasons already discussed. Differences between ears, even so-called normal ears, do exist in some cases, no doubt. Even a trained ear may vary from night to night owing to nervous fatigue or a slight cold. But experience for many years with the average results of a number of *trained* observers in measuring reverberation has shown a surprising conformity in threshold sensation. One cannot help but wonder if the differences noted by some experimenters is not due largely to *unskilled* observation. The sense of hearing must be trained just as the sense of taste in the case of a tea taster. It is assuredly impossible to call upon inexperienced students or members of the staff to make a few casual observations with any hope of consistency.

The values of the minimum audible intensity obtained by Fletcher and Wegel are exceedingly interesting, as they are free from most of the sources of error we have noted. They are obtained, however, from the averages of forty-one persons, both men and women, many, if not most of whom, could hardly have been experienced in detecting the very faint sound near the threshold of audibility. The inevitable result with persons untrained to use their ears is to indicate an end-point long before it is properly reached. This would give the final averages too great a threshold value or too small a sensitivity. The trend of their curve bears this out, as it lies quite close to the values of Rayleigh and of Lane, whose results, as we have seen, are undoubtedly too large owing to other causes.

To sum up, all other investigations in this subject may be grouped in three classes, those carried on out-of-doors, those confined in a closed room with neglect of boundary conditions, and those resulting from too great an admixture of unskilled observation. Each of these has inherent sources of error, tending in the first and third instances to produce too large results, and in the second, too small. The errors in the first case are difficult to eliminate; not so in the other two. It has been the object of this research to undertake such a determination in a closed room with two observers of long experience, using an entirely new method which takes account of the interference and reverberation. The results seem to justify the expectations.

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THE SALAMANDERS OF THE FAMILY HYNOBIIDAE.

By Emmett Reid Dunn.

THE SALAMANDERS OF THE FAMILY HYNOBIIDAE.[1]

BY EMMETT REID DUNN.

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THE Salamander family *Hynobiidae* is a group of some five genera and twenty-one species. They are animals of northern Asia, the Urals and the mountains of Sze-chuan marking their extreme outposts to the West and to the South. A species of one of the genera (*Ranodon olympicus* Gaige) has been recorded from North America, but I have shown that this animal is not a *Ranodon,* and should be considered as the type of a distinct genus *Rhyacotriton,* which belongs to the *Ambystomidae,* and is most closely related to the genus *Dicamptodon.* (Proc. New England Zool. Club, VII, p. 55, 1920).

The relationships of the family are with the large river-salamanders, the *Cryptobranchidae.* If the *Hynobiidae* are not directly ancestral to these, at least they represent very well the less modified type from which such flattened, aquatic forms must have arisen. That the *Cryptobranchidae* are known from as far back as the Oligocene compels us to assume a still earlier era as the time of origin of the *Hynobiidae.* The characters on which this relationship is based serve also to distinguish between the *Hynobiidae-Cryptobranchidae* on the one hand and the remaining groups of salamanders on the other. To express this relationship I have proposed the superfamily name *Cryptobranchoidea.* (Amer. Nat. LVI, p. 418, 1922). These are certainly not derived from any existing group of salamanders, nor is it possible to point out any groups descended from them.

From a phylogenetic standpoint *Hynobius* occupies a central position, the other four genera showing various secondary modifications.

Within the genus *Hynobius* itself the Korean form, *H. leechii,* apparently represents the stock from which the other forms have descended. The other continental species, *chinensis, turkestanicus,* and *keyserlingii,* each bear marks of specialization. These four forms occupy separate areas. To the eastward, however, in the islands, the situation is more complex, as many as four forms of *Hynobius* occurring on Hondo.

The species of Sakhalin, *Hynobius cristatus,* seems to have been derived from *keyserlingii* of the opposite mainland.

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[1] Contributions from the Department of Zoölogy, Smith College, No. 105.

On Hondo, *peropus* of the high interior is very similar to the Korean *H. leechii*, while the very specialized form on Yezo, *retardatus*, has evident affinities with *peropus*. These two may well be the oldest *Hynobius* stock of Japan, as none of the other groups have reached Yezo.

A rather isolated group, (*stejnegeri* of Kiusiu, *kimurai* of Hondo, and *naevius* of Hondo and Shikoku), has affinities remotely with *leechii* rather than with any of the other Japanese forms, and may represent another invasion. The Formosan species, *sonani*, is much like *naevius*. It may be a member of the *naevius* group, or the likeness may be parallelism. Indeed this series of four species may have come from some mainland form rather different from *leechii*.

A third group, the animals of the islands in the Korean straits, connects *leechii* with *nebulosus* of Kiusiu, with *vandenburghi* of Hondo, and with the as yet undescribed form from Shikoku. Apparently this group represents the most recent invasion.

Pachypalaminus boulengeri, is, save for its peculiar feet, very like *Hynobius vandenburghi* of the same general region. This may be taken in the absence of other evidence to indicate genetic relationship.

Batrachuperus sinensis has the same foot modification, and what are essentially weakened prevomers, and weakened vomerine dentition. It has probably arisen from some form of *Hynobius*, many of its characters, as color, loss of the fifth toe, and development of the labial folds, suggesting *H. keyserlingii*.

Ranodon sibiricus has the same dentition as *Batrachuperus* but none of the other characters of that animal. It has probably arisen independently from some species of *Hynobius*. *H. turkestanicus*, as the only *Hynobius* of that region, may be the closest relative.

Onychodactylus, with two closely allied species, is developed for mountain brook life, and the lungs have disappeared. The prevomers are shortened, and larvae and breeding adults have claws. Otherwise it is not very different from *Hynobius*, from some species of which it has doubtless arisen. No modern species of *Hynobius*, however, is at all close to *Onychodactylus*.

The course of evolution of the family has been about as follows: The primitive stock was a series of essentially similar species spread over northern Asia. Owing, perhaps, to different immigrations, either in space, or in time, or in both, more than one form of *Hynobius* is found in Hondo, in Shikoku, and in Kiusiu. From this series of essentially similar species modified forms have arisen in several places, (Semiryechensk, Sze-chuan, Ussuri, Hondo). All these modified forms seem adapted to a more aquatic mode of life. A similar process in the more remote past produced the *Cryptobranchidae*.

Owing, perhaps, to their dependence on external fertilization, this group seems never to have produced forms adapted to a less aquatic mode of life. In this the *Hynobiidae* offer a great contrast to the *Plethodontidae* in which all the forms practise internal fertilization, and in which many or most of the species are terrestrial.

From a geographic standpoint the inferences to be drawn from the distribution of the *Hynobiidae* are: that Sakhalin has been connected with the mainland rather than with Yezo; that the Tsugaru Strait is of great age, Yezo having been separated from Hondo for considerable time; that the other islands of Japan have been connected with each other and with Korea both before and after the separation of Yezo; and that the present Korean Strait is of recent age.

Before embarking upon a systematic survey of the family it may be well to review briefly the significant features of the anatomy and the variations thereof.

The prearticular bone is distinct in the lower jaw of *Hynobiidae* and *Cryptobranchidae* and fused with the angular in the jaw of all other salamanders. Larvae of *Onychodactylus fischeri* have a horny edge on the dentary as well as teeth. This structure is seen elsewhere among salamanders only on the dentary of the *Sirenidae*.

A free lachrymal, extending from the orbit to the naris, is a feature of all Hynobiid skulls examined by me. (*B. sinensis, O. fischeri, H. leechii, H. peropus, H. kimurai, H. retardatus, H. vandenburghi*). Wiedersheim figures a lachrymal in *R. sibiricus* and in *H. naevius* and mentions its presence in *H. keyserlingii*. But Schlegel figures *H. naevius* as without a lachrymal and depicts a similar condition in *O. japonicus,* while Okajima failed to find a lachrymal in the latter animal. I can only comment that the bone is hard to see at times, being overlapped by both maxilla and nasal, and underlaid by the prefrontal.

In both *Hynobiidae* and *Cryptobranchidae* the nasals meet in the middorsal line and the nasal spines of the premaxillae are very short. This is in decided contrast to the condition of these elements in *Ambystomidae, Salamandridae,* and *Plethodontidae,* where the nasals are nearly always separated by the long spines of the premaxillae. A large premaxillary fontanelle is present in *Onychodactylus:* a smaller one in *Pachypalaminus,* according to a sketch kindly furnished me by Dr. Van Denburgh; and small ones are found in *Batrachuperus,* and, according to Wiedersheim's figure, in *Ranodon*. Except in young *H.*

retardatus this fontanelle is absent in *Hynobius* and in the *Cryptobranchidae.*

The prevomers of *Hynobius, Pachypalaminus,* and *Onychodactylus* are broadly in contact on the median line, and extend to a greater or less extent posteriorly. The dentition consists of a short outer branch, curving forward and in, and a longer inner branch extending backward and inward.

The length of this inner branch depends upon the length of the prevomer. Both bone and tooth row are shortest in *Onychodactylus.* In *Batrachuperus,* where the prevomers do not meet on the median line, the inner branch is absent, and apparently the same condition is found in *Ranodon.*

Both columella and operculum are present and free in *Hynobius leechii.* The *Cryptobranchidae,* and probably *Ranodon* and *Onychodactylus* as well, lack the operculum.

The hyobranchial apparatus of the family agrees with that of the *Cryptobranchidae* in having the ceratobranchial and epibranchial of the first arch fused into a cartilaginous rod, and in having the ceratobranchial and epibranchial of the second arch present and free. The *Salamandroidea* (*Ambystomidae, Salamandridae,* and *Plethodontidae*), have the elements of the first arch usually separate, and always lack the epibranchial of the second arch.

No family characters have yet been observed in vertebrae of salamanders but I think it worth while to mention the complete agreement of vertebrae of *Cryptobranchus, Onychodactylus,* and *Hynobius,* and the fact that no such agreement is found on comparison with vertebrae of *Ambystoma,* of *Triturus,* or of *Desmognathus.*

The ribs of *Hynobius,* of *Onychodactylus,* and of *Cryptobranchus* are widened proximally, but are not forked. Ribs of *Salamandroidea* are forked.

The number of costal grooves varies from eleven to fifteen, thirteen being the usual number. Individual variation in number amounts to two or three costal grooves.

Occasional specimens of *Ranodon* and of some *Hynobius* have a lateral fold of skin between the insertions of the legs, such as is very prominent in the *Cryptobranchidae.*

The legs are always well developed. The appressed toes may overlap along the sides, or be separated by as many as four costal folds. This varies individually, as well as with age and sex. Young have proportionately longer legs than adults as determined by this criterion. Among the adults the males have longer legs than the females.

Larval *Onychodactylus* have a fold of skin along the hind border of the limbs as do the *Cryptobranchidae*.

According to Noble (Bull. Amer. Mus. Nat. Hist., XLVI, 1, pp. 1–87, 1922) the *Cryptobranchidae* differ from other salamanders in having the pubotibialis muscle fused with the puboischiotibialis, and in having the iliotibialis not in contact with the ilioextensorius. My own dissections of *H. leechii* and of *O. japonicus* show agreement with the *Cryptobranchidae* in that they too have the pubotibialis fused with the puboischiotibialis, but in the other character they agree with the majority of salamanders and the two muscles in question are in contact nearly throughout their length. Dr. Noble holds that the *Cryptobranchidae* are primitive in having these muscles separated, but I think this view untenable, as they are in contact in *Hynobius* and in *Onychodactylus*, and I would suggest that their separation in *Cryptobranchidae* is a consequence of the dorso-ventral flattening which these animals have undergone, the flattening and widening of the thigh having drawn these muscles somewhat apart.

I find the following elements in the carpus of *H. leechii:* radiale, ulnare (to which is fused the intermedium), two centralia, four carpalia, and a bit of cartilage corresponding to the tubercle under the base of the first finger which according to Kehrer is a rudiment of the prepollex. In the tarsus I find tibiale, intermedium, fibulare, two centralia, six tarsalia and a bit of cartilage corresponding to the tubercle under the base of the first toe. These small bits of cartilage are distal to and in line with the radiale and first carpale and the tibiale and first tarsale respectively. All the other elements as well as the digits converge strongly towards the ulnar or the fibular. This state of affairs is quite in consonance with the findings of Wiedersheim, Kehrer, and Schmalhausen, in *H. keyserlingii* and in *Ranodon*. Their conclusions were that the original Tetrapod appendage was heptadactylous, and that these Asiatic salamanders had the most primitive carpus and tarsus found in living Tetrapods. The tubercles at the bases of the first digits are of almost universal occurrence in Hynobiids, and Kehrer's idea that they represent prepollex and prehallux is worthy of consideration.

The fifth toe is lacking in *Batrachuperus*, in *H. keyserlingii*, and in *H. cristatus*. It is nearly always absent in *H. kimurai*, and may occasionally be missing or rudimentary in other species of *Hynobius*.

Batrachuperus and *Pachypalaminus* have the under surfaces of the hands and feet and the tips of the digits covered with thickened, horny epidermis.

The larvae of both species of *Onychodactylus* possess claws, and these are present also in the breeding adults of the Japanese species.

The tail varies from short and thick, to very flat, and to long and slender. *H. kimurai* offers the extreme development of the first type; *Ranodon, Batrachuperus, Pachypalaminus, H. vandenburghi,* and *H. retardatus* are examples of the second; *Onychodactylus* has the third type of tail.

H. leechii and its allies with a moderately long tail, cylindrical at the base, and flattened towards the tip, perhaps represent the primitive tail form. The proportionate length of the tail increases with age.

The larvae of *Hynobius* and of *Batrachuperus* have a dorsal fin on the body. This is not present in larvae of *Onychodactylus*, the fin being restricted to the tail, and as this is the case in most mountain brook forms, it is probably confined to the tail in *Ranodon* also.

The gill filaments or fimbriae of *Hynobius* larvae arise from the ventral edges of long filament bearers or rami. These are very short in *Onychodactylus* larvae and absent in larvae of *Batrachuperus* so that the filaments seem to arise directly from the dorsal part of the septum.

Onychodactylus has no lungs. The other forms have the lungs well developed. However, the ypsiloid cartilage, so generally lacking or aborted in lungless salamanders is normally developed in *Onychodactylus*.

The tongue is sessile, but free at the lateral and posterior edges, save in *Batrachuperus*, where it is very small, apparently not free at the edges, and seems to be retractile into a sort of sheath. The hyobranchial skeleton of *Batrachuperus* does not differ from that of *Hynobius*.

Males of the family lack the papillae which line the cloaca in males of *Ambystomidae, Salamandridae, Plethodontidae, Proteidae,* and *Amphiumidae*, and which secrete the spermatophores by means of which internal fertilization is accomplished in these families. The spermatheca, found in females of the families just mentioned, is lacking in females of the *Hynobiidae*. In both male and female characters the *Cryptobranchidae* agree with the *Hynobiidae*. The fertilization of the former is well known to be external. The fertilization of the latter is known from two observations (Dybowski 1870, *H. keyserlingii;* Kunitomo 1910, *H. nebulosus*) both of which indicate external fertilization; and these, together with the anatomical evidence, may well mean that this method is in vogue throughout the family.

The vent of female *Hynobius* is a simple longitudinal slit; but males of this genus, probably at the breeding season, develop additional slits and reduce the original one, so that the end result is a transverse slit. The process of change has not been observed, but as a completely graded series can be found to connect the most extreme types, it seems logical to suppose that there is an actual change in the shape of the vent in individuals. All young have the vent of the female type. The simplest type of male vent is a longitudinal slit with a small papilla at the anterior end. This papilla gives the vent a Y-shaped appearance, the two arms of the Y being very short; next in order of complexity is the above condition with the addition of two lateral slits, directed obliquely forward, nearly as long as the median slit, and meeting it or the arms of the Y near the point of forking. A third type lacks the median slit, but otherwise is like the form just described and is somewhat M-shaped, or X-shaped, the exact form depending on the length and obliquity of the two lateral slits, which may be almost parallel or may make an angle of almost 180 degrees. These various conditions are subject to many minor changes. They are not specific, as I have observed the same series of conditions in several different species of *Hynobius*.

In *Onychodactylus* two pairs of slits directed obliquely forward meet the longitudinal slit in both sexes.

In *Batrachuperus* there is an irregular star-shaped or cruciform arrangement of the slits.

Color pattern when present is in the form of a light dorsal band, but this is often visible only on the middorsal line of the tail. This light band is most prominent in *Onychodactylus* and in *H. keyserlingii*. In most of the species there is no definite color pattern, and the animals are marbled or spotted with dark and light in varying proportions.

The habits of *Batrachuperus* and of *Pachypalaminus* are unknown. Of *Onychodactylus* we know only that it breeds in the spring and that it frequents mountain brooks. *Ranodon* lives in mountain brooks, breeds in the spring, and lays eggs in spindle-shaped sacs attached to the under side of stones in streams. *Hynobius* lives on land, goes to ponds in the spring to breed, and lays eggs in spindle-shaped sacs attached to objects in the water. The eggs are pigmented, of small vitelline content, and the larvae emerge in a very immature condition. Save for the external fertilization, *Hynobius*, in its habitat reactions and breeding habits, resembles the common American *Ambystoma maculatum*.

Measurements are in millimeters. The head is taken from the tip of the snout to the gular fold; the body from thence to the posterior angle of the vent; and the tail from that point to the tip. When I have a small series of a species I give measurements of all. In the case of very large series I give measurements of a selected group of specimens, including all the largest males and females, the smallest transformed young, and the largest and smallest larvae.

In the inclusion of references in the synonymy I have used my own judgment and left out nominal references which added nothing to the previous knowledge. I have included all unverified references on the chance that they may contain useful information which I have been unable to procure. Such are many of the Russian citations. The literature of this group is extraordinarily difficult, much of it being couched in Russian and in Japanese, contained in journals which are peculiarly hard to come by, and whose contents are not listed in the Zoological Record. As examples I mention the important papers of Nikolski (1918), Abé (1921), and Maki (1921), all three of which I saw for the first time when this paper was in galley proof. I therefore crave indulgence for unintentional omissions.

Another difficult matter has been the allocation of the proper names to some of the Japanese forms. The old terms *naevius* and *nebulosus* have been applied in their time to many species, and while the strict application of these two names is easy enough, it is by no means easy to tell what others than Schlegel meant by them. It is a pleasure to me to acknowledge my debt to the work of K. Tago. He, with his wide field experience, and large series of specimens, stated that he knew personally four *Hynobius* from Hondo. I, with a small series, came to the same conclusion, and have been able to identify them with Tago's four. His descriptions of the forms in nature are singularly charming and I have quoted largely from them.

In this paper I have included what I consider to be the essentials of a systematic treatise. These are: a consideration of the anatomy as used in classification; an account of the forms in the group and their distribution; an account of the relationships of these forms to one another and of their probable origin and dispersal; an account of their habits; references to the names by which they have been known and to the papers which contain information concerning them; references to the collections in which are deposited the specimens upon which the accounts are based. I have combined the introduction and the conclusions because the general student will read only this part and no useful purpose is served by sandwiching it around the systematic account of the animals.

This paper is based upon the examination of 1128 specimens of the family, representing 4 genera and 15 species. I have not been able to examine specimens of *H. chinensis*, *H. turkestanicus*, *H. cristatus*, or of *Pachypalaminus boulengeri*. 1001 of these specimens are in the collection of the California Academy of Sciences and I am greatly indebted to the kindness of Dr. Van Denburgh for the opportunity of examining them, and for some notes which he sent me on the unique type of *Pachypalaminus*. I am also indebted to Dr. Leonhard Stejneger for permission to examine the *Hynobiidae* in the collection of the National Museum; to Dr. Thomas Barbour for the loan of material from the Museum of Comparative Zoology; to Dr. G. K. Noble for the loan of the American Museum specimens; to Dr. H. H. Wilder for the type of *H. kimurai;* to Mrs. Arthur Johnson for copies of the original descriptions of *H. cristatus* and *H. chinensis:* to Dr. H. M. Parshley for casting a critical eye over the whole; to Mr. Tadanori Ono for skilful rendition of Japanese into English; to Mr. Austin Clarke and Miss Alexandria Gerecht for similar help with the Russian; and to Mr. Shunlchi Ono for several of the most important Japanese papers.

Family HYNOBIIDAE Cope.

1850. *Molgidae* Gray, Cat. Bat. Grad. Brit. Mus., p. 14.
1858. *Ellipsoglossidae* Hallowell, Journ. Acad. Nat. Sci. Philadelphia, (2), 3, p. 339.
1859. *Hynobiinae* Cope, Proc. Acad. Nat. Sci. Philadelphia, p. 125.
1866. *Hynobiidae* Cope, Journ. Acad. Nat. Sci. Philadelphia, (2), 6, p. 107.

Range: From the Urals to Kamchatka, Sakhalin, and the Islands of Japan; Formosa; Sosova River, Verkoyansk, and Anadyr to Semiryechensk and Sze-chuan.

Diagnosis: A family of salamanders which complete the transformation; haemal arch complete in tail vertebrae; ribs not forked proximally; prearticular bone present; first epibranchial and first ceratobranchial fused into a cartilaginous rod; second epibranchial present; nasals meeting in middorsal line; nasal spines of premaxillae very short; no fronto-squamosal arch; pterygoids present; prevomer long or short; no parasphenoid tooth patches; lungs present or absent; ypsiloid cartilage present; pubotibialis muscle fused with the puboischiotibialis; fertilization external; males without papillae in cloaca; females without spermatheca; no excessive dorso-ventral flattening; iliotibialis muscle in contact with ilioextensorius; vomerine teeth not anterior in position, not parallel to maxillary teeth; 5 genera, 21 species.

Key to Genera of Larvae.

Gill rami short or absent.
  Horny claws, dorsal fin present only on tail............*Onychodactylus*
  Horny soles, dorsal fin present on body................*Batrachuperus*
Gill rami long.
  No claws or soles, dorsal fin present on body................*Hynobius*

Key to Genera of Adults.

A. Vomerine teeth in more or less continuous angular series.
  B. Lungs present; no claws or horny soles; vomerine teeth usually in V-shaped series; tail never long and cylindrical............*Hynobius*
  BB. Lungs present; horny soles on feet; vomerine teeth in V-shaped series; tail flat..................................*Pachypalaminus*
  BBB. Lungs absent; claws present in some specimens; vomerine teeth in nearly straight line across mouth; tail long and cylindrical. *Onychodactylus*
AA. Vomerine teeth in two small widely separated series.
  B. Toes 4; horny soles present; labial folds much developed. *Batrachuperus*
  BB. Toes 5; no horny soles; labial folds not developed........*Ranodon*

Genus HYNOBIUS Tschudi.

1838. *Pseudosalamandra* Tschudi, Mem. Soc. Sci. Neuchâtel, pp. 56, 91 type *Salamandra naevia* Schlegel).
1838. *Hynobius* Tschudi, Mem. Soc. Sci. Neuchâtel, pp. 60, 94 (type *Salamandra nebulosa* Schlegel).
1839. *Molge* Bonaparte, Icon. Fauna Ital., 2, fasc. 26 (type *Salamandra naevia* Schlegel), (not *Molge* Merrem 1820, Syst. Amph., p. 185).
1848. *Hydroscopes* Gistel, Naturg. Thierr., p. XI (substitute for *Pseudosalamandra* Tschudi).
1854. *Ellipsoglossa* Duméril and Bibron, Erpet. Gén., 9, p. 97 (type *Salamandra naevia* Schlegel).
1870. *Salamandrella* Dybowski, Verh. Zool.-Bot. Ges. Wien, 20, p. 237 (type *Salamandrella keyserlingii* Dybowski).
1870. *Isodactylium* Strauch, Mem. Acad. Sci. St. Petersbourg, (7), 16, 4, p. 55 (type *Isodactylium schrenckii* Strauch.).

(*Hynobius* is used instead of *Pseudosalamandra* because Cope (1859), who first combined the two, selected *Hynobius*).

RANGE: From the Urals to Kamchatka, Sakhalin, and Islands of Japan; Formosa; from the River Sosova, Verkoyansk, and Anadyr, to Turkestan and Hupeh.

DIAGNOSIS: *Hynobiidae* with no foot modifications; lungs present; premaxillary fontanelle absent or very minute; gills of larvae with long filament bearers; vomerine teeth usually in V-shaped series; tail variable, never long and cylindrical; 16 species.

Key to Adults of Hynobius.

A. Mainland forms. Vomerine series medium.
B. No light dorsal band, toes 5.
C. Very small scattered dark dots on light ground, 14 costal grooves, Turkestan. . . . . *turkestanicus*
CC. Crowded spots or mottling, Eastern Asia.
D. Costal grooves 13, Korea. . . . . *leechii*
DD. Costal grooves 11, Hüpeh, China. . . . . *chinensis*
BB. A light dorsal band, toes 4, costal grooves 13–15, Siberia. . *keyserlingii*
AA. Island forms.
B. Inner branch of vomerine series equals or exceeds in length the distance between outer edge of naris and opposite naris; tail very short and thick; *naevius* group.
C. Toes 5.
D. Light markings both above and below, Kiusiu. . . . . *stejnegeri*
DD. Light markings on sides and on belly, Hondo. . . . . *naevius*
CC. Toes 4, light markings above and on sides, Hondo. . . . . . *kimurai*
BB. Inner branch of vomerine series equals or exceeds in length the distance between outer edge of naris and opposite naris; tail tapering, Formosa. *sonani*
BBB. Inner branch of vomerine series does not equal distance between outer edge of naris and opposite naris, but equals or exceeds in length the distance between outer edge of naris and peak of opposite vomerine series; tail more or less flat; *nebulosus* group.
C. Sides of tail black.
D. Tail very flat, males not black, Hondo. . . . . *vandenburghi*
DD. Tail not flat, males black, Tsu-shima South. . . . . *tsuensis*
CC. Sides of tail not black.
D. Middorsal line of tail usually yellow, spots on body black.
E. Tail flatter, spotting coarser, ground color lighter, Kiusiu. *nebulosus*
EE. Tail less flat, spotting finer, ground color darker, Iki-shima. *ikishimae*
DD. Middorsal line of tail not yellow; very fine stippling of dark brown on light brown ground color; Tsu-shima North. . . . . . *tagoi*
BBBB. Length of inner branch of vomerine teeth does not equal distance between outer edge of naris and peak of opposite series; *peropus* group and *cristatus*. (*H. leechii* bridges gap between the *peropus* and *nebulosus* groups.)

C. Pale, with darker marbling; 11–13 costal grooves; vomerine series medium to short; toes usually 5; tail keeled above; Hondo. . *peropus*
CC. Pale, with a dark lateral band; 11–12 costal grooves; vomerine series very short; toes 5; tail very flat; Yezo. . . . .*retardatus*
CCC. Black; 14 costal grooves; vomerine series medium; toes 4; tail flat; Sakhalin. . . . .*cristatus*

HYNOBIUS LEECHII Boulenger.

1887. *Hynobius leechii* Boulenger, Ann. Mag. Nat. Hist. (5), 19, p. 67; Stejneger, 1907, U. S. Nat. Mus. Bull., 58, p. 29, pl. 4, f. 8; Abé, 1921, Tokyo Zool. Mag. 34, p. 330.

TYPE: In British Museum, collected by Mr. J. H. Leech.
TYPE LOCALITY: Gensan (Wonsan), Korea.
RANGE: Korea.

DIAGNOSIS: Vomerine series short to medium; tail somewhat flattened; usually 13 costal grooves; toes 5; sides of tail not black; middorsal line of tail not yellow; grayish brown with black mottling.

DESCRIPTION: Cal. Acad. Sci. No. 31856, adult male, Fusan, Korea, collected May 8, 1911, by Victor Kuhne; 14 costal grooves, one costal fold between appressed toes; head width $5\frac{2}{3}$ in length from snout to vent; head length $3\frac{1}{2}$ in length of body; head oval as seen from above; eye as long as its distance from tip of snout; outline of upper jaw straight as seen from side; angle of jaw back of hind angle of eye; lower eyelid fitting under upper in front and behind; a groove from eye to gular fold; a groove from this down behind angle of jaw; limbs well developed; fingers 2, 3, 4, 1 in order of length; toes 3, 4, 2, 5, 1 in order of length; a tubercle under base of first finger and first toe; tail shorter than body, cylindrical at base, trigonal in middle, flattened at tip, ending in an obtuse point; vent a slit forking forward, a small tubercle in fork; vomerine teeth beginning behind inner border of nares, a short forward and incurving branch, and a long straight branch extending back and a little in; series separated from its fellow by length of outer branch, length of inner branch not quite equal to the distance between outer border of naris and peak of opposite series; lead black above; light gray below a line joining insertions of legs; total length 105 mm., head 14, body 49, tail 42.

Variation: A female, Cal. Acad. Sci. No. 35958, same data, has 13 costal grooves, 3 costal folds between the appressed toes; head width $5\frac{1}{3}$ in length from snout to vent; head length $3\frac{2}{7}$ in length of body; tail somewhat thicker; vent a slit; inner branch of vomerine series equal in length to distance between outer border of naris and peak of opposite series; yellowish brown above, light gray below, heavy mottling of irregular black spots above; total length 101 mm., head 14, body 46, tail 41.

A young specimen, Cal. Acad. Sci. No. 32126, same data, has 12 costal grooves; appressed toes meeting; head width 4½ in length from snout to vent; light grayish brown above, with mottling of dark brown; scattered white flecks; cream color below with gray mottling; tail much flattened, finned above; traces of recently resorbed gills at sides of neck; total length 47 mm., head 8, body 21, tail 18.

A larva, Cal. Acad. Sci. No. 32154, Chiksan, Korea, shows a light grayish brown dorsal surface, the sides and belly being white. Back and sides are marked with small irregular black spots. The dorsal fin reaches forward past the middle of the back. Total length 53 mm., head 8, body 23, tail 22.

A female from Wonsan, Korea, Cal. Acad. Sci. No. 32168, has the inner branch of the vomerine series equal in length to the distance between the outer edge of the naris and the median line; this is the case in three males from Chiksan (Cal. Acad. Sci. 32158–32160), and in seven specimens from Kong-ju (Cal. Acad. Sci. 32161–32167). It is closely approached by some of the Fusan specimens.

Among 317 adults of this species, two (32048, 31934) have no fifth toe; eight (31854, 31860, 31972, 32076, 32096, 35964, 35970, 31949) lack this toe on one foot; 3200 has both fifth toes rudimentary; 31964 has the fifth toe of the left foot rudimentary.

The costal grooves of 17 males are: 13 in 13 specimens, 13–14 in one, and 14 in three.

The count for 13 females is: 13 in twelve specimens, and 14 in one. Of 11 young one has 12 costal grooves, the rest have 13. Thus some 85% have 13 costal grooves.

In the young the legs are proportionately longer. Thus of 11 specimens the appressed toes meet in five, and are separated by one costal fold in six. This proportionate leg length is somewhat less in adult males, for five out of seventeen have the appressed toes meeting, in ten animals they are separated by one costal fold, and in two individuals by two. Females have still shorter legs, as out of fourteen adults seven have the appressed toes separated by three costal folds, three by two costal folds, and four from Kong-ju have only one costal fold between the appressed toes.

REMARKS: I have used this species for most of the anatomical work I have done on *Hynobius*.

This animal is about intermediate between the forms of the *nebulosus* group (which inhabit Hondo, Kiusiu, Shikoku, and the islands in the Korean straits), and *peropus* of the high interior of Hondo. It may therefore be taken to represent the mainland stock from which both these island groups have arisen. Its intermediate nature is seen in its dentition, its coloration, and its proportions. It could be confused more readily with *tagoi*, *ikishimae*, and *peropus*, than with *tsuensis*, *nebulosus* and *vandenburghi*.

All the *nebulosus* group have vomerine series as long as or longer than the maximum length found in *leechii*. *Tagoi* has a rather individual coloration which should distinguish it easily. *Ikishimae* has a longer vomerine series and nearly always has a yellow stripe on the middorsal line of the tail which *leechii* never has. *Peropus* has a vomerine series as short as the shortest in *leechii*, fewer costal grooves, longer legs, and a different coloration.

Besides the following localities Abé (1921) has recorded it from Hokkansan and Heijo.

Specimens seen: 347, as follows;

Korea: Kang-wom-do Province: Wonsan (= Gensan), 1 (Cal. Acad. Sci. 32168)
Tschhung-Tschhong-Do Province: Kong-ju 7 (Cal. Acad. Sci. 32161–32167).
Kwi-Do Province: Chiksan 34 (Cal. Acad. Sci. 32158–32160, 32138, 32127–32137 larvae, 32139–32157 larvae).
Kjong-Sang-Do Province: Fusan 305 (Cal. Acad. Sci. 31841–32126, 32374, 35958–35975).

MEASUREMENTS OF *Hynobius leechii*.

* = costal grooves. / = costal folds between appressed toes.

Specimen	Locality	Sex	Total Length	Head	Body	Tail	*
Cal. Acad. Sci. 35969	Fusan	male	109	14	47	48	13/2
“ 32070	“	“	108	14	54	40	13/0
“ 31856	“	“	105	14	49	42	14/1
“ 31923	“	“	102	13	48	41	13/1
“ 32018	“	“	100	14.5	46.5	40	13/0
“ 32067	“	“	101	13.5	49.5	39	13/1
“ 31874	“	“	95	14.5	43.5	37	13/0
“ 31975	“	“	94	13.5	46.5	34	13/2
“ 32036	“	“	93	13	42	38	13/1
“ 32037	“	“	93	13	41	39	13/0

Specimen	Locality	Sex	Total Length	Head	Body	Tail	*
Cal. Acad. Sci. 32095	Fusan	female	108	14	54	40	13/3
“ 31942	“	“	106	14	49	43	13/3
“ 32110	“	“	101	14	47	40	13/2
“ 35958	“	“	101	14	46	41	13/3
“ 31905	“	“	101	13.5	43.5	44	13/3
“ 32008	“	“	100	14	47	39	13/2
“ 32053	“	“	100	13	48	39	13/3
“ 31903	“	“	100	14	46	40	13/2
“ 32055	“	“	100	13	47	40	13/3
“ 32069	“	“	98	15	49	34	13/3
“ 32161	Kong-ju	male	95	13	52	30	14/1
“ 32162	“	female	108	15	51	42	/1
“ 32163	“	“	107	14	51	42	13/1
“ 32164	“	male	89	12	43	34	14/1
“ 32165	“	“	90	13	42	35	13/0
“ 32166	“	female	73	11	33	29	13/1
“ 32167	“	male	83	11.5	38	33.5	13/1
“ 32168	Wonsan	female	90	12	43	35	14/1
“ 32158	Chiksan	male	90	12.5	42.5	34	13–14/1
“ 32159	“	“	79	11	39	29	13/1
“ 32160	“	“	79	11	38	30	13/1
“ 32138	“	young	53	8	25	20	13/0
“ 31917	Fusan	“	60	10	28	22	13/1
“ 32024	“	“	60	9	26	25	13/1
“ 32119	“	“	59	9	26	24	13/1
“ 32686	“	“	58	8.5	27	22.5	13/1
“ 32083	“	“	57	8.5	26.5	22	13/1
“ 31979	“	“	55	9	25	21	13/1
“ 31901	“	“	53	9	25	19	13/0
“ 31873	“	“	48	8.5	23.5	16	13/0
“ 32126	“	“	47	8	21	18	12/0
“ 31937	“	“	40	8	23	9	13/0
“ 32156	Chiksan	larva	61	9	26	26	
“ 32137	“	“	56	9	22	25	
“ 32146	“	“	43	7	17	19	
“ 32154	“	“	53	8	23	22	
“ 32144	“	“	57	9	22	26	

HYNOBIUS CHINENSIS Günther.

1889. *Hynobius chinensis* Günther, Ann. Mag. Nat. Hist., (6), 4, p. 222.

TYPE: In British Museum, collected by Mr. Pratt.
TYPE LOCALITY: Ichang, Province Hupeh, China.

Range: Known only from type locality.

Diagnosis: A *Hynobius* with short vomerine series; compressed tail; 11 costal grooves; 5 toes; no dorsal light stripe on body or tail.

Description: Having seen no specimens of this form I quote the original description. "Allied to the Japanese *Hynobius nebulosus*, but with the series of vomerine teeth much shorter, extending backwards only to the middle of the eyeball. General habit short and stout; head large, nearly as broad as long, its length being rather more than one fourth the length of the trunk. Tail compressed in its whole length, but without crest; body with eleven lateral folds. The limbs meet when appressed; fifth toe well developed; no carpal or tarsal tubercles. Skin smooth; gular fold indistinct. Nearly uniform horny black, the lower parts brownish, finely marbled with darker."

	(*H. chinensis*)	(*H. leechii*, Cal. Acad. Sci. 32113, Fusan, Korea.)
Total length	85 mm.	81 mm.
From snout to cloaca	46 "	46 "
Length of head	11 "	11 "
Width of head	10 "	9 "
Fore limb	15 "	13 "
Hind limb	16 "	15 "
Tail	39 "	35 "

Remarks: "Two specimens were collected by Mr. Pratt at Ichang." This animal seems rather allied to *H. leechii* of Korea than to *H. nebulosus*. The slight proportional differences, a less number of costal grooves in the Chinese species, and a broader head than *H. leechii* are about all that distinguish it. *H. leechii* has 13–14 costal grooves.

Hynobius turkestanicus Nikolski.

1909. *Hynobius turkestanicus* Nikolski, Charikov Trav. Soc. Nat., 43, p. 73; 1918, Faune de la Russe. p. 245. f. 47, pl. 4, f. 4.

Type: Zool. Mus. Petrograd No. 2404, collected by V. Nikolski.

Type Locality: Russian Turkestan ("Pamirs of Samarkand," Nikolski 1918).

Range: Known only from type locality.

Diagnosis: A *Hynobius* with medium vomerine series; tail keeled above and below; 14 costal grooves; 5 toes; no dorsal light stripe on body or tail; light with darker dots.

DESCRIPTION: "A *Hynobius* with head depressed, longer than broad; snout rounded, the length equalling interorbital width; palatine series as long as ¾ width of tongue, not broader than long; length of eye 1½ times in interorbital width; eye prominent; no labial lobes; body cylindric, scarcely depressed; head length 4½ in length of body and head (tail omitted); snout to gular fold 3¼ in distance from gular fold to anus; hind feet laid forward not reaching front feet laid back by more than length of head; fingers and toes well developed; fifth toe 1½ in length of fourth; carpal and tarsal tubercles indistinct; tail as long as gular fold to anus; tail keeled above and below, tip acuminate; anus a longitudinal slit, no transverse groove, no raised edges; body smooth, a deep middorsal groove forking on occiput, a longitudinal groove crossing parietal region of head; 14 transverse grooves on the sides reaching belly; paratoids on sides of head; a second gular fold between angles of jaws; a deep horizontal groove in side of head beginning at gular fold; body reddish brown above with small brown spots; below immaculate yellowish brown; total length 90 mm., head 10, body 40, tail 40."

Nikolski (1918, f. 47) shows the inner row of vomerine teeth equalling the distance from outer edge of naris to median line plus width of naris, and thus like those of *H. leechii*. Pl. IV, f. 4, a photograph of the type, shows it to be not fully transformed, and closely resembling *Ranodon sibiricus* in habit and markings.

REMARKS: This animal is not very different from *H. leechii* save in color. In this character it approaches *Ranodon sibiricus* rather closely.

It is particularly unfortunate that no specimens of this species are available because its relations with *Ranodon sibiricus* are of the greatest interest. The legs are shorter and there are more costal grooves, so that it is not particularly close anatomically, but geographically it is the only *Hynobius* at present available out of which *Ranodon* could have been evolved, so that the animal should be examined very carefully for any similarity.

HYNOBIUS KEYSERLINGII (Dybowski).

1853. *Salamandra terrestris* (?) Zerrener. Erdkunde Gouv. Perm., p. 321.
1859. *Triton* nov. sp., Maack, Putesh. Amur, p. 154.

1870. *Salamandrella Keyserlingii* Dybowski, Verh. Ges. Wien, 20, p. 237, pl. 7; Strauch, 1870, Mem. Acad. Sci. St. Pétersbourg, (7), 16, 4, p. 110; Wiedersheim, 1876, Morph. Jahrb. 2, p. 426, pl. 29, f. 4–5, (extremities); 1877, Morph. Jahrb., 3, pl. 21, ff. 39, 51, pl. 24, ff. 85–6, pl. 25, f. 98 (skull); Boulenger, 1882, Cat. Bat. Grad. Brit. Mus., p. 34; Mueller, 1887, Verh. Ges. Nat. Basel, 8, (2), p. 249; Kulagin, 1888, Izv. Mosc. Obsch. Lyub. Est., 56, (2), p. 34; Slovtzov, 1892, Pozv. Tyumensk okr., p. 75; Kastschenko, 1896, Izv. Tomsk Univ., 10, p. 1 (habits); Bedriaga, 1898, Wiss. Res. Przewalski Zentralasiat. Reise, Zool., 3, (1), Amph., p. 3; Wolterstorff, 1898, Blatt. Aqua. Terr., 9, p. 91; Anikin, 1902, Otc. O. Komand. Narymsk. Kr., p. 104; Kastschenko, 1902, Izv. Tomsk Univ., p. 20; Nikolski, 1905, Zap. Imp. Akad. Nauk. S. Peterburg (8), 17, p. 436; Elpateevski, 1905, Trud. Troitzko-Savsk. Otd. Geog. Obstch. 9, p. 43; 1907, Op. Pres. Zemn. Ross. Imp., p. 180; Stejneger, 1907, Bull. U. S. Nat. Mus., 58, p. 37, pl. 5, ff. 7–8, ff. 33–38; Schmalhausen, 1910, Mosk, Dnevn., 12, p. 289 (extremities); 1910, Anat. Anz., 37, p. 431 (extremities); Cugunov, 1911, Ann. Mus. Zool. St. Pétersbourg, 16, p. 220; 1913, Ann. Mus. Zool. St. Pétersbourg, 18, p. 250; 1914, Ann. Mus. Zool. St. Pétersbourg, 19, p. 532; Nikolski, 1915, Trud. Troitsko-Savsk. Otd. Geog. Obstch. 15, p. 37; 1918, Faune de la Russe, p. 236, f. 44.

1910. *Hynobius Keyserlingii* Boulenger, Les Batraciens, pp. 49, 96.

1870. *Isodactylium Schrenkii* Strauch, Mem. Acad. Sci. St. Pétersbourg, (7), 16, p. 56, pl. 2, f. 1, (Ussuri R., Shilka R., Lake Baikal); Sabaneef, 1872, Bull. Soc. Nat. Moscou, 44, p. 275; 1874, Pozv. Sred. Urala, p. 190; Kehrer 1886, Ber. Nat. Ges. Freiburg, I, p. 5, pl. 4, f. 5–6 (extremities); Shitkov, 1895, Zool. Anz., 18, p. 165 (development); 1899, Zool. Anz., 22 (extremities); 1900, Dnev. Otd. Ichtiol. Akklim. Zhiv., 2, p. 41 (habits); 1902, Izv. M. Obstch Lyub. Est. Antr. i T. D., 98, Tr. Zool. Otd., 13, Dnevn Zool. Otd., 3, (4), p. 5 (extremities).

1870. *Isodactylium wosnessenskyi* Strauch, l. c., p. 58, pl. 2, f. 2a–d, (Javina, Kamchatka); Wiedersheim, 1876, Morph. Jahrb., 2, p. 426, pl. 29, f. 6 (extremities).

1882. *Salamandrella wosnessenskyi* Boulenger, Cat. Bat. Grad. Brit. Mus., p. 34; Kulagin, 1888, Izv. M. Obstch. Lyub. Est., 56, (2), p. 74.

1896. *Salamandrella uralensis* Nikolski, Ann. Mus. Zool. St. Pétersbourg, 1, p. 13, (nomen nudum).

1905. *Salamandrella Keyserlingii tridactyla* Nikolski, Zap. Imp. Akad. Nauk. S. Peterburg, (8), 17, p. 491 (Vladivostok).

1921. *?Hynobius doii* Abé, Tokyo Zool. Mag., 34, p. 330 (Manchuria).

Type: None mentioned.

Type Locality: Southwest corner of Lake Baikal.

RANGE: From the River Sosova, the Ural Mts., and the Kirgis Steppe, to Verkoyansk, Anadyr, Kamchatka, and south into Manchuria, and Mongolia.

DIAGNOSIS: A *Hynobius* with moderately long, flattened tail; moderate vomerine series; 13–15 costal grooves; 4 toes; a light dorsal band and darker sides.

DESCRIPTION: U. S. Nat. Mus. No. 36414, adult male, Paratunka, Kamchatka; 14 costal grooves; 1 costal fold between appressed toes; head width 5 in length from snout to vent; head length 3½ in length of body; head oval; eye shorter than its distance from tip of snout; outline of upper jaw convex as seen from side; angle of jaw back of hind angle of eye; lower eyelid fitting under upper in front and behind; a groove from hind angle of eye not quite reaching gular fold; a groove from this down behind angle of jaw; limbs well developed; fingers 2, 3, 1, 4 in order of length; toes 3, 2, 4, 1 in order of length; tail longer than body, much flattened, a dorsal keel throughout, a ventral keel in distal two-thirds; vent formed by three slits converging with apex forwards, median longest; vomerine series beginning behind inner edge of nares; a branch of 5 teeth running in and forward to level of anterior edge of nares; a branch of 8 teeth running in and back, making an acute angle with the outer branch, and extending farther back than it; series separated from its fellow posteriorly by width of nares; length of inner branch equals distance between outer edge of naris and median line; dull grayish brown, lighter below; an irregular black line from behind eye along sides of back to base of tail where it broadens into irregular marbling on sides of tail; a middorsal black line; dorsal area of a richer brown than sides; total length 132 mm., head 15, body 52, tail 65.

Variation: A female, U. S. Nat. Mus. No. 22594, from Rakovaya Bukhta, Avatcha Bay, Kamchatka, has 4 costal folds between the appressed toes; head width 7 in length from snout to vent; head length 4 in length of body; tail shorter than body, no keels; vent a simple slit; 13 teeth in the inner branch of the vomerine series, whose length is a little more than equal to the distance between the outer border of the nares and the median line; darker, with irregular black marbling on sides of body as well as on sides of tail; total length 124 mm., head 15, body 60, tail 49.

Variations in proportions may be noted in the table of measurements. Males are longer legged than females.

HABITS: I quote first from Dybowsky (1870) who says, in his original description; "I have found this species at the southwest corner of Baikal, especially in the marshy coastal meadows of the Kultusnaya and Pachabicha valleys, where it is very abundant. On the tenth of May 1869 I found grown animals, about 130 mm. long, laying eggs in pools and ponds. The females lay their eggs in string-like bunches and glue them to stalks of plants or other objects in the water near the surface; the males emit their sperm over the eggs while moving hither and thither very actively. The animals are very shy and at the slightest disturbance they dive under and either creep into the mud or cling motionless flat on the bottom. In either case they can easily be caught with nets. After the egg-laying, about the first half of June the adults leave the water and live in low, swampy meadows; either in moss or under pieces of wood or else in rotten logs. The immature animals stay in such places throughout the summer.

"Reckoning by the sizes of the hundreds of specimens we caught, *Salamandrella Keyserlingii* reaches its full development and puberty in the third year. The food of this animal consists of earthworms and of insects. Pregnant females kept in the aquarium will not lay eggs; and young experimentally kept in water soon perish; but I have kept both young and adults alive for months in damp moss, and fed with earthworms."

Shitkov (1895), speaks of the habits near Ekaterinburg in the following terms: "The eggs are laid by the females in especial gelatinous sacks, which are fastened by one end to a plant or other object in the pond, and not far (2–3 cm.) from the surface; the other end of the sack hangs freely in the water, so that it will be rocked up and down by the slightest movement of the water. The sack is generally 15 cm. long, much longer than the breadth, (the breadth of the empty, collapsed sack is 2 cm.), bowed, and thickly wrinkled on the inner side. The wall of the sack is of the same origin as the albuminous coating of the egg. It is certain, however, that each egg has its own shell, which I, (although I myself owing to my late arrival in Ekaterinburg have seen no new-laid eggs), conclude from the fact that the larvae just after their emergence are still surrounded by a transparent gelatinous covering, from which they free themselves immediately after their emergence. Each sack contains 50–60 eggs."

"According to Mr. A. Hackel the adults enter the water on the twenty-first of April, and lay their eggs on the same night."

"The larvae which were in a sunny aquarium emerged after 14 days; those in an aquarium which faced the North in 23 days. At the time of my arrival in Ekaterinburg (the last of May) the larvae in the aquarium of Mr. Hackel were already 3 cm. long and had both pairs of limbs. At the same time I found sacks in the open, in ponds near Lake Schartasch about six versts from Ekaterinburg, which still contained larvae. These were ripe for emergence. They were distinctly backward in their development compared with those shown me, which is probably to be explained by the cold weather of last May in East Russia."

"All the larvae whose emergence I had the opportunity to observe, agreed in breaking out of the lower part of the sack. Larvae from the same sack, were often not equally developed, as appeared best in the gills. This unequal development seems to me accounted for by the position of the sack, one end of which is placed in the influence of the warmer weather, while the other is much deeper in the water.

"The newly emerged larva is about 10 mm. long and is marked by the length (1 mm.) of the balancer. The gills of the more advanced ones bear two or three filaments apiece; the gills of the less advanced only two on the second gill. The fore-limbs are visible as little knobs."

The egg sack, as figured by Shitkov, bears a striking resemblance to that of *H. naevius*, as figured by Tago (1907).

Kastschenko (1896), "found the animals in May near Tomsk on the ground near a pond under the dead leaves of the last fall.

"They do not enter the water save for breeding.

"The eggs are laid near Tomsk in the middle of May. They are always in a transparent sausage-shaped bag, which is fastened to plants under water, near the surface, usually 1–2 vershok (2–4 inches) from it, and not deeper than $\frac{1}{4}$ the depth of the water. Two bags are together, probably laid by one female. The gelatinous walls of the bag can support all the movements of the water, but when taken out they separate on account of their own weight. The bag is somewhat spiral, so that there are narrow and wide places.

"They are fastened to any under water plant or dead branch fallen from the surrounding trees.

"The end of larval life and the time of leaving the water is at Tomsk during the first days of August."

REMARKS: I am unable to distinguish between the Kamchatkan specimens and those from southeastern Siberia. There seems to be no break in the distribution, and I have seen no evidence of local differences.

This animal is quite distinct and there is no danger of confusion with *H. leechii* of Korea. It is the only member of the genus with a light dorsal band. Its relations seem remotely with *leechii* of Korea and less remotely with *cristatus* of Sakhalin. It has almost certainly given rise to the latter.

The animal has been recorded from Sze-chuan by Bedriaga (1898), but this record refers in reality to *Batrachuperus sinensis;* and Nikolski's (1905) record for Sakhalin probably refers to *H. cristatus.*

Abé (1921) mentions a *Hynobius doii* from Manchuria. This name is based on two larvae and probably refers to *keyserlingii.*

Besides the localities from which I have seen specimens, it has been recorded from the following;

Russia proper: Nishne-Tagilsk (Nikolski 1905); Middle region of Urals from Orenburg district to Bogoslovsk, especially Ekaterinburg and Chasninsk (Sabaneef 1871); Baschkiri Urals (Zarudny 1895).

Tobolsk Gov.: Tyumen, Tobolsk (Slovtzov 1892); Sosova River (Gondatti 1888); Surgut (Cugunov 1914); River Pyderata (Nikolski 1918).

Tomsk Gov.: Narimsk (Anikin 1902); Tomsk (Kastschenko 1896); Ischmorsk, Krasnaya (Kastschenko 1902).

Yenisseisk Gov.: Minussinsk (Martianov 1881); Ilanskaya (Cugunov 1911).

Yakutsk Gov.: Verkoyansk (Wolterstorff 1898); Aldan River, Yakutsk, Lena River above Schigansk (Nikolski 1905).

Irkutsk Gov.: Irkutsk, Zima (Cugunov 1913); Kultuk (Maack 1859); Irkut River, Lower Tunguska River (Nikolski 1905); Southwest corner of Lake Baikal (Dybowsky 1870).

Transbaikalia Gov.: Dauri, Ust-Strielka (Maack 1859); Schilka R., (Strauch 1870); Ingoda R. (Dybowsky 1870); Tschita (Nikolski 1915).

Maritime Gov.: Khabarovka (Mueller 1887); Amur R. (Kulagin 1888); Agdeki, Kazlovkaia, Vladivostok (Nikolski 1905); Imansk, Schmakovkaia (Nikolski 1918).

Kamchatka Gov.: Tolbatcha, Uson Volcano, (von Ditmar 1890); Anadyr, (Nikolski 1905); Konina (Kulagin 1888); Javina (Strauch 1870).

Mongolia: Northern Mongolia (Bedriaga 1898).

It has also been recorded from the following, which I am unable to place under special provinces: Kirgis Steppe, Dolguloch (Nikolski 1905); Raktiya R., in western Siberia (Kulagin 1888); Zej River (Dolbuka) (Nikolski 1918); Njado-dla-pa-mal (Nikolski 1918).

Specimens seen: 19, as follows;

Lake Baikal 2 (Mus. Comp. Zool. 1288); I-mien-po, Manchuria 1 (U. S. Nat. Mus. 53366); Lake Xanka 1 (Cal. Acad. Sci. 14578);

Gichiga 2 (Amer. Mus. 1452–3); Kamchatka 5 (U. S. Nat. Mus. 31712–6); Avatcha Bay, Kamchatka 1 (U. S. Nat. Mus. 22594); Paratunka, Kamchatka 7 (U. S. Nat. Mus. 23521, 36414–9).

MEASUREMENTS OF *Hynobius keyserlingii*

* = costal grooves. / = costal folds between appressed toes.

Specimen		Locality	Sex	Total length	Head	Body	Tail	*
M. C. Z. 1288		Lake Baikal	♂	115	13	48	54	14/2
"		"	♂	68	9	34	25	14/4
Cal. Acad. Sci. 14578		Lake Xanka	♂	75	9	34	32	14/2
U. S. N. M. 53366		I-mien-po	♀	95	12	35	38	
Am. M. N. H. 1452		Gichiga	♀	75.5	10.5	35	30	14/
"	1453	"	♀	93	11.5	44.5	37	14/4
U. S. N. M.	22594	Avatcha Bay	♀	124	15	60	49	14/4
"	23521	Paratunka	♀	111	14	52	45	14/3
"	36414	"	♂	132	15	52	65	14/1
"	36415	"	♂	116	14	51	51	14/3
"	36416	"	♂	130	16	53	61	13/2
"	36417	"	♂	128	15	58	55	14/1
"	36418	"	♂	111	14	56	51	14/1
"	36419	"	♀	114	13	59	42	15/4
"	31712	Kamchatka	♀	112	14	55	43	15/3
"	31713	"	♀	117	14.5	52.5	48	14/4
"	31714	"	♀	120	15	59	46	14/3
"	31715	"	♂	110	15	49	46	14/2
"	31716	"		99	13			
Types of *I. wosnessenskyi* (Strauch 1870) in Petrograd Museum		Javina		121	14	48	59	14–15/
				108	13	41	54	14–15/
				98	12	43	43	14–15/
				92	12	42	38	14–15/
				85	11	38	36	14–15/
I. schrenkii Type (Strauch 1870) in Petrograd Museum		Siberia		107	14	44	49	14/

HYNOBIUS CRISTATUS (Andersson).

1905. *Salamandrella keyserlingii* Nikolski (part), Zap. Imp. Akad. Nauk, S. Peterburg, (8), 17, 1, p. 436.

1917. *Salamandrella cristata* Andersson, Medd. Göteborg Mus. Zool. Afdel. 10.

TYPE: Göteborg Museum No. 302, collected by Mr. L. Munsterhjelm.

TYPE LOCALITY: Sakaehama, east coast of Sakhalin.

RANGE: Sakhalin.

DIAGNOSIS: A *Hynobius* with long, flat tail; moderate vomerine series; 13 costal grooves; 4 toes; black.

DESCRIPTION: Quoted from Andersson (1917). "Vomero-palatine teeth in an angular series; the median arms which converge behind form an acute angle, and equal one-half the width of the tongue; the lateral branches, of about half the length of the inner ones, form right angles with the latter. Head flat and broad; nostrils about halfway between eye and tip of snout, their distance from each other more than the interorbital width. Eye rather large, the upper eyelid considerably longer than the interorbital space. Body moderately elongate; distance from tip of snout to gular fold contained three times in distance from gular fold to anterior border of vent. Adpressed limbs just meeting; fingers and toes four on each foot, short, broad, and depressed; toes with a rudiment of web at the base; outer finger very small, second and fourth of about similar size, third much the longest and broadest; of toes, the outer (fourth) as long as but somewhat narrower than the innermost (first), both much smaller than second and third, of which the latter is somewhat longer and narrower than the former. No trace of a fifth toe. Tail long, as long as head and body, very compressed in its whole length, tapering, obtusely pointed at the end, crested by a high dorsal and a lower ventral fin, both bordering the whole tail. Thirteen distinct costal grooves, twelve abdominal ones, the inguinal not continuing across the belly; median dorsal groove not distinct. A large, very distinct parotoid gland; a distinct postocular groove continues along the upper and posterior margin of the gland to the gular fold, which is very distinct. Vent a longitudinal slit with a broad triangular flap, directed forwards, protruding between the anterior parts of the very swollen lips, and forming a kind of roof over the external opening. Colour black, except on the chin and throat to the gular fold, which parts are brownish white, and on the breast and under parts of limbs, which are brown.

Dimensions of the single specimen, which is a female. (Ba. ex. 302). Total length 110 mm., head 14, gular fold to anterior angle of vent 41, anterior angle of vent to tip of tail 55."

REMARKS: I am inclined to believe that the sex is wrongly determined. The modification of the vent described above is a male character.

This animal occupies territory between the ranges of *H. keyserlingii* and *H. retardatus*. I have seen no specimen of *cristatus* but it seems to be more allied to *keyserlingii*, on account of the dentition, the costal groove count, and the feet, which are the same as those of the Siberian form and very different from those of the Yezo animal. On the other hand the tail is quite like that of *retardatus*. But the Yezo species seems to me to be an end form, very specialized, derived from *peropus* of Hondo, and while the matter cannot be regarded as settled, it is far more probable, both anatomically and geographically to regard *cristatus* as derived from *keyserlingii*, from which it differs, as Andersson says "very clearly in the structure of the tail, which is higher, longer, more compressed, and crested; also the colour is quite dissimilar. In addition to this the vent has a triangular flap, the limbs are longer, and the body shorter in proportion to the head."

Nikolski's reference (1905) to a specimen of *keyserlingii* in the Petrograd museum from Sakhalin (No. 1941, collected by Dr. Saprunenko in 1890), probably refers in reality to *cristatus*.

HYNOBIUS PEROPUS Boulenger.

1882. *Hynobius nebulosus* Boulenger (part ?), Cat. Batr. Grad. Brit. Mus. (2), p. 32; Okada 1891, Cat. Vert. Japan, p. 65; Stejneger 1907, Bull. U. S. Nat. Mus. 58, p. 30, ff. 23–26; Sternfeld, 1916, S. B. Ges. Nat. Fr. Berlin, p. 172; Abé 1921, Tokyo Zool. Mag., p. 330.

1882. *Hynobius peropus* Boulenger, loc. cit., p. 33, pl. 2, f. 1; Stejneger, loc. cit., p. 32, pl. 4, ff. 4–7; Tago, 1907, Tokyo Zool. Mag. 19, p. 202; Hashimoto, 1910, Tokyo Zool. Mag., 22, p. 497.

1883. *Hynobius lichenatus* Boulenger, Ann. Mag. Nat. Hist., (5), 12, p. 165, pl. 5, f. 1, (Aomori, Mottsu Prov.); Stejneger loc. cit., p. 36, pl. 4, ff. 1–3; Tago, loc. cit., p. 203; Hashimoto, loc. cit., p. 497.

1907. (July 22) *Hynobius nigrescens* Stejneger, loc. cit., p. 34, ff. 27–32 (Sendai, Rikuzen Prov.); Tago, loc. cit., p. 204; Hashimoto, loc. cit., p. 497; Abé, loc. cit., p. 330.

1907. ("July") *Hynobius fuscus* Tago, loc. cit., p. 204, 233, (Nikko, Shimozuke Prov.); Hashimoto, loc. cit., p. 499.

1921. *Hynobius naevius* Abé (part?), loc. cit., p. 329.

TYPE: In British Museum. Adult male, collected by Mr. A. Adams.

TYPE LOCALITY: "China or Japan."

RANGE: From northern extremity of Hondo to the Provinces of Izu and Yamato.

DIAGNOSIS: A Hynobius with thick tail, which may be keeled above;

short or very short vomerine series; 11–13 costal grooves; toes usually 5, sometimes 4; grayish brown, marbled with darker, and with light marking on sides.

DESCRIPTION: Cal. Acad. Sci. No. 26685, adult male, Inawashiro, Iwashiro Prov., Hondo, collected by V. Kuhne in 1910.

13 costal grooves; appressed toes meeting; head width 6 in length from snout to vent; head length 3½ in length of body; head oval; eye as long as its distance from tip of snout; outline of upper jaw straight as seen from side; angle of jaw back of hind angle of eye; lower eyelid fitting under upper in front and behind; a groove from eye to gular fold; a groove from this down behind angle of jaw; limbs well developed; fingers 2, 3, 4, 1 in order of length; toes 2, 3, 4, 1 in order of length; a tubercle under base of first finger and toe; tail shorter than body, cylindrical at base, flattened at tip; vent a slit forking forwards with a small tubercle in apex; vomerine teeth beginning behind inner edge of nares; a forward, incurving branch of 4 teeth; a branch of 13 teeth running back and in; series separated from its fellow anteriorly by twice length of outer branch, and posteriorly by length of outer branch; length of series equals distance between outer edge of nares to median line plus width of nares; brownish gray, lighter below; heavily mottled with black above, and with dark and light on sides; total length 108 mm., head 15, body 52, tail 41.

Variation: A female, Cal. Acad. Sci. No. 26703, same data, differs in having 12 costal grooves; appressed toes separated by one costal fold, head width 5⅔ in length from snout to vent; head length 3⅓ in length of body; toes 3, 4, 2, 1, 5 in order of length, fifth well developed vent a simple slit; total length 106 mm., head 15, body 50, tail 41.

A series of 23 specimens from Inawashiro, Cal. Acad. Sci. Nos. 26685–26707, shows that of 12 males two have 13 costal grooves, nine have 12, and one has 11; of 11 females six have 13 costal grooves and five have 12. Four, all females, have the appressed toes separated by one costal fold; the rest have them meeting.

Four have a fold of skin on the sides between the legs as in *Ranodon sibiricus*. One has no fifth toe, one has this toe rudimentary, and four have it shorter than do the remaining specimens.

Two specimens from Hakone differ somewhat.

The male, Cal. Acad. Sci. No. 16020, has 13 costal grooves, two costal folds between appressed toes; rather uniform brownish or pinkish gray above, lighter below; total length 109 mm., head 14, body 50, tail 45.

The female, Cal. Acad. Sci. No. 16019, an adult containing large eggs, agrees with the male, except that the vomerine series is shorter, equalling in length the distance between the outer edge of the nares and the median line; rather darker, brownish gray, more mottled above and below, but very nearly uniform; total length 91 mm., head 11.5, body 41, tail 38.

In spite of the shorter legs, slightly shorter head, and paler color I regard these as the same species as the Inawashiro ones.

An adult female from Inawashiro, containing large eggs, Cal. Acad. Sci. No. 26697, measures 96 mm.

I refer to this species larvae from Northern Hondo which agree in general appearance with larvae of *retardatus* differing in the absence of the dark band on the sides and of the circular spots on the tail.

If my opinion that *peropus*, *nebulosus* as described by Stejneger, *lichenatus*, *nigrescens*, and *fuscus* are the same is correct, the following are additional variations.

The tail may be longer than head and body (*nebulosus* Stejneger, *nigrescens*, *fuscus*), the vomerine series may be somewhat shorter than in the female from Hakone (*lichenatus*), the animal may reach a total length of 135 mm. (*nebulosus* Stejneger), 150 mm. (*nigrescens*), or 160 mm. (*fuscus*).

HABITS: I quote from Tago (1907) his passages concerning the habits of this animal in the region around Nikko.

"Shobu Beach is that part of Lake Chuzenji where the Yugawa River joins the lake. There is an inlet leading to an ice-pond at this point, which I happened to pass during a rain in late October, 1904. The trees were completely bare after a storm the previous day and the foliage everywhere was under my feet instead of over my head. I saw a small dark creature moving among the leaves in the inlet and caught three large, fat monsters measuring 16 cm. each and after careful search found two more of the same size which probably go out into the lake in the spring to breed.

"Tada Lake, which has many streams entering it but no outlet, is located north of Yumoto where I kept my headquarters for this research. Most of my time from early morning until night was spent around this lake which is situated between Onsen Cliff and the back of Yumoto Mt. Evidently volcanic action some generations ago has closed one end of the valley and streams from every direction have gathered to make a lake at this point. It is elongated, lying at the foot of the mountain and at one end the water is very deep, gradually becoming shallow and narrow at the other end which is dammed up. At this shallow end an uninterrupted growth of firs and spruces makes the water dark even in daytime, and here and there branches fallen in heavy snowstorms check the course of the stream and the water spreads over sandy places and is barely deep enough to cover one's heels.

"It was the middle of July, 1903, and the water was warm and filled with moss and weeds just washed from the mountain-side, and I happened to notice an abundance of tadpoles and caught some more from curiosity than with a view to studying them.

"There were two sorts of tadpoles among them; one small and with small gills was *Bufo*, and the other large and brown with triple gills proved to be *Hynobius fuscus*, which I discovered later from my experience.

"I caught several of the latter, some of which I preserved and others I put into water in order to feed them with algae and egg yolk. About three months later these lost their gills and became land animals, while those in the lake reached their fullest development in August and toward the end of September they lost their gills and in every rain they came on land and went into the bushes.

"Suganuma is a large lake about ten miles round located in a thick wood on the way from Nikko to Joshu by way of Nonjo summit. It is about 3000 feet above sea-level and the temperature there very seldom goes above 16 degrees even on the hottest summer days. The lake runs down to Katashinagawa, a branch of the Tone, and the other end, which has been dammed with rocks and sand, may have been a case of volcanic action on Mt. Shirane.

"Like most of the lakes in this region, running along a valley, it has considerable variation in its width, and is generally divided into three sections with different names; Irinuma (entrance of the lake); Nahanuma (middle section); Kitamata (northern section). The last extends from north to south, the other two run east and west.

"In the middle of July, 1903, I came here for the first time in search of specimens and found eggs of *Hynobius fuscus*. It was at the northeast corner of Irinuma, where the Kiyomizu river enters the lake. I was fishing with a net and saw something white in the water about three or four feet down which looked like a sinker hanging on a fallen tree near the beach, and when it was taken from the water there were 30 or 40 eggs surrounded by a gelatinous mass which appeared like a bunch of cotton batting. The eggs were rather flat, and bicolored, brown and yellowish brown. The former was animal pole and the latter yolk. The gelatin like the wings of a bird formed two connected bags. As I was familiar with the description of *Hynobius fuscus* by

Mr. S. Ikeda in a previous volume I knew they were their eggs, and convinced myself by rearing them that they were the same species as the tadpoles I studied at Tada Lake.

"There are two natural hatcheries of this animal at Suganuma, besides the place just mentioned, one in Nahanuma, and the other at the southeastern corner of Kitamata.

"The adults of this species are found at Ojiri on Kitamata, and on the south shore of Nahanuma, where the beach faces to the west. They live along the lake, generally from three to five feet from the water so that floods do not disturb them, under rocks or decayed trees, and seem to live in colonies although they are often found singly.

"They generally protect themselves from attacks of enemies very cleverly, with more than two entrances or holes into their dwelling place, and make so many passage ways in the ground that in case of attack they can hide or escape. However when the rock or log that covers their place is suddenly removed they simply shrink and try to hide by putting only their head under rocks or in the ground while their body is utterly exposed so that we can easily catch them.

"They go out near the water at night for food, which consists of worms, insects, eggs of tiny animals, fishes and their eggs, and so forth, while the larva eats both animal and vegetable food; algae that grow on dead leaves and bushes in the water, copepods and frog tadpoles and larval insects.

"In May after the snow is gone from the mountain sides at Nikko they come out of their winter retirement to the lake to breed. The breeding takes place in the darkness of night and cannot be described In summer they stay around the lake or hunt for food along the valleys, so that we can catch them by moving rocks or piles of dead leaves. They retire late in September or early in October for a long winter rest under decayed trees, in the ground or under piles of dead leaves. Sometimes they go many feet deep into the ground following along tree roots.

"The hatching process is similar to that of *H. naevius*. Sometimes a green alga gets into the eggs."

REMARKS: This animal is related on the one side to *H. leechii* of Korea and on the other to *H. retardatus* of Yezo.

The Yezo form is sufficiently specialized to be quite distinct but the Korean form is quite close, and the color is perhaps the most trustworthy feature by which they can be separated. The lichen-like light patches on the sides of *peropus*, and the absence of marked dark spotting on light ground on the back is quite different from the state of affairs in *leechii*. Unfortunately *leechii* varies far more in color than does *peropus* but in all its variation it never reproduces the coloration of its ally in Hondo.

This is beyond question the form described almost simultaneously by Tago as *fuscus* and by Stejneger as *nigrescens*. That the two were the same was admitted by Tago, and the two type localities, Nikko and Sendai, are mentioned by both authors.

The range of variation in the specimens I have seen covers the descriptions of *lichenatus* and of *peropus*. Tago with his large series and wide field experience could find no specimens of these two forms and regarded them as very doubtfully distinct. He says "the only difference between *fuscus* and *lichenatus* is the rudimentary fifth toe of the latter." This is a purely individual variation, as series of several species of *Hynobius* show, although three forms apparently always have only four toes. I refer Abé's "abnormal" specimen of "*naevius*" from Gumma to this form. It had short vomerine series, and a long tail.

Stejneger's specimen of *nebulosus* from Nikko had 13 costal grooves, thus differing from his specimen of *nigrescens* from Nikko which had 11 costal grooves. As the Inawashiro series shows, this is individual variation. I have examined this Nikko specimen of "*nebulosus*" (Tokyo Col. Sci. No. 51) and find that it is unquestionably the present form. Boulenger (1882), and Okada (1891) record *H. nebulosus* from Nikko, Sternfeld (1916) records it from Shimozuke and Abé (1921) from Shinano. All four doubtless refer to the present form.

Besides the localities mentioned in the list of specimens, it has been recorded from the following places:

Hondo: Mottsu Province: Aomori (Boulenger 1883, *lichenatus*).
- Rikuzen Province: Sendai (Stejneger 1907, *nigrescens:* Tago 1907, *fuscus*).
- Ozen Province: Shonai (Abé 1921).
- Echigo Province: (Abé 1921).
- Hida Province: (Abé 1921).
- Ishikawa Prefecture: (Abé 1921).
- Gumma Prefecture: (Abé 1921).
- Kotsuke Province: Lake Suganuma (Tago 1907, *fuscus*).
- Shinano Province: Mt. Togashi (Tago 1907, *fuscus*).
- Yamato Province: (Stejneger 1907, *peropus*).

Specimens seen: 39, as follows;

Hondo: Rikuoku Province: Mimaya, 7 larvae (Cal. Acad. Sci. 26716, 25395a–f); Ozorezan, 1 larva (Cal. Acad. Sci. 26708).
- Rikuzen Province: Shiogama, 5 larvae (Cal. Acad. Sci. 25669a–e).

Iwashiro Province: Inawashiro, 23 (Cal. Acad. Sci. 26685–26707).
Shimozuke Province: Nikko, 1 (Tokyo Col. Sci. 51).
Izu Province: Hakone, 2 (Cal. Acad. Sci. 16019–16020).

MEASUREMENTS OF *Hynobius peropus*.

* = costal grooves
a = 5th toe rudimentary
c = 5th toe short
e = fold of skin on side
/ = costal folds between appressed toes
b = vomerine series short
d = 5th toe absent

Specimen	Locality	Sex	Total length	Head	Body	Tail	*	Remarks
Brit. Mus.	?	♂a	109	12	47	50		*peropus* Type
"	Aomori	bc	74	9	29	36	11/	*lichenatus* "
Tokyo Coll. Sci.	Sendai	♂	150	20	54	76	11	*nigrescens* "
" " "	Nikko	♀	108	15.5	43.5	49	11/	" paratype
" " "	"	♂	135	16	57	62	13/0	*nebulosus* Stejneger 1907
	"		146	19	61	88	13/	*fuscus*, averages Tago 1907
	Sendai	♂	126	15	52	59	11/0	Abé 1921
	"	♂	127	17	53	57	11/0	"
	Shinano		120	15	49	56	12/0	"
	Shonai	♀	111	14	47	50	11/0	"
	Ishikawa		123	16	50	57	11/0	"
	Hida	♀	130	19	53	58	11/0	"
	Echigo	♀	115	16	45	53	11/0	"
	Gumma	?	115	15	50	50	13/	"
Cal. Acad. Sci.								
16019	Hakone	♀b	90	11	41	38	13/2	
16020	"	♂	109	14	50	45	13/2	
26707	Inawashiro	♂a	101	15	48	38	11/0	
26706	"	♂	98	14.5	46.6	37	12/0	
26705	"	♀	100	14.5	35.5	40	12/0	
26704	"	♀	101	14	46	41	13/0	
26703	"	♀	106	15	50	41	12/1	
26702	"	♀	99	14	45	40	13/0	
26701	"	♀	86	13	45	28	13/0	
26700	"	♂	89	13.5	42.5	33	12/0	
26699	"	♂c	81	12	38	31	12/0	
26698	"	♀e	94	14	44	36	12/0	
26697	"	♀	96	14	45	37	13/1	contains large eggs
26696	"	♀	97	13.5	49.5	34	13/0	
26695	"	♀	100	14	47	39	12/1	

Specimen	Locality	Sex	Total length	Head	Body	Tail	*	Remarks
Cal. Acad. Sci.								
26694	Inawashiro	♂	105	14	49	42	13/0	
26693	"	♀	90	13.5	41.5	35	12/0	
26692	"	♂c	85	15	43	27	12/0	
26691	"	♂c	107	15	49	43	12/0	
26690	"	♂	95	14	42	39	12/0	
26689	"	♂e	96	14	44	38	12/0	
26688	"	♀c	93	13	49	31	13/1	
26687	"	♂e	106	15	58	43	12/0	
26686	"	♂	101	14.5	47.5	39	12/0	
26685	"	♂d	108	15	52	41	13/0	
26708	Ozorezan	larva	34	7	21	6	12/0	
26716	Mimaya	"	51	8	22	21	12/0	
25395	"	"	49	7.5	21.5	20	12/0	
25395	"	"	50	8	22	20	12/0	
25395	"	"	39	6	15	18		
25669	Shiogama	"	35	6	14	15		
25669	"	"	30					

HYNOBIUS RETARDATUS Dunn.

1892. *Hynobius nebulosus* Sclater Batr. Indian Mus., p. 36 (Yezo).

1907. *Hynobius fuscus* Tago, (part), Tokyo Zool. Mag., 19, pp. 204, 233.

1910. *Hynobius lichenatus* Hatta, Tokyo Zool. Mag., 22, p. 88; Hashimoto, 1910, Tokyo Zool. Mag., 22, p. 497; Kuhne, Hatta, and Hashimoto, 1910, Tokyo Zool. Mag., 22, p. 559; Abé 1921, Tokyo Zool. Mag., 34, p. 320; Inukai 1922, Journ. Coll. Agric. Hokkaido Imp. Univ., 10, 5, p. 107.

1910. *Hynobius nigrescens* Hashimoto, loc. cit., p. 497.

1923. *Hynobius retardatus* Dunn, Proc. California Acad. Sci., (4), XII, 2, p. 27.

TYPE: Cal. Acad. Sci. No. 35928, recently transformed male, collected Aug. 30, 1911, by Kuhne.

TYPE LOCALITY: Noboribetsu, Iburi Province, Yezo.

RANGE: Island of Yezo.

DIAGNOSIS: A *Hynobius* with long, flat tail, very short series of vomerine teeth; 11–12 costal grooves; toes five, long; grayish brown, a darker lateral stripe; reaching a large size before transforming.

DESCRIPTION: Cal. Acad. Sci. No. 35928, Noboribetsu, Yezo, recently transformed, male. Costal grooves 12; appressed toes overlapping; head width 5 in length from snout to vent; head length 3 in length of body; head a blunt oval from above; eye as long as its distance from tip of snout; lower eyelid fitting under upper in front and behind; upper jaw straight as seen from side; angle of jaw back of hind angle of eye; a groove from eye to gular fold; a groove from this down behind angle of jaw; limbs well developed; fingers 2, 3, 4, 1 in order of length; toes long, 3, 4, 2, 1, 5 in order of length; a tubercle at base of first finger and toe; tail longer than body, flattened, finned above; vent a longitudinal slit; vomerine teeth in two arched series convex forwards, about 17 in series, beginning behind outer edge of nares, curving forward and in to meet its fellow, not curving back at all in median line; grayish brown, a darker lateral stripe, lighter below.

Total length 103, head 15, body 43, tail 45.

Variation: Cal. Acad. Sci. No. 35927, same data, recently transformed male; head width $5\frac{1}{5}$ in length from snout to vent; head length $3\frac{2}{3}$ in length of body; angle of jaw under hind angle of eye; tail shorter than body; vomerine teeth in two arched series between nares, 13 teeth in series, on right side a row of 5 teeth behind nares; dark brownish above, lighter below.

Total length 96, head 12, body 44, tail 40.

A recently transformed female (?) Mus. Comp. Zool. No. 5122, Yezo, is similar to the described specimens; the vomerine teeth are in two arched series between the nares, about 12 teeth to a series, series meeting in median line, not extending back of nor in front of nares.

Total length 89, head 13, body 39, tail 37.

Larvae: These are brownish above with black spots; on the tail there are a few circular black spots; sides light with dark marbling; belly light; dorsal fin to middle of back; no fold of skin along leg.

Cal. Acad. Sci. No. 25982–9, Sapporo, Yezo.

25982	total	45	head	8	body	24	tail	13
25983	"	54	"	8	"	23	"	23
25984	"	50	"	7	"	23	"	20
25985	"	38	"	8	"	21	"	9
25986	"	55	"	9	"	24	"	22
25987	"	35	"	8	"	20	"	5
25988	"	60	"	9	"	26	"	25
25989	"	30	"	8	"	22		

Nemuro, Yezo, Cal. Acad. Sci. 25990, total 41, head 7, body 19, tail 15.

This specimen has 5 toes on one side and 4 on the other. All these larvae have the appressed toes meeting and 12 costal grooves.

An adult male, U. S. Nat. Mus. No. 64902, Kutara Lake, Iburi

Province, has 11 costal grooves, the vent is Y-shaped, the vomerine teeth are in adult form for *Hynobius*, consisting of an outer branch of 6 teeth making a right angle with an inner branch of 10 teeth, series separated from its fellow by width of nares, extending further back in the middle than on the sides, length of inner branch not equalling distance between outer edge of nares to median line; total length 134 mm., head 18, body 45, tail 48.

Abé (1921) gives measurements of two adult males from Sapporo:

Total length 110	head 15	body 45	tail 50
" 140	" 19	" 55	" 66

A large larva, U. S. Nat. Mus. No. 64903 from the estuary of the Shirisetunai River, near Lake Shikotsu, Iburi Province, agrees with the Cal. Acad. Sci. specimens. Total length 102 mm., head 14, body 43, tail 45.

Remarks: The specimens I have examined from Yezo are all one species and this species is very distinct from any of the Hondo *Hynobius* I have seen.

None the less it has been considered to be *fuscus*, *lichenatus*, or *nigrescens*. Irrespective of whether one considers, as I do, that *lichenatus*, *fuscus*, and *nigrescens* are all synonyms of *peropus*, or not, there are good reasons why this Yezo animal cannot be identified with any of them.

In the first place *lichenatus* was described from a fully developed specimen 74 mm. long. This is a length at which the Yezo animal is still a larva. The color of *lichenatus* as described by Boulenger is very different from that of the Yezo form. The toes of *lichenatus* are short. The reasons for identifying the Yezo *Hynobius* with *lichenatus* are the shortness of the vomerine series of *lichenatus*, its 11 costal grooves, and its northern range.

Nigrescens as described by Stejneger is a very different animal from the Yezo one; in color, in shape of tail, in length of toes, and in longer vomerine series it is sufficiently different.

Fuscus was described by Tago at nearly the same time as Stejneger described *nigrescens*. According to Tago the two are the same. It is hard to believe that his description of Nikko specimens could apply to the present form.

Hashimoto (1910) apparently made no comparison with actual specimens. He came to the conclusion that *lichenatus* and *nigrescens* occurred on Yezo by comparing his specimens with Boulenger's and Stejneger's descriptions. His opinion that he had two species was apparently not very firm as the next month there appeared "The Reptiles and Amphibians of Hokkaido" by Kuhne, Hatta, and himself, in which only *lichenatus* is recorded from Yezo. Abé (1921) after examining Hashimoto's specimens concludes that there is only one species on Yezo.

My conclusion, then, is that the present form is amply distinct from any of the Hondo species. It may, of course, be proven eventually that two species occur on Yezo, and that the other is the same as one of the Hondo forms, but I take the liberty of expressing polite doubt.

Its relationships are with what I have called *peropus* and are shown in the shortness of the vomerine series, shortness of body, color of larva, and to some extent in the color of the adult. This relationship is rather remote, the fact being that *retardatus* seems to represent an end form, and while *peropus* seems to represent the stock from which *retardatus* was derived, a great deal of differentiation has taken place.

Inukai (1922) says that this animal is neoteinic and "brings offsprings in the larval stage," "as quite recently made known by Dr. Sasaki."

The breeding season of 1922 was April 7–21, and the eggs were laid in long sacs as described for *keyserlingii*, 25–37 in a sac.

Besides the specimens listed below it has been recorded from Kashiuchi and from Yubari, both in the high eastern part of the Hirahara district of Ishikari Province (Hashimoto 1910).

Specimens seen: 14, as follows;

Yezo: no definite locality, 1 (M. C. Z. 5122).
    Nemuro Province: Nemuro, 1 (Cal. Acad. Sci. 25990).
    Iburi Province: Noboribetsu, 2 (Cal. Acad. Sci. 35927–35928); Kutara Lake, 1 (U. S. N. M. 64902); estuary of Shirisetunai R., near Lake Shikotsu, 1 (U. S. N. M. 64903).
    Ishikari Province: Sapporo, 8 (Cal. Acad. Sci. 25982–25989).

Hynobius sonani (Maki).

1921. *Salamandrella sonani* Maki, Zool. Mag. Tokyo, 34, p. 636.

1921. *Hynobius formosanus* Maki, loc. cit., p. 637 (Mt. Noritaka, 7000 ft. alt., collected May, 1919, by Mr. Sonan).

1921. *Hynobius arisanensis* Maki, loc. cit., p. 637 (Mt. Ari, 6600 ft. alt., collected March 1921, by Mr. Ando).

Type: Adult female, collected May 1919, by Mr. Sonan.

Type Locality: Mt. Noritaka, Formosa, at altitude of 10,000 ft.

Range: Mountains of Formosa.

DIAGNOSIS: A *Hynobius* with very long vomerine series; tail tapering; 12–13 costal grooves.

DESCRIPTION: (I endeavor to combine Maki's remarks about these Formosan salamanders in such a way as not to obscure the points which he considers to differentiate his three species). Head round, flattened; nostril nearer tip of snout than eye (*formosanus, arisanensis*); nostril equidistant between eye and tip of snout (*sonani*); distance between nares greater than between eyelids; eyelid longer than interorbital space (*sonani*); eyelid equal to interorbital space (*formosanus*); eyelid almost equal to interorbital space (*arisanensis*); costal grooves 13 (*sonani*), 12 (*formosanus, arisanensis*); vomerine tooth row equals $\frac{3}{4}$ width of tongue (*formosanus*), $\frac{4}{5}$ (*sonani*), $\frac{6}{7}$ (*arisanensis*); inner rows close together in *formosanus*, well separated in *arisanensis*, intermediate in *sonani;* fifth toe absent (*sonani*), rudimentary, tubercle-like (*arisanensis*), smaller than first (*formosanus*); "yellow with round dark spots on back and sides" (*sonani*); "tawny yellow, brown spots, small spots on belly" (*formosanus*); "dark brown, with small yellowish-brown spots, somewhat cream colored on belly" (*arisanensis*);

total length	113	head	16	body	50	tail	49	male *arisanensis*
"	107	"	15	"	48	"	44	" "
"	98	"	13	"	45	"	40	female *formosanus*
"	90	"	13	"	41	"	36	" *sonani*

REMARKS: Maki had four salamanders from Formosa. On these he established three species. His characters appear to me individual variation. The males (*arisanensis*) are much darker than the females (*formosanus, sonani*), thus agreeing with the species from Tsu-shima South, etc. In any single character his four specimens form a rather graded series. I have put the description in such a way, however, that others may apply the criteria he used.

The relationships of this form would seem at first sight to be with the *naevius* group. It is more probable, however, that it has been derived independently of that series of forms from some mainland species, especially as the general affinity of the Formosan fauna is far greater for the Chinese forms than for the Japanese. I should not be surprised at the discovery of an animal of the *naevius* group on the mainland, for it is very difficult to derive it from *leechii*. In this case the resemblances would be explained without recourse to parallelism, or to migration from Formosa to Kiusiu.

Hynobius stejnegeri Dunn.

1896. *Hynobius naevius* von Martens, Preuss. Exped. Ost.-Asien, Zool., I, p. 384; Stejneger, 1907, U. S. Nat. Mus. Bull., 58, p. 26, ff. 17–22; Abé 1921, Tokyo Zool. Mag., 34, p. 329.

1923. *Hynobius stejnegeri* Dunn, Proc. California Acad. Sci., (4), XII, 2, p. 28.

Type: U .S. Nat. Mus. No. 23901, adult female, collected by Mr. Nakagawa, 1884.

Type Locality: Kumamoto, Province Higo, Kiusiu.

Range: Kiusiu.

Diagnosis: A Hynobius with short, thick tail; very long vomerine series; 13–14 costal grooves; 5 toes; light markings present on whole surface of body.

Description: U. S. Nat. Mus. No. 23901, adult female, Kumamoto, Kiusiu; 14 costal grooves; 3 costal folds between appressed toes; head width 6 in length from snout to vent; head length 3⅓ in length of body; head oval; eye as long as its distance from tip of snout; outline of upper jaw straight as seen from side; angle of jaw back of hind angle of eye; lower eyelid fitting under upper in front and behind; a groove from eye to gular fold; a groove from this down behind angle of jaw; limbs well developed; fingers 2, 3, 1, 4 in order of length; toes 3, 4, 2, 5, 1 in order of length; a tubercle under base of first finger and toe; tail shorter than body, thick throughout, ending bluntly, somewhat flattened in distal half; vent a simple slit; vomerine series beginning behind inner edge of nares, a forward incurving branch of 6 teeth, a branch of 18 teeth running straight back and a little in, separated from its fellow by about length of outer branch, length of inner branch equalling distance from outer edge of one naris to inner edge of the opposite naris; chocolate brown, lighter below; thickly marbled with largish lighter blotches, which have a reddish brown tinge on the back and are white on the sides and the belly; total length 87 mm., head 14, body 47, tail 26.

Variation: Stejneger (1907) mentions two other specimens in the collection of the Science College, Tokyo. One of these had 13 costal grooves, and total length 107 mm., head 14, body 44, tail 49. It was "paler (more faded?) with less coarse marblings, . . . and the underside nearly uniform pale brownish." These came from the same locality as the type. Abé (1921) gives measurements of two additional specimens.

Chikusen	♀	Total length 122	head 17	body 62	tail 43	14 costal grooves
Kumamoto	?	" 96	" 14	" 46	" 36	13 " "

REMARKS: This animal has been considered the same as *H. naevius*. But the constant difference in color is amply sufficient to distinguish the Kiusiu form. It is doubtless close to *naevius*, and with it and the Hida species, *H. kimurai*, they form a very natural group of vicarious forms. In color *kimurai* and *stejnegeri* are almost identical, but *kimurai* has only four toes, a longer series of vomerine teeth and a longer body.

The record of von Martens (1876) for *naevius* from Nagasaki doubtless refers to the present form. Abé (1921) records it from Chikusen Province.

Specimens seen: one, the type.

HYNOBIUS NAEVIUS (Schlegel).

1838. *Salamandra naevia* Schlegel, Fauna Japonica, Rept., p. 122, pl. 4, ff. 4–6, pl. 5, ff. 9–10; 1844, Abbild. Amphib., p. 122, pl. 39, f. 4; Wiedersheim, 1876, Morph. Jahrb., 2, p. 428 (extremities).

1838. *Pseudosalamandra naevia* Tschudi, Mem. Soc. Sci. Neuchâtel, pp. 56,91.

1838. *Pseudosalamandra naevia* Tschudi, Classif. Bat., pp. 56, 91.

1839. *Molge naevia* Bonaparte, Icon. Fauna Ital., 2, fasc. 26.

1854. *Ellipsoglossa naevia* Duméril and Bibron, Erpet. Gén., 9, p. 99, pl. 101, f. 5; Wiedersheim, 1877, Morph. Jahrb. 3, p. 417, pl. 23, f. 64–67, (skull and Hyobranchial apparatus); Druner, 1904, Zool. Jahrb., 19, p. 665, pl. 25 (Hyobranchial apparatus).

1859. *Hynobius naevius* Cope, Proc. Acad. Nat. Sci. Philadelphia, p. 125; Boulenger, 1882, Cat. Bat. Grad. Brit. Mus., p. 32; Namie, 1903, Tokyo, Zool. Mag., 15, p. 286; Tago, 1904, Tokyo Zool. Mag., 16, p. 44; Stejneger, 1907, Bull. U. S. Nat. Mus., 58, p. 26, pl. 3, ff. 1–3; Tago, 1907, Tokyo Zool. Mag., 19, p. 200, 229.

TYPE: In the Rijksmuseum, Leiden, collected by Siebold. (No type was specified and specimens from the original series have been widely distributed).

TYPE LOCALITY: Mountainous parts of Hondo and Shikoku.

RANGE: Hondo (more specifically from Uso to Kazusa and to Shimozuke, 0–1500 ft. alt.); Shikoku.

DIAGNOSIS: A *Hynobius* with short, thick tail; very long vomerine series; 14 costal grooves; 5 toes; light markings only on sides and belly.

DESCRIPTION: Mus. Comp. Zool. No. 7365, adult female, (Cotype, ex. Leiden Museum via British Museum), "Japan," 14 costal grooves; 2 costal folds between appressed toes; head width 6 in length from snout to vent; head length $3\frac{1}{2}$ in length of body; head a broad oval; eye as long as its distance from tip of snout; outline of upper jaw straight as seen from side; angle of jaw back of hind angle of eye; lower eyelid fitting under upper in front and behind; a groove from eye to gular fold; a groove from this down behind angle of jaw; limbs well developed; fingers 3, 2, 1, 4 in order of length; toes 3, 4, 2, 5, 1 in order of length; a tubercle under base of first finger and toe; tail shorter than body, thick throughout, ending bluntly, somewhat flattened in distal half; vent a simple slit; vomerine series beginning behind inner edge of nares; a forward incurving branch of 8 teeth; a long straight branch of 27 teeth extending back and slightly in; one tooth on each side in from hind end of series separated from its fellow by half width of nares; series separated from its fellow by length of outer branch anteriorly, and by width of nares posteriorly; length of inner branch equalling distance between outer edge of one naris and the inner edge of the other; blackish, on sides and beneath with large whitish marblings; total length 123 mm., head 18, body 64, tail 41.

Variation: I append some measurements taken from Boulenger (1882) and from Stejneger (1907). These are in all probability from animals of the original series.

Brit. Mus.		male	total length	131	head	14	body	64	tail	53
"		female	"	122	"	14	"	60	"	48
Copenhagen	34	"	"	116	"	16.5	"	54.5	"	45
"	35	"	"	128	"	19.5	"	60.5	"	48
"	36	"	"	143	"	18.5	"	63.5	"	64

HABITS: I quote the following from Tago (1907).

"In less elevated regions from the coast to 500 meters above sea level, wherever there is a valley, stream, or pond surrounded by hills and cliffs with cedar, elm, or chestnut trees, we can trace their haunts among moss, dead leaves around swamps, in bushes, under roots, rocks, or in deserted nests of water rats or moles. In short they like damp, shady places where they are free from sunshine.

"It is hard for us to find them among trees or rocks, but after heavy summer rain storms we can very often find them picking up worms and insects driven out of the ground, or sometimes they are caught by a net in the river where they are driven out themselves by the flood.

"When a *Hynobius naevius* is fully grown it starts a life on land and at night, in rain, or on damp days it goes out of its hiding place after food; which consists of worms, insects, and tiny fishes. It enjoys this life all through summer until late in the fall, and about October it retires for a long winter rest, hiding himself in the ground near water. Once I found one of them in this concealment in a cave six inches deep in the ground by a spring in a cedar grove.

"Early in March, on coming out of a long hibernation it takes up a strenuous life again. The first thing it does is to seek its mate, going along water-ways, and looking for a suitable place for breeding its offspring. They seem to be very particular in the selection of such a place as we very often find several of them in one place at this season, which is the latter part of March.

"March 2nd, 1904, at 8.30 in the morning, I found a group of their eggs in a small pond under a cedar grove which covered a low cliff called Takanosu of Iwabune, Tochigi Prefecture. The lower bushes were mostly of Yamabuki and of young elm trees, and the water was 9½ inches deep and the pond was open to the north. There were some thirty or forty eggs contained in a transparent gelatinous mass shaped like a lead weight [sinker] with both ends bent in and attaching to the twig in the pond, and measuring 60 mm. in diameter.

"The eggs varied in size from 2.5 to 3 mm. in diameter, and their animal poles were yellowish brown, and the vegetative poles were yellow.

"Also I found two groups of eggs with their mother under dead leaves not far from the same place in a narrow ditch 6 inches wide in rice fields (swamp).

"In a shallow stream hardly 3 inches deep, running to the north through bushes among young cedar trees I found another two groups with their mother. Most of the eggs were out of water and were in an early stage of development.

"In a narrow stream, hardly above our heels and about a foot wide, that ran northeast through dark bushes in a wood I found another two groups with eleven adults. There were many chestnut, elm, etc., with a few cedar trees in the woods.

"In a spring beneath large cedar trees, (the upper part full of clear water that runs some distance into a narrow stream, and the lower, which is wider and swampy, being dug out in connection with the spring for rice growing), I noticed many egg-masses, and after taking out twenty buckets of water I found them hung on dead branches and roots. Toward the center, where the water was deeper I noticed that the eggs were supported by a large cedar branch in such a way that they would not sink deeper than 6 inches. Many eggs and several adults were the result of my efforts.

"A score of large cedar trees at Iwabune shrine always keep the yard in shade, and there is a large stone bowl inside the wooden fence. This bowl is filled with water by occasional rains and in the fall by dead leaves. Here I found a quantity of eggs with the mother.

"Shakutakano in Ibaraki Prefecture is indeed a place of natural beauty. Hill after hill of considerable height contrasts with valleys winding far below, and a crystal stream echoes through the region. A mile upstream, the valley becomes narrow and the water scanty, and large rocks and trees here and there make the place dark. There is a tree there which gives a queer uncomfortable odor, and there I found many egg-masses, some of which were just out of the cloaca of the mother, and the sack which contains them was wrinkled up instead of stiffened by assimilated water.

"The eggs hatch after 12–15 days. After 15–16 days the larva grows more lively and comes out from a corner of its shell, or sometimes it breaks the shell in the center. It is then about 15 mm. long. On both sides of the cheeks it has double balancers, and triple external gills, but mouth and gill slits do not open at this stage. There is a stump which later becomes the front leg. The tail is about half as long as head and body, and has a fin above and below, the upper fin reaching the middle of the body. It is yellowish brown above and yellow below. There are brown spots. It holds on to things by means of the balancers. About ten days after hatching the front legs begin to grow, at first oar-shaped, the middle part gradually increasing. It takes fourteen days before the hind legs begin to develop. Next, mouth and gill slits open, and the gills grow large and long.

"Probably in August or September the metamorphosis takes place and land life begins."

REMARKS: This strongly marked form is related only to *kimurai* and to *stejnegeri*. Of these *stejnegeri* is marbled with lighter both above and below; *kimurai* is marbled above and on the sides, the belly being uniform; *naevius* is marbled on the sides and the belly, the dorsal surface being immaculate.

Tago (1907) records this animal from parts of Hondo north of Tokyo, as follows: Uso Province, Kariwano; Iwashiro Province, Hautayama; Shimozuke Province, Tochigi Machi, Iwabune; Fukushima Prefecture, Ogawa; Ibaraki Prefecture, Shakutano; Kazusa Province, Koshikiya.

Okada records it from Kyoto, but this probably refers to *kimurai*, and the reference has been placed in the synonymy of that species.

Mention will be made below of larvae from Miyazu, Tango Province, doubtfully referred to *kimurai*. If Tago is correct in his statement that *naevius* occurs only northward in Hondo, then the Kyoto and Miyazu animals may well be *kimurai*, and I have given them the benefit of the doubt.

Besides Schlegel's original reference, Namie (1903) records it from Shikoku. (Schlegel's type locality for this form is very vaguely given. He says of *naevius* merely that it lives in the same places as *unguiculata* (*O. japonicus*), for which he gives the following: "In abundance in mountainous parts of Nippon and Sikok, especially Sagami, Sinano, Tanba, Tazima, and Tosa." This may or may not be wholly applicable to *naevius*).

Specimens seen: one, cotype. "Japan."

HYNOBIUS KIMURAI Dunn.

1891. *?Hynobius naevius* Okada, Cat. Vert. Japan, p. 65; Abé, 1921, Tokyo Zool. Mag., 34, p. 329.

1907. *Salamandrella* n. sp. Tago, Tokyo Zool. Mag., 19, p. 237; Maki, 1921, Tokyo Zool. Mag. 34, p. 636.

1923. *Hynobius kimurae* Dunn, Proc. California Acad. Sci., (4), XII, 2, p. 27 (typ. err.).

TYPE: Mus. Comp. Zool. No. 8546, adult female. Collected by Dr. Kimura and Dr. H. H. Wilder, 1820.

TYPE LOCALITY: Mt. Hieizan, Omi Province, Hondo.

RANGE: West Central Hondo.

DIAGNOSIS: A *Hynobius* with short, thick tail; very long vomerine series; 13–15 costal grooves; usually 4 toes; brown, with light flecks over dorsal surface.

DESCRIPTION: Mus. Comp. Zool. No. 8546, adult female, Mt. Hieizan, Hondo; 15 costal grooves; 4 costal folds between appressed toes; head width 6 in length from snout to vent; head length 3¾ in length of body; head oval; eye as long as its distance from tip of snout; outline of upper jaw straight as seen from side; angle of jaw back of hind angle of eye; lower eyelid fitting under upper in front and behind; a groove from eye to gular fold; a groove from this down behind angle of jaw; limbs well developed; fingers short, 2, 3, 1, 4 in order of length: toes 3, 4, 2, 1 in order of length; a tubercle under base of first toe and finger; tail shorter than body, thick throughout, ending bluntly, slightly flattened in last half; vent a simple slit; vomerine series beginning behind inner border of nares, a forward incurving branch of 5 teeth, a branch of 17 teeth running straight backwards, not meeting its fellow, separated all along by width of nares, length equalling distance between outer edges of nares; bluish brown above, lighter below; small light flecks over whole dorsal surface: total length 121 mm., head 17.5, body 65.5, tail 39.

Variation: A male, Cal. Acad. Sci. No. 27258 Hida Province, Hondo, differs in having 14 costal grooves; 3 costal folds between appressed toes; head width 5 in length from snout to vent; head length $3\frac{2}{5}$ in length of body; vomerine series beginning behind middle of nares; main branch S-shaped, of 25 teeth, meeting its fellow behind; length equalling distance between outer edges of nares plus width of nares; total length 108 mm., head 15, body 51, tail 42. Tago (1907) gives averages of 13 males and 22 females from Takaramura, Yoshishiro district, Hida Province, as follows:

Male	Total 110	head 15	body 50	tail 45
Female	“ 105	“ 14.5	“ 47.5	“ 43

I refer the three specimens of *H. nacrius* mentioned by Abé (1921), from Western Hondo to this species, not without misgivings. The Etchu specimen had 4 toes. Measurements as follows:

Etchu Prov.	♀	total 120	head 14	body 56	tail 50	13 costal grooves
Fukui Pref.	♀	“ —	“ 14	“ 47	“ —	13 “ “
Gifu Pref.	?	“ —	“ 15	“ 53	“ —	13 “ “

Maki (1921) says “recently Prof. Watase found one at Kuroba, Toyama.”

I refer two larvae, Cal. Acad. Sci. No. 16322–3, from Miyazu, Tango Province, Hondo, to this species with considerable doubt. They may belong to *H. naevius*. They have 13 costal grooves and one costal fold separates the appressed toes. They have four toes. The color is rather light brown with two black cross-bands on body, one just back of legs, and two or three on tail.

16322	Total 48	head 6	body 20	tail 22
16323	“ 35	“ 4	“ 16	“ 15

Tago (1907) says that in his series, referred to above, one out of twenty has five toes.

Habits: According to a letter from Mr. Yasutaro Mayeda, quoted by Tago (1907), the habits at Kami, Takaramura, Yoshishiro district, Hida Province, are as follows: “Above the reservoir where the stream is 6–7 feet wide and 6–7 inches deep there is a small spring near the foot of the mountain. About three weeks of every year the animals gather round this spring by hundreds from every direction to breed in the water. During the summer they go into the mountainside and seek damp places under bushes or fallen trees, and through fall, winter, and spring they burrow and are very seldom seen. Thus it is easy to catch twenty of them in the breeding season while it is hard to get even a few at other times of the year. They are called Oka Imori (Land Lizard) in this part of the country." The breeding season is not definitely stated but is evidently spring.

REMARKS: This animal is close to *H. naevius* but quite distinct. In color it resembles rather *H. stejnegeri* from Kiusiu. Tago intended to describe it as new, but I have been unable to find that he ever did so.

It is apparently a mountain form and ranges south and west of *H. naevius* which is according to Tago restricted to low altitudes in Northern Hondo. The larvae from Miyazu are quite a problem. Tago (1907) records *naevius* from Kariwano, Uso Province on the west coast. If altitude is the distributional factor involved the Miyazu specimens might well be *naevius*, and they agree with *naevius* in proportions rather than with *kimurai*. But they are well advanced and have only four toes, and in the uncertainty I think it better to leave them with the less known form. I think it probable that the record of *naevius* from Kyoto, (Okada 1891) refers to *kimurai*.

Specimens seen: 4, as follows;

Hondo: Hida Province, 1 (Cal. Acad. Sci. 27258); Mt. Hieizan, Omi Province, 1 (M. C. Z. 8546); Miyazu, Tango Province, 2 (Cal. Acad. Sci. 16322-16323, larvae).

HYNOBIUS TAGOI Dunn.

1907. *Hynobius nebulosus* Stejneger (part), U. S. Nat. Mus. Bull., 58, p. 31.
1923. *Hynobius tagoi* Dunn, Proc. California Acad. Sci., (4), XII, 2, p. 29.

TYPE: California Acad. Sci. No. 26563, adult male, collected October, 1910, by Victor Kuhne.

TYPE LOCALITY: Tsu-shima, North Island.

RANGE: Known only from the type locality.

DIAGNOSIS: Vomerine series medium; tail somewhat flattened; 13 costal grooves; toes 5; sides of tail not black; middorsal line of tail not yellow; grayish brown with fine darker brown stippling.

DESCRIPTION: Type; 13 costal grooves; one costal fold between appressed toes; head width $5\frac{1}{3}$ in length from snout to vent; head length $3\frac{1}{3}$ in length of body; head an oval from above; eye as long as its distance from tip of snout; outline of upper jaw concave as seen from side; angle of jaw back of hind angle of eye; lower eyelid fitting under upper in front and behind; a groove from hind angle of eye to gular fold; a groove from this down behind angle of jaw; limbs well developed; fingers 2, 3, 4, 1 in order of length; toes 3, 4, 2, 5, 1 in order of length; a tubercle under base of first finger and of first toe; tail shorter than body, cylindrical at base, flattened at tip, just past level reached by appressed toes trigonal with a low dorsal keel, scarcely widened anywhere; vent a longitudinal slit, slightly forking forwards, so that there is a small triangular flap at apex; vomerine series beginning behind and in from inner border of nares, a short forward and incurving branch of 4–5 teeth, a long straight branch of 17 teeth extending back and slightly in, a short branch of 2 teeth curving in to meet its fellow of opposite side; series separated from its fellow in front by 1½ length of outer branch, and posteriorly by length of outer branch; length of inner branch equals distance between outer edge of naris and the peak of the opposite vomerine series; brownish gray above, with fine crowded darker stippling, middorsal line of tail slightly lighter; below light, nearly covered with faint gray marbling; total length 116 mm., head 15.5, body 51, tail 49.5.

REMARKS: The specimen mentioned by Stejneger (1907, p. 31) as collected by Holst in Tsu-shima, Oct. 6, 1891, seems to be this form rather than the Tsu-shima South species. He describes the color of the specimen, a female, Brit. Mus. No. 92.1.11.18, as "yellowish brown with numerous obscure dusky spots; underneath uniform pale brownish, this pale effect being produced by a uniform dusting of minute, round, yellowish specks on a dark brownish gray ground."

"Total length 117, snout to vent 68, snout to gular fold 16, vent to tip of tail 49."

This species is a member of the group comprising the animals of Tsu-Shima South, Iki-shima, *nebulosus* of Kiusiu, *vandenburghi* of Hondo, and the undescribed form from Shikoku. From them all it differs in color. I cannot find a match for it in a series of 168 adults from the South island. In that species the sides of the tail are always black, the ground color is yellow, the black spotting is coarse and irregular, and unless the animal is uniform black (as in some old males) the middorsal line of the tail is bright yellow. The Iki-shima form has a shorter head, the color is much darker, although the spots are fine as in *tagoi*, and again unless the animal is uniformly dark the middorsal line of the tail is yellow. *Nebulosus* of Kiusiu is similar to the South Tsu-shima form, but the sides of the tail are not black, while the Hondo form is abundantly distinct by reason of its very flat tail.

Specimens seen: One, the type.

HYNOBIUS TSUENSIS Abé.

1907. *Hynobius nebulosus* Tago (part), Tokyo Zool. Mag., 19, p. 201, 233; Sternfeld 1916, Sitz. Ber. Gesch. Nat. Fr. Berlin, p. 172.
1921. *Hynobius tsuensis* Abé, Tokyo Zool. Mag., 34, p. 331.
1923. *Hynobius* ***bicolor*** Dunn, Proc. California Acad. Sci., (4), XII, 2, p. 28 (Tsu-shima, South Island).

TYPE: In Tokyo Zoölogical Museum (?).

TYPE LOCALITY: "Tsu-shima."

RANGE: Known only from Tsu-shima, South Island.

DIAGNOSIS: Vomerine series medium; tail somewhat flattened; costal grooves 13–14; toes 5; sides of tail black; middorsal line of tail yellow; yellow with black mottling; males black.

DESCRIPTION: Cal. Acad. Sci. No. 26447, Type, adult male; 13–14 costal grooves; one costal fold between appressed toes; head width 5¼ in length from snout to vent; head length 3⅓ in length of body; head an oval as seen from above; eye as long as its distance from tip of snout; outline of jaw straight as seen from side; angle of jaw back of hind angle of eye; lower eyelid fitting under upper in front and behind; a groove from eye to gular fold; a groove from this down behind angle of jaw; limbs well developed; fingers 2, 3, 4, 1 in order of length; toes 3, 4, 2, 5, 1 in order of length; a tubercle under base of first digit of hand and foot; tail as long as body, cylindrical at base, flattened at tip, just past level reached by appressed toes trigonal with a low keel above, scarcely widened anywhere; vent a longitudinal slit, forking forwards so that there is a small triangular flap at apex; vomerine series beginning behind inner border of nares; a short forward and incurving branch of 6 teeth; a long branch of 20 teeth extending straight back and a little inwards; series separated from its fellow anteriorly by 1½ length of outer branch, posteriorly by ½ length of outer branch; series extends from level of anterior edge of nares to hind quarter of eye socket; length of inner branch equals distance from outer edge of naris to peak of opposite series; black above, with irregular lighter flecks; middorsal line of tail light; gray beneath; sides of body marbled gray and light; sides of tail black above, shading into gray below; total length 129 mm., head 17, body 56, tail 56.

Variation: A female, Cal. Acad. Sci. No. 26491, same data, differs as follows; 14 costal grooves; 3 costal folds between appressed toes; head width 6 in length from snout to vent; head length 3⅔ in length of body; tail shorter than body; vent a simple slit; yellow above with small irregular black spots which tend to connect with each other; middorsal line of tail yellow; sides of body and of base of tail marbled yellow and black; sides of distal half of tail black; total length 123 mm., head 17.5, body 62.5, tail 43.

A young specimen, Cal. Acad. Sci. No. 26519, same data, has the appressed toes meeting; the coloration is scarcely different from that of the described female; total length 49 mm., head 8, body 21, tail 20.

A larva, Cal. Acad. Sci. No. 26562, same data, has external gills pigmented, and rather flat and wide; the dorsal fin extends a little forward of the middle of the body; pigmentation rather uniform and confined to the dorsal surface, save on the tail which is more coarsely spotted all over; there is a fold of skin along the leg from the fifth toe up; total length 49 mm., head 8, body 20, tail 21.

Among the adults the males have longer tails and the black spotting of the dorsal surface tends to obscure the yellow ground color, by coalescence of the spots. Certain males, e.g. No. 26479, are quite black above. Males also have the legs proportionately longer as do the young. Of 18 males one has the appressed toes meeting, 16 have them separated by one costal fold, and two by two; while of 20 females the appressed toes are separated by one costal fold in two, by two in eight, and by three in ten; of fifteen young ten have the appressed toes meeting, in four they are separated by one costal fold, and in one by two.

A few females are almost as dark as the described male. The costal groove count is 14 in five out of 13 males, 13–14 in one and 13 in the other seven; it is 14 in two out of eleven females and 13 in the nine others.

The fifth toe is rudimentary on the left side in No. 26463, a male; another male, No. 26492 is a partial albino.

These are the only noticeable abnormalities in a series of 184.

Remarks: I have included two of the three references in the literature to *H. nebulosus* from Tsu-shima in the synonymy of this species. Neither are accompanied by descriptions. The third is accompanied by a description but apparently refers to *H. tagoi* from the North Island.

I think Abé's *H. tsuensis* refers to this form. He says Tago made a plate of it. Tago took it at Iwahara, on the southern island.

The *Hynobius* of South Tsu-shima differs strikingly in color from its close ally *H. tagoi* of North Tsu-shima, resembling rather *H. vandenburghi* of Hondo, from which it can easily be told by the very flat tail of the latter. The black spotting is much coarser than in the forms from North Tsu-shima, from Iki-shima, and from Kiusiu, and none of these three have the black tail of *tsuensis*.

Specimens seen: 184, as follows;

Tsu-shima, South Island: 183 (Cal. Acad. Sci. 26380–26547, 168 adults: 26548–26562, 15 larvae).

"Tsu-shima" (a specimen of *tsuensis*): 1 (M. C. Z. 5123).

Measurements of *Hynobius tsuensis*.

* = costal grooves. / = costal folds between appressed toes.

Specimen		Locality		Sex	Total length	Head	Body	Tail	*
Cal. Acad. Sci.	26447	Tsu-shima,	S. I.	male	129	17	56	56	13–14/1
M. C. Z.	5123	“	?	“	132	16.5	57.5	58	14/1
Cal. Acad. Sci.	26479	“	S. I.	“	126	16.5	59.5	50	/1
“	26495	“	“	“	126	17	58	51	14/1
“	26460	“	“	“	126	16	57	53	/1
“	26458	“	“	“	125	15.5	57.5	52	/1
“	26501	“	“	“	125	17	56	52	/1
“	26467	“	“	“	125	16	57	52	/1
“	26464	“	“	“	125	17	55	53	/1
“	26403	“	“	“	122	16.5	55.5	50	14/2
“	26486	“	“	“	119	16	55	48	/1
“	26439	“	“	“	119	15	54	50	14/1
“	26408	“	“	“	118	17	56	45	13/1
“	26399	“	“	“	117	16	56	45	13/1
“	26380	“	“	“	116	16	52	48	14/1
“	26420	“	“	“	113	16	50	47	13/0
“	26387	“	“	“	110	15.5	57.5	37	13/2
“	26450	“	“	“	108	16	50	42	/0
“	26419	“	“	“	99	14	46	39	13/1
“	26423	“	“	“	95	13.5	49.5	32	13/1
“	26424	“	“	“	96	14	46	36	13/1
“	26491	“	“	female	123	17.5	62.5	43	/3
“	26413	“	“	“	129	16.5	61	51.5	13/3
“	26414	“	“	“	126	17	61	48	14/3
“	26429	“	“	“	125	17.5	59.5	48	13/2
“	26432	“	“	“	123	17	61	45	13/3
“	26443	“	“	“	121	16	56	49	/2
“	26466	“	“	“	119	16	57	46	/3
“	26421	“	“	“	119	16	52	51	13/1
“	26302	“	“	“	118	16	62	40	13/3
“	26487	“	“	“	117	15.5	59	42.5	/3
“	26451	“	“	“	115	15	54	46	/3

Specimen		Locality	Sex	Total length	Head	Body	Tail	*
Cal. Acad. Sci.	26507	Tsu-shima, S.I.	female	116	17	58	41	/2
"	26513	" "	"	116	17	58	41	/2
"	26404	" "	"	116	16	54	46	13/2
"	26505	" "	"	115	16	55	44	/2
"	26465	" "	"	110	15	57	38	/2
"	26449	" "	"	112	15	53	44	/2
"	26384	" "	"	111	15.5	54.5	41	13/2
"	26405	" "	"	105	14	48	43	13/3
"	26396	" "	"	95	13	44	38	13/1
"	26500	" "	not sexed	100	13.5	45.5	41	/1
"	26454	" "	"	99	14	50	35	/2
"	26459	" "	"	91	14	46	31	/0
"	26498	" "	"	82	12.5	39.5	30	/0
"	26511	" "	"	78	12.5	38.5	27	/0
"	26515	" "	young	77	11	35	31	/0
"	26514	" "	"	74	12	33	29	/0
"	26472	" "	"	65	9.5	29.5	26	/1
"	26473	" "	"	60	10	27	23	/0
"	26534	" "	"	60	9.5	25.5	25	/1
"	26527	" "	"	58	8.5	25.5	24	/1
"	26543	" "	"	57	9.5	27.5	22	/0
"	26536	" "	"	55	9	24	22	/0
"	26519	" "	"	49	8	21	20	/0
"	26480	" "	"	48	8	20	20	/0
"	26562	" "	larva	49	8	20	21	
"	26561	" "	"	44	8	18	18	
"	26560	" "	"	42	8	17	17	
"	26550	" "	"	41	7	17	17	
"	26548	" "	"	41	8	16	17	

HYNOBIUS IKISHIMAE Dunn.

1923. *Hynobius ikishimae* Dunn, Proc. California Acad. Sci., (4), XII, 2, p. 28.

TYPE: California Acad. Sci. No. 26314, adult male, collected October, 1910, by Victor Kuhne.

TYPE LOCALITY: Iki-shima.

RANGE: Known only from type locality.

DIAGNOSIS: Vomerine series medium; tail somewhat flattened; 13 costal grooves; toes 5; sides of tail not black; middorsal line of tail yellow; grayish brown with black mottling; males darker than females.

DESCRIPTION: California Acad. Sci. No. 26314, adult male; collected October, 1910, by Victor Kuhne, Iki-shima; costal grooves 12; 2 costal folds between appressed toes; head width 6 in length from snout to vent; head length 3$\frac{4}{5}$ in length of body; head oval from above; eye as long as its distance from tip of snout; outline of upper jaw straight as seen from side; angle of jaw back of hind angle of eye; lower eyelid fitting under upper in front and behind; a groove from hind angle of eye to gular fold; a groove from this down behind angle of jaw; limbs well developed; fingers 3, 2, 4, 1 in order of length; toes 3, 4, 2, 5, 1 in order of length; a tubercle under base of first finger and of first toe; tail shorter than body, cylindrical at base, flattened at tip, trigonal just beyond level reached by appressed toes, no keel; hind part of tail widened somewhat; vent a slit forking forward so that there is a small triangular flap at apex; vomerine series beginning behind and in from inner border of naris; a short forward-and-incurving branch of 5 teeth, a long straight line of 20 teeth extending back and slightly in; series separated from its fellow by twice length of outer branch in front and by 1$\frac{1}{2}$ times outer branch behind; series reaches posterior border of eyesockets; length of inner branch equals distance from outer border of naris to a point halfway between peak of opposite series and inner edge of opposite naris; uniform dark grayish brown above; lighter below a line joining insertions of legs; tail dark grayish brown above, lighter below, rather bicolored; total length 118 mm., head 15, body 57, tail 46.

Variation: A female, Cal. Acad. Sci. No. 26310, same data, has 13 costal grooves; one costal fold between appressed toes; head width 6$\frac{1}{3}$ in length from snout to vent; head length 4$\frac{1}{7}$ in length of body; vomerine tooth series beginning at level of nares, meeting its fellow behind by an inner branch of two teeth on each side; dull grayish brown above with spots of black; gray below; middorsal and midventral lines of tail bright yellow; total length 101 mm., head 14, body 58, tail 29.

A young specimen, Cal. Acad. Sci. No. 26214, same data, has 13 costal grooves; appressed toes meeting; head width 4 in length from snout to vent; head length 2$\frac{5}{7}$ in length of body; grayish brown with an indication of an irregular dorso-lateral row of black spots and a lateral darker stripe; a bright streak on median line of tail above and below; these bordered by a black streak, and middle of sides of tail lighter; light gray below; total length 38 mm., head 7, body 19.5, tail 11.5.

The males are generally darker, with less prominent tail stripe. The tail is flattest in some males.

Of nineteen males one has 12, one 12–13, fifteen have 13, one 13–14, and one 14 costal grooves. In nine the appressed toes are separated by 2, and in ten by 3 costal folds.

Of twenty females all have 13 costal grooves. In six the appressed toes are separated by 3, and in fourteen by 4 costal folds.

In ten young specimens the number of costal grooves is 13. The appressed toes meet in two, and are separated by one costal fold in two, by two in three, and by three in three.

This indicates a greater relative leg length in the young and in the males.

The fifth toe is lacking in No. 26205, lacking on the right side in 26161, rudimentary in 26207 and 26212, and rudimentary on side in 26156, 26157, 26186 and 26201. This is from a series of 179 in the Cal. Acad. Sci. all with the same data as the type.

The length of the vomerine series is quite constant. The distance between the inner branches is rather variable, as is the meeting posteriorly of the two series.

Remarks: The species from Iki-shima is closest to the Kiusiu *nebulosus*, both anatomically and geographically. It differs from *nebulosus* in the greater fineness of its black spotting, and in the tail being trigonal in cross section rather than oval as in *nebulosus*. It is, however, very close to it, and as I have seen no specimens of *nebulosus* it may be that I am employing too fine discrimination. However, none of the Iki-shima specimens can be matched by Schlegel's (1838) figure of *nebulosus*, and the present form is certainly distinct from the Hondo and Tsu-shima species, so that I think myself justified in naming it.

Specimens seen: 179, as follows;
Iki-shima, 179 (Cal. Acad. Sci. 26149–26327).

MEASUREMENTS OF *Hynobius ikishimae*.

* = costal grooves. / = costal folds between appressed toes.

Specimen	Locality	Sex	Total length	Head	Body	Tail	*
Cal. Acad. Sci.							
26314	Ikishima	male	118	15	57	46	12/2
26160	"	"	113	15	62	36	13/3
26166	"	"	117.5	15	57.5	45	13/3
26298	"	"	110	15	55	40	13/3
26149	"	"	110	14	56	40	13/2
26316	"	"	109	14	57	38	13/3

Specimen	Locality	Sex	Total length	Head	Body	Tail	*
Cal. Acad. Sci.							
26190	Ikishima	male	105	14	59	32	13/2
26285	"	"	109	15	54	40	12–13/2
26151	"	"	108	15	55	28	13/2
26168	"	"	107	15	52	40	13/2
26281	"	"	107	14.5	51.5	41	13/3
26153	"	"	106	14.5	52	39	13/3
26320	"	"	105	14	51	40	13–14/2
26325	"	"	102	14	51	37	13/3
26319	"	"	98	14	54	30	
26318	"	"	99	13.5	51.5	34	13/3
26200	"	"	101	14.5	48.5	28	13/2
26256	"	"	100	13	49	38	14/3
26321	"	"	93	13	47	33	13/2
26158	"	female	107	14	58	35	13/3
26164	"	"	103	14	49	30	13/4
26310	"	"	101	14	58	29	13/4
26150	"	"	111	13.5	57.5	40	13/4
26257	"	"	111	13.5	57.5	40	13/4
26248	"	"	110	13.5	57.5	39	13/4
26165	"	"	111	14	56	41	13/4
26296	"	"	110	13.5	56.5	40	13/3
26283	"	"	109	15	52	42	13/4
26327	"	"	105	14	55	36	13/4
26292	"	"	105	13.5	54.5	37	13/4
26308	"	"	102	13	55	34	13/4
26312	"	"	100	13	53	34	13/4
26324	"	"	102	14	51	37	13/4
26161	"	"	102	13.5	51.5	37	13/3
26265	"	"	101	12.5	53.5	35	13/4
26155	"	"	97	13	51	33	13/3
26294	"	"	104	13.5	49.5	41	13/4
26305	"	"	102	13.5	49.5	39	13/3
26244	"	"	95	12.5	46.5	36	13/3
26187	"	young	84	11.5	39.5	33	13/3
26277	"	"	83	12	39	32	13/2
26188	"	"	76	12	39	25	13/2
26198	"	"	74	11	37	26	13/3
26171	"	"	68	10	35	23	13/3
26220	"	"	67	10	32	25	13/2
26202	"	"	47	8	33	6	13/1
26206	"	"	53	8	25	20	13/1
26238	"	"	42	7	19	16	13/0
26214	"	"	38	7	19.5	11.5	13/0

HYNOBIUS NEBULOSUS (Schlegel).

1838. *Salamandra nebulosa* Schlegel, Fauna Japonica, Rept., p. 127, pl. 4, ff. 7–9; 1844, Abbild. Amphib., p. 126, pl. 40, ff. 7–10; Wiedersheim 1876, Morph. Jahrb. 2, p. 428 (carpus and tarsus).

1838. *Hynobius nebulosus* Tschudi, Mem. Soc. Sci. Neuchâtel, pp. 60, 94; Gray 1850, Cat. Bat. Grad. Brit. Mus., p. 30; Boulenger 1882, Cat. Bat. Grad. Brit. Mus. (2), p. 32; Stejneger 1907, U. S. Nat. Mus. Bull., 58, p. 30, pl. 3, f. 4–6; Tago 1907, Tokyo Zool. Mag., 19, p. 201, 233; Kunitomo 1910, Anat. Hefte, 40, p. 193 (habits); 1911, Anat. Hefte 44, p. 457 (embryology); Abé 1921, Tokyo Zool. Mag., p. 330.

1854. *Ellipsoglossa nebulosa* Duméril and Bibron, Erpet. Gén., 9, pl. 100; Wiedersheim 1877, Morph. Jahrb. 3, p. 417 (skull); Edgeworth 1920, Journ. Anat., 54, p. 225, ff. 71–76, (hyobranchial apparatus); 1923, Journ. Anat., 57, p. 97, ff. 11–13 (hyobranchial apparatus).

TYPE: In Leiden Rijksmuseum, collected by von Siebold. (No type was specified and specimens from the large original series are in the British Museum, the Senckenberg Museum and the Copenhagen Museum).

TYPE LOCALITY: "Mits jama" near Nagasaki, Kiusiu.

RANGE: Kiusiu.

DIAGNOSIS: Vomerine series medium; tail flattened; 13 costal grooves; toes usually 5; sides of tail not black; middorsal line of tail yellow; yellow with black marbling; sexes alike in color.

DESCRIPTION: Since I have seen no specimens of this form I quote Schlegel's original description.

"Purely aquatic, this species has some analogy with the black salamander of North America. It is almost as big as the crested salamander, but it is stouter, the head is small and rounded, the feet are less developed and with short fingers, the tail is large, shorter than the body, thick at the base and compressed towards the tip. On the flanks there are very extensive and deep transverse folds, a similar fold surrounds the throat like a half collar. Of a more or less deep brownish yellow, all the upper parts are clouded with deep and very fine marblings; these extend sometimes on the under side. Some have the edges of the tail with a yellow streak, others have the colors very deep; in a word, one observes in this species, just as in most others, many accidental variations. The tongue is very large and long but only free at its edge. The eyes are smaller than usual and directed a little forward. The skeleton much resembles that of *S. naevia;*

the palatine teeth are exactly alike; but there are only 17 trunk vertebrae; and it has a shorter skull.

"I can note no external difference between the sexes; the young, which soon lose the gills, resemble the adults save in size."

Variation: Stejneger (1907) mentions a specimen in the Copenhagen Museum (No. 33) from the original series. This animal, a male, had the fifth toe rudimentary. I append measurements of two males of the original series taken from Stejneger (1907) and from Boulenger (1882).

Brit. Mus.	Total length	99	head	12	body	44	tail	43
Copenhagen Mus. 33	"	95	"	12	"	42	"	41

Abé (1921) records a female from Oita Prefecture. It had total length 128 mm., head 18, body 53, tail 57, costal grooves 13, appressed toes meeting.

HABITS: Schlegel (1838) says "purely aquatic," and that Siebold "saw larvae of this species in a spring at the base of Mt. Ho Kwa San near Nagasaki. They swim with much agility so that one might mistake them at first for little fishes."

Kunitomo (1910) gives an extended account of the breeding habits near Nagasaki, from which I quote the following.

"I knew already in the year 1904 that *Hynobius* was present in the Nagasaki-Yamasatomura Mts., and in the Spring of 1905 I found the pool where every year they lay their thousands of eggs. The mountain on which this pool is found is about 600 m. high and unwooded, although bushes and much grass grows there. The west side is a sheer cliff from the peak down about 150 m., and at the bottom of this cliff lies a pool, which is about 3 m. broad, 4 m. long and 30 cm. deep. So little water reaches the pool from the cliffside, except in continuous rainy weather, that it has no outlet. When it does not rain at all for many weeks it dries up completely."

"*Hynobius nebulosus* belongs to those Urodeles which lay their eggs very early in the Spring, before the other Japanese salamanders, for its spawning time is between March 10 and April 3, while the local *Triton* breeds during May and June, and *Cryptobranchus* during August and September. During the aforementioned period, if the weather progresses so favorably that the animals come out of hibernation, they crawl out here and there and betake themselves to a nearby pool, spring, or brook. According to whether the favorable weather comes earlier or later during the aforementioned period, the animals also appear earlier or later.

"Now March is often so cold in Japan, that frost covers the ground in the morning, and on rare occasions it snows.

"The weather at this season is very variable.... Often it turns suddenly warmer and great quantities of rain fall, thereupon the *Hynobius* emerge and seek a pool, where they can spawn, and if these weather conditions endure about three days, egglaying takes place, even though it turns cold again afterwards. If, however, the cold sets in again after a very short time, before the Liebesspiel has begun, (after about a day), the animals remain motionless at the bottom as if sleeping, and await a second warm rain. If the weather is very unfavorable for their purposes, e.g. if the pool dries up entirely, they crawl back on land and hide themselves in their former dwelling places. If the pool does not dry up, but there comes a second warm rain, then they swim vivaciously hither and thither, newly awaked, and with them others, which have just come in from the land. Then they carry on their Liebesspiel; thereafter they lay the eggs.

"This process is repeated three or four times each Spring during the spawning period, if the weather is sufficiently favorable. Naturally, each repetition is by new animals....

"Warm weather alone is not sufficient for the process, but there must be at the same time a heavy rain. If this is not the case, even if it is warm enough, no animals appear and no eggs are laid.

"As early dates for the egglaying I can give the following dates for the last four years:

1905:	March	15:	Water Temperature	8	degrees	Cent.
1906:	"	25:	" "	10	"	"
1907:	"	28:	" "	7	"	"
1908:	"	10:	" "	9	"	"

and as late dates:

1907:	April	6:	Water Temperature	18	degrees	Cent.
1908:	"	1:	" "	18	"	"

"In 1905 and 1906 they laid only once and in small numbers.... When the *Hynobius* males in warmer localities awake from hibernation, they crawl at once to the nearest place of water and there await the females. The latter come at the same time or at most a few days later, and the animals get into a state of sexual excitement, provided the favorable weather conditions hold. The Liebesspiel then takes place in the following manner.

"The males and females assemble in groups under a stone, rock or in the shadow of grasses, and swim forwards and back, up and down, hither and thither, and the males especially make a quivering motion with their tails. Finally after ten to twenty hours the females lay the folded spindle-shaped jelly masses. Then the males swim once more hither and thither over the strings of eggs (egg sacks), and it looks as if they squirted out semen over the jelly masses.

"In most cases only one end of this spindle-shaped string sticks fast to a stone, rock or branch dipping into the water, on the base of a tree, or on grass, while the other end is free in the water and can move up or down according to the water level. If the water almost dried up, I often saw the string hanging from a stone or a twig.

"Unfortunately, I have not yet absolutely determined whether the spermatophore of the male is taken up by the cloaca of the female or whether fertilization takes place in some other way, for the act took place at night. However, it is indisputable that the union of sperm and egg takes place in the water and outside the body of the female. . . . I believe that the sperm of *Hynobius* either remains only a short time in the female cloaca and enters the egg capsule at the moment of egg-laying, or that the sperm is emitted by the male directly over the eggs after they have been laid. The sperm must be emitted by the males shortly after the egglaying, for two hours after the eggs were laid I could observe the attraction cone on the surface of the egg, which is the first indication of the union of sperm and egg.

"A brooding habit by the male . . . is not seen in *Hynobius*. Both males and females, shortly after the completion of the spawning, crawl back on land.

"The egg sack is a spindle-shaped outer layer about 17 cm. long, in which the eggs, each protected by its own capsules, and numbering about 50, are kept. A few hours after the egglaying, it is slim, folded, and has an opalescent appearance. But there is no curling. Presently it begins to swell, for the egg capsule increases much in size by inhibition of water, so that after 3–5 days one can hardly see any folds. This strains the egg sack and the free end curls up. Then it has a thick, skein-like appearance. After a few more days fine particles of dust settle on the jelly mass, and it takes on a dirty light yellow color.

"If it does not rain for a long time and the water in the pond dries up, the sack lies on the bottom and the outer layer dries but the eggs in it do not die immediately, but live quite a while, (10–15 days) without taking harm, and continue developing, because there is plenty of water in the real egg capsule.

"In most cases the end of the egg sack is stuck to some object. . . .

If it is laid freely in the water, it drops to the bottom. I often noticed that a part of the eggs, in such a free laid sack, are spoiled."

The eggsack is 13–17 cm. long, 1.5–1.9 cm. thick in the middle, and contains 37–70 eggs. "The number of the eggs varies with the size of the sack. A pair of sacks is laid by each female and the number of eggs is not always the same in both."

"The single egg is covered by two coats, and the interval between them is filled by a jelly mass."

The egg is 3–3.2 mm. in diameter. The inner coat fits tight to the egg at first, but later swells to a diameter of 6 mm. The outer capsule is at first 4 mm. in diameter but swells eventually to an oval whose diameters are 10 x 13 mm.

Remarks: Very close to *H. ikishimae.* I have seen no specimens of the Kiusiu form, but from the figures and measurements it seems to differ from the species of Iki-shima in having a longer head, lighter ground color, and flatter tail. The figure of *nebulosus* in Schlegel (1838) shows an animal much resembling *tsuensis* of Tsu-shima South, but differing in having a flatter tail with mottled sides.

This seems the proper place to mention a specimen taken by Tago (1907) at Ishizuchi yama, Umegahira, province Iyo, Shikoku. He says "this animal may belong to some different species as I noticed some differences in color, spots, and also a condition of the eggs while in its body." He did not, however, describe it, but left it under *H. nebulosus,* a course which I am compelled to imitate, although I am convinced that the Shikoku animal will be found to differ both from the Hondo and from the Kiusiu forms.

Hynobius vandenburghi Dunn.

1903. *Hynobius nebulosus* Yamada, Tokyo Zool. Mag., 15, p. 73 (habits); Tago 1904, Tokyo Zool. Mag., 16, p. 73; 1907, Tokyo Zool. Mag., 19, pp. 201, 233; Okajima 1911, Anat. Hefte 45, p. 1 (otic apparatus); 1911, Anat. Anz. 38, p. 27 (otic apparatus); Okajima and Tsusaki 1921, Zeitschr. f. Anat. u. Entw. 60 (Scleral cartilages); Tsusaki 1922, Fol. Anat. Jap. 1, p. 1 (Hyobranchial apparatus).

1921. *Hynobius naevius* Abé (part?), Tokyo Zool. Mag., 34, p. 329.

1923. *Hynobius vandenburghi* Dunn, Proc. California Acad. Sci., (4), XII, 2, p. 28.

Type: California Acad. Sci. No. 26714, adult male, collected May 1907.

Type Locality: Nara, province Yamato, Hondo.

RANGE: Southern Hondo.

DIAGNOSIS: Vomerine series medium; tail very flat; 14–13 costal grooves; toes 5; sides of tail black; middorsal line of tail yellow; yellow with black mottling; males and females similarly marked.

DESCRIPTION: Adult male, Cal. Acad. Sci. No. 26714, Nara, Province Yamato, Hondo, collected by V. Kuhne, May, 1907; 14 costal grooves; 1 costal fold between appressed toes; head width $6\frac{1}{2}$ in length from snout to vent; head length $3\frac{2}{3}$ in length of body; head oval; eye as long as its distance from tip of snout; outline of upper jaw straight as seen from side; angle of jaw back of hind angle of eye; lower eyelid fitting under upper in front and behind; a groove from eye to gular fold; a groove from this down behind angle of jaw; limbs well developed; fingers 2, 3, 4, 1 in order of length; toes 3, 2, 4, 1, 5 in order of length; a tubercle under base of both inner and outer fingers and toes; tail shorter than body, trigonal at base, much flattened in last two-thirds, not keeled or finned; vent formed of three slits meeting, the posterior one quite short, the whole figure wider than long; vomerine teeth beginning behind inner edge of nares; a forward, incurving branch of 4 teeth; a branch of 17 teeth running straight back and in; series separated from its fellow anteriorly by twice width of nares, meeting its fellow behind; length of series equalling distance between outer edge of nares and peak of opposite series; color light with heavy dark mottling; upper edge of tail light; sides of tail black; belly light; total length, 99 mm., head 12.5, body 46, tail 40.5.

Variation: an adult female, Cal. Acad. Sci. No. 25949, Central Hondo, has 13 costal grooves; head width $4\frac{4}{5}$ in length from snout to vent, head length 4 in length of body; vent a longitudinal groove, the front half of which opens to the inside; vomerine teeth as described but 5 teeth in the outer and 25 in the inner branch; total length 110 mm., head 13, body 53, tail 44.

The tail may be longer than the body; two males (Cal. Acad. Sci. No. 25948, 25950, Central Hondo) have the vomerine series as long as the distance between the outer edge of the nares and halfway between the inner edge of the opposite nares and the peak of the opposite series; No. 25950 shows the extreme breeding development of the male vent; the longitudinal groove has disappeared and the vent is formed by two grooves meeting in an angle, apex forward, a tubercle indenting the apex of the angle; the coloration of this specimen is light yellowish brown, almost obscured by heavy black mottling, light gray below, sides of tail black, upper and lower edges of tail light. In three of the five specimens I have seen two costal folds separate the appressed toes, the exceptions being the type and No. 25949, which last is the only one to have 13 costal grooves, the rest having 14. Measurements of three males:

"Japan"

Cal. Acad. Sci. No.	35391	total	114 mm.	head	14	body	49	tail 51
" "	25948	"	120	"	14	"	52	" 54
" "	25950	"	118	"	14	"	51	" 53

HABITS: Yamada (1903) made some observations on this species which he says is called "Hatake dojo" by the Japanese, a name meaning "Field Loach" owing to a real or fancied resemblance to the "Dojo," or Loach, a kind of small fish. He says that the toes have "a certain clasping power at the extreme end so that they can climb up an almost perpendicular surface." They are often found "under a pile of old bricks and stones when there is sufficient dampness." Their food is "small insects or worms, especially the latter." "Disposition like a crab, as they like to be under cover of some object." "When they see a worm their eyes stand out and slowly getting near to it they suddenly seize it just as a frog does. The breeding season is May, when they go into a ditch where they find sufficient water. Their eggs are white and small, and are wrapped in chain-like groups by a gelatinous substance, the strings being about the size of a pencil. When the young are hatched they are about $\frac{1}{4}$ inch long. It is doubtful whether they have balancers or not." "Very often while young they change their skins, which habit becomes less frequent as they grow old." "As people use this animal for medicinal purposes it is quite hard to get them."

REMARKS: This animal differs from its near allies sufficiently to merit recognition. From *tsuensis* of Tsu-shima South, it can be told by the much flatter tail, and by the fact that the males are not black; from *nebulosus* of Kiusiu by the flatter tail, which has black sides; from the Iki-shima and Tsu-shima North forms, by the flatter tail and by the coarser marbling. It is not close enough to any other forms to cause confusion. These five, however, have the vomerine series practically identical. The Hondo form has an extremely flat tail, while the other four differ mainly in color, and although they can easily be told apart it is difficult to formulate descriptions as there is much variation. The five form a closely allied group of which the Hondo form is the most specialized.

I refer Abé's "*H. naevius*" from Bingo to this form. It had short vomerines, lacked the tail, the appressed toes did not meet, and there were 13 costal grooves. It was a female with head length 17 mm., and body length 36 mm.

Pachypalaminus boulengeri is probably closest to the present animal.

The range according to Tago (1907) is mainly south of Tokyo. Definite localities are few. Besides Nara, in Yamato Province, Tago (1907) records it from Unebi in the same Province, Okajima (1911) from Asakawamura, Musashi Province, and Abé (1921) from Bingo Province.

Specimens seen: 5, as follows;

Hondo: Province Yanato; Nara 1 (Cal. Acad. Sci. 26714).
Central Hondo 3 (Cal. Acad. Sci. 25948–50).
Japan: 1 (Cal. Acad. Sci. 35391).

Genus PACHYPALAMINUS Thompson.

1912. *Pachypalaminus* Thompson, Proc. California Acad. Sci. (4), 3, p. 183, (type *Pachypalaminus boulengeri* Thompson).

RANGE: Same as that of type species.

DIAGNOSIS: *Hynobiidae* with horny covering to palms and soles and over tips of digits; premaxillary fontanelle large; lungs ?; gills of larvae ?; vomerine teeth in V-shaped series; tail very flat; one species.

PACHYPALAMINUS BOULENGERI Thompson.

1912. *Pachypalaminus boulengeri* Thompson, Proc. California Acad. Sci. (4), 3, p. 184, pl. 14.

TYPE: Cal. Acad. Sci. no. 33192, adult male.

TYPE LOCALITY: Odaigahara Mt., Yamato, Hondo, Japan.

RANGE: Known only from type locality.

DIAGNOSIS: A salamander with the dentition of *Hynobius;* a very flat tail; "palms, soles, and inferior surface and tips of fingers and toes covered with a tough brown corneous modification of the epidermis," (as in *Batrachuperus*); no markings.

DESCRIPTION: "Head large, depressed, and as broad as long; snout long and rounded; nostril situated a trifle nearer to the orbit than to the tip of the snout; eyes rather large and prominent; orbit contained one and one third times in the length of the snout. Series of palatine teeth interrupted, not forming a reentrant angle; apices of the two salient angles on a line with the centers of the choanae; the length of the inner side of one of the angles equal to the interval between the choanae; the length of the outer side equal to one third this interval. Tongue circular, strong and fleshy, filling the floor of the mouth, the surface finely and longitudinally plicate; two fairly deep sulci with a general antero-posterior trend, their outline that of two laterally directed obtuse angles, enclosing about one half the central area of the tongue. The gular fold moderately developed. Body depressed; distance from the snout to the gular fold contained nearly three times in the distance from the latter to the cloaca; median dorsal groove, markedly deepened over the pectoral and pelvic regions; thirteen well developed costal folds, including the one flexed to enter the axilla and the one reaching the groin; the nine middle folds continued across the abdomen. Vent (of male) three slits meeting in front, the medium longitudinal and longest, the two others obliquely directed forwards, forming an angle; the borders swollen. Limbs stout, when appressed the digits overlap for about two millimeters. Digits well developed. Tail a trifle longer than the distance from the gular fold to the cloaca, strongly compressed, deepened and fleshy in the posterior half; not keeled; the tip rounded. Skin smooth; numerous mucous glands on snout, around nostrils and eyes, and on upper and lower lips; paratoids distinct; an irregular horizontal groove from eye to gular fold, joined by a short vertical one posterior to angle of mouth. Color in spirits slate, a trifle paler beneath." Total length 161 mm., head 23 mm., body 69 mm., tail 69 mm.

REMARKS: Thompson says "the larvae possess stout claws." This is not quite clear, because the type is the only specimen known. Larvae of *Batrachuperus* possess the peculiar foot coverings of the adults, and as the same modification exists in *Pachypalaminus,* one might expect its larvae likewise to resemble the adults in this respect.

The relationships are clearly with *Hynobius.* Among the species of that genus it most resembles *H. vandenburghi* of the same region in Hondo. It may well be a very local aquatic modification of that form.

It certainly has nothing to do with *Batrachuperus,* in spite of the similar modifications of the feet, which may be considered as parallelism which have appeared independently in animals of common ancestry.

Dr. Van Denburgh has kindly examined the skull of *Pachypalaminus* for me and sent me a sketch of it. I find that it differs from that of *Hynobius* in having a large premaxillary fontanelle.

Genus ONYCHODACTYLUS Tschudi.

1838. *Onychodactylus* Tschudi, Mem. Soc. Sci. Neuchâtel, pp. 57, 92 (type *Onychodactylus schlegeli* Tschudi).
1839. *Dactylonyx* Bibron, in Bonaparte, Icon. Fauna Ital., 2, fasc. 26, (substitute for *Onychodactylus* Tschudi).
1854. *Onychopus* Dumeril and Bibron, Erpet. Gen., 9, p. 113 (substitute for *Onychodactylus* Tschudi).
1886. *Geomolge* Boulenger, Proc. Zool. Soc. London, p. 416 (type *Geomolge fischeri* Boulenger).

RANGE: From Khabarovka, Maritime Gov., to Wonsan, Korea. Hondo and Shikoku.

DIAGNOSIS: *Hynobiidae* with claws in larvae and in some adults; no lungs; premaxillary fontanelle large; gills of larvae without filament bearers; vomerine teeth in a nearly transverse and continuous series; tail long and nearly cylindrical; two species.

ONYCHODACTYLUS JAPONICUS (Houttuyn).

1782. *Salamandra japonica* Houttuyn, Verh. Genootsch. Wetensh. Vlissingen, 9, p. 329, pl. facing p. 336, f. 3.
1787. *Lacerta japonica* Thunberg, Svensk. Vet. Akad. Nya Handl., 8, pl. 124, pl. 4, f. 1 ("Fakoneberget").
1839. *Onychodactylus japonicus* Bonaparte, Icon. Fauna Ital., 2, fsc. 26; Gray 1850, Cat. Batr. Grad. Brit. Mus., p. 33, pl. 3, f. 15; Troschel 1877, Arch. Naturg., 1, p. 199, pl. 15 (elaborate descriptions of adult and larva); Hilgendorf 1880, Sitz. Ber. Naturf. Berlin, p. 121; Boulenger, 1882, Cat. Batr. Grad. Brit. Mus., (2), p. 35; Cope 1889, Bull. U. S. Nat. Mus. 34, pl. 46, f. 6 (carpus and tarsus); Okada 1891, Cat. Vert. Japan, p. 65; Tago 1903, Tokyo Zool. Mag., 15, p. 464; 1904, Tokyo Zool. Mag. 16, p. 43; Stejneger, 1907, Bull. U. S. Nat. Mus. 53, p. 42, pl. 5, f. 1–4, pl. 6; Tago, 1907, Tokyo Zool. Mag., 19, p. 238 (habits); Okajima, 1908, Zeitschr. Wiss. Zool. 91, p. 351 (osteology); 1909, Zeitschr. Wiss. Zool. 94, p. 171 (sense organs); 1909, Anat. Anz., 34, p. 182 (hyobranchial apparatus); Dunn, 1918, Bull. Mus. Comp. Zool., 52, p. 454; Okajima and Tsusaki 1921, Zeit. Anat. Entw. 60 (scleral cartilages); Okajima 1922, Fol. Anat. Japon., 1, p. 196 (hyobranchial apparatus).
1798. *Lacerta thunbergii* Donndorf, Zool. Beytr., 3, p. 132 (based on Thunberg 1787).

1820. *Molge striata* Merrem, Tent. Syst. Amph., p. 185 (based on Houttuyn and Thunberg).

1838. *Salamandra unguiculata* Schlegel, Fauna Japon., Rept., p. 123, 129, pl. 5, ff. 1–6 (Hondo and Shikoku); Geerts, 1883, Nouv. Arch. Mus. Paris, (2), 5, p. 275; Goppert, 1896, Morph. Jahrb. 25, p. 16, ff. 15–16 (claws).

1838. *Onychodactylus schlegeli* Tschudi, Mem. Soc. Sci. Neuchâtel, p. 92 based on Schlegel 1838); Duméril and Bibron 1854, Erpet. Gén., 9, p. 114, Atlas, pl. 93, f. 1.

1841. *Onycopus sieboldii* Duméril and Bibron, Erpet. Gén., 8, p. 4 (lapsus).

1879. *Onychodactylus* sp. Parker, Trans. Linn. Soc. London, (2), 2, part 3, No. 5, p. 188, pl. 19, ff. 1–3 (skull and hyobranchial apparatus).

Type: Not known to exist. Collected by Thunberg.

Type Locality: Not definitely stated but evidently the "Fakoneberget" (mountains of Hakone, Izu Prov.), where Thunberg collected the types of his *Lacerta japonica*.

Range: Mountains of Hondo and of Shikoku.

Diagnosis: An *Onychodactylus* with 13–14 costal grooves, generally no gap between vomerine series, a very marked light dorsal band.

Description: Cal. Acad. Sci. No. 26711, adult male, Sawatari, Prov. Kotsuke, Hondo, Sept., 1910; 14 costal grooves; appressed toes meeting; head width 6 in length from snout to vent; head length $3\frac{2}{3}$ in length of body; head oval; eye as long as its distance from tip of snout; outline of upper jaw straight as seen from side; angle of jaw back of hind angle of eye; both eyelids fitting under a fold of skin behind; a groove from eye to gular fold, a groove from this down behind angle of jaw; limbs well developed; fingers 3, 2, 4, 1 in order of length; toes 4, 3, 5, 2, 1 in order of length; a tubercle under base of first finger and toe; a fold of skin along outer edge of fifth toe and lower leg; tail longer than head and body, cylindrical at base, flattened towards tip; vent a longitudinal slit with a pair of shorter slits meeting it so as to form an arrowhead appearance, point forwards; Vomerine teeth, 15 in series, beginning behind inner edge of nares, curving forward and in to level of anterior edge of nares, thence back and in to meet its fellow, not so far back in middle as on sides; yellowish brown, a dorsolateral black stripe on body and tail, much black spotting on dorsal surface; sides below the stripe cream color clouded with darker, nearly uniform cream color below; total length 151 mm., head 15, body 56, tail 80.

Variation: An adult female, Cal. Acad. Sci. No. 16021, Yokohama, Hondo, collected by J. C. Thompson; 13 costal grooves; stout claws on fingers and toes; tail longer than body; vent a longitudinal slit;

black above, lighter below a light dorsal streak from head to tip of tail, total length 136 mm., head 15, body 58, tail 63.

The larvae have dorsal fin only on tail; there is a fold of skin along the outer edge of arms and legs. There is great variation in color, some larvae having a marked dorsal light stripe, while others do not. All have black claws.

Schlegel (1838) says that the larvae and the breeding males and females have claws. Troschel, in 1877, claimed that the claws are confined to larvae and males, as his series of females had no claws. Stejneger (1907) shows that breeding females are provided with claws, and suggests that males have claws the year round and females have them only in the breeding season. Three clawless males, Cal. Acad. Sci. No. 26709–11, from Sawatari, Kotsuke Prov., Hondo, taken in September seem to indicate that Schlegel was right and that claws are seasonal and possessed at times by both sexes.

The vomerine teeth do not always meet in the median line, being sometimes separated by the width of the nares.

The fibulo-tarsal dilation of the male is supposed to be present only in the breeding season, but was very noticeable in the clawless, September males mentioned above.

The anus of the breeding female is very similar to that of the male, but never so developed, and the posterior pair of lateral slits join the longitudinal one in the middle of its length, instead of near its front end.

Tago's smallest larvae were 25–30 mm. long. His smallest transformed specimens were 60–65 mm. long. He says they grow to 180 mm. Troschel (1877) records larvae 87 mm. long. The males are in general the larger.

HABITS: Schlegel (1838) says; "The Japanese call this salamander *San-sjo-uwo*, but in the mountains where it lives, they call it by the Chinese name of *Kao chou youen he yu*, which means black fish of the mountain springs.

"It is found particularly in mountainous country.

"Those which come from Mount Facone are very celebrated, and are sold in Pharmacies as *Fakone-no-san-sjo-uwo*. They attribute to them medicinal qualities analogous to those of the giant salamander, and their use is particularly recommended as a vermifuge for children.

"This species frequents the sources of brooks and damp places in the mountains. It usually stays in the cracks of cliffs near these places and only comes out in the evening to seek its food, which is small mollusks, terrestrial and aquatic insects and worms. So, during the night, the Japanese, with torches, go to hunt these animals, which they catch in abundance on damp, moss-covered rocks.

"It is said that they stay by preference near waterfalls, and that they like to climb on the damp walls of the cliffs whence they precipitate themselves into the water. They catch them then with lines and put them alive into terra-cotta pots. In preparing them for medical use they pierce the head without gutting them and thread them on a thin bamboo so as to form packets of 10–20. When dried they are brown and oily to the touch.

"The larvae appear at the end of April. Mr. Siebold has seen them at this time during his stay in the Fakone Mts."

Tago (1907) describes the habits as follows: "This animal always likes a cool, shady place where they do not have the direct light of the sun, and with considerable dampness and clear water."

He describes the junction of a small river with the Sukumo River at Kiwadasawa, Sagami Province. "At this junction in the woods hundreds of these animals gather every year.

"Nearly a mile up this stream beyond a rapid there is a large rock on the left covered with wild wistaria and other vines and called *Kuro yuwa* or *Tate yuwa* (Black Rock or Standing Rock) and many years ago, so I understand, the water was so high that it covered half of this rock and made a fall to the valley behind it.

"Every spring when the azaleas begin to bloom hundreds of these animals, male as well as female, gather on this rock and the inhabitants catch them by the light of torches.

"In Nikko they call this animal *Hibihari* and young ones are found in the ponds of Nikko shrine, Umagayeshi, around Chugushi Lake, Yuhake, and other waters in the mountains, but adults seem to be restricted to two or three places, and it is difficult to catch them unless you know a definite place where they live. The most convenient places to find them are in woods near lakes, where it is shady and damp and usually with northern exposure. Every year I used to catch them at Suganuma, Katashinamura Tome, Gumma Prefecture.

"The lake is about ten miles in circumference, running north and south, irregular in width, its three wide spaces being called Irinuma, Nahanuma, and Kitamata. Covered with thick spruce of natural growth, and with the mosses and detritus of several centuries it is naturally kept shady and damp and suited to the happiness and comfort of this animal, and we can find them always under these shady places, but in the breeding season they come out to the beach at a point where it is open to the north.

"In the lake we find both this animal and *Hynobius fuscus* which generally keep apart, the former in shady damp places, with northern exposure, while the latter keep themselves more or less in the sun with eastern or western exposure, and both under cover of rocks and decayed roots, although the former does not burrow as much as the latter, and seems to go into the water at night as they are found in some of the lake fishes — in fact, some had more than two in their stomachs."

The young "generally live in clear water less than six inches deep, in a small stream or around the lake where there are plenty of bushes on the bank and lots of pebbles on the bottom of the water, or sometimes in cold water less than 10 degrees Cent. in hot summer which is found on a north mountain slope.

"The best way to catch them in these circumstances is to dry out the water first and to move pebbles and rocks at the bottom although very often we use a net.

"As soon as it loses gills and starts to live on land they go into the bushes or woods and hunt for a place most suited for their comfort and happiness.

"Even in those places described above they live in most cases singly among mosses, decayed trees, under rocks and even in a decayed hole in a tree near the water, where they hide themselves most of the day and go out at night after food, which mostly consists of live fish, tadpoles, worms, insects, and their eggs as well as young ones.

"Although they dislike a great deal of sun and dry places they wander far from their lairs at night or during a rain, and often go into the water, or go up to a mountain summit by a stream, and occasionally are caught in a path.

"Their winter is spent in the ground, under the root of a tree, or rocks or in the hollow of a dead tree, and sometimes they come out in groups.

"In the spring early in May when the wild azaleas starts to bloom at Hakone they come out of their long winter retirement and start for the distant breeding place along water-ways. On a warm, rainy night with south wind they come out to the lover's lane to find a mate and from every corner of stream, spring, or pond they gather to a comparatively large river or lake such as that at Kiwadasawa above the Sukumo river at Kuroyuwa (Black Rock) or Suganuma described above and also in Iwaki, my native place, at a junction of two rivers.

"There is no way to describe accurately their breeding as this is done in the darkness of night.

"A female taken at this season had 30–40 eggs; whether these eggs are thrown out as they are, or as larvae after being kept some time in the oviduct, I regret very much that I have not had an opportunity to decide.

"It is very doubtful as to their medical worth, although they have been sold for years under this impression as *Magotaromushi* and at Hakone, Amaki, and Kuroyama, district of Nikko, they catch them by means of weighted baskets in the breeding season when they migrate along the valley, and preserve them by broiling after killing them in salt water."

REMARKS: This animal is extremely close to the Korean species. Far from being generically distinct it is doubtful if any constant character separates the two species. The Korean form generally has a gap between the vomerine series, one more costal groove, and the light dorsal band is not very distinct. This color character is the best means of distinguishing between the two. So far, none of the mainland adults have been found with claws. But few mainland adults have been found. The larvae of the two forms cannot be separated, save those Japanese specimens which have less than 14 costal grooves and those mainland ones which have more.

There is considerable similarity between Schlegel's and Tago's account of the habits and one wonders whether Schlegel also did not have reference to Kuroyuwa near Hakone.

Schlegel's account might be bodily transferred to some of the American mountain brook species.

It is noteworthy that this animal and *fischeri* apparently have developed the legs rather than the tail for swimming. For the disproportionate development of the outer toes is always found in aquatic animals which use the legs for propulsion. This development is unique among salamanders, so that for comparison one must refer to Plesiosaurs.

Besides the localities mentioned in the list of specimens it has been recorded from the following places:

Hondo: Mottsu Province, Kitagami Mts. (Tago 1907).
Rikuoku " " " " "
Rikuzen " " " " "
Iwaki " (Okada 1891); Abukuma Mts. to Takahara (Tago 1907).
Hitachi Province, northern boundary of Tsukuba Mts. (Tago 1907).
Sagami Province, Hatta, Oyama, (Hilgendorf 1880), Kiwadasawa, Hatashiku (Tago 1907).

Idzu Province, Amaki Mts. (Tago 1907).
Shimozuke " Nikko (Tago 1907).
Kotsuke " Suganuma (Tago 1907).
Shinano " (Schlegel 1838).
Iga " (Geerts 1883).
Ise " (Tago 1907).
Tamba " (Schlegel 1838).
Tozima " (Schlegel 1838).
Border between Mimasake and Hoki (?), (Tago 1907).
Shikoku: Tosa Province (Schlegel 1838), Yamaba (Tago 1907).

Tago (1907) says of its altitudinal range "Thus in warm regions they live high up in mountains or valleys of 500–1000 ft., while in northern states we find them very often on a hill or in a stream only 300–500 ft. above the sea."

Specimens seen: 98, as follows;
Mimaya, Rikuoku Province, 1 larva (Cal. Acad. Sci. 26715).
Sawatari, Kotsuke Province, 5 (2 larvae), (Cal. Acad. Sci. 26709–13).
Tokyo (?), 1 larva (M. C. Z. 1864) (larva).
Yokohama (?), 1 larva (Cal. Acad. Sci. 16021).
Hakone Lake, Sagami Province, 87 (45 larvae): (M. C. Z. 2594, 9 larvae), (M. C. Z. 2595, 6 larvae), (M. C. Z. 2586, 4 adults), (U. S. N. M. 34218–46, 34249–57, 34258–87 larvae).

The localities Tokyo and Yokohama should be taken with a grain of salt, more especially as the animals are articles of commerce.

MEASUREMENTS OF *Onychodactylus japonicus.*

* = costal grooves. / = costal folds between appressed toes.

Specimen	Sex	Total length	Head	Body	Tail	*	
U. S. N. M. 34220	female	159	15	62	82	14/0	clawed
" 34221	male	168	16	63	89	14/0	"
" 34227	female	128	15	58	55	13/0	"
" 34256	male	142	14	55	73	13/0	"
(These are the largest and smallest adults seen.)							
Cal. Acad. Sci. 16021	female	136	15	58	63	13/0	clawed
" 26709	male	140	15	55	70	13/0	no claws
" 26710	"	144	15	55	74	13/0	"
" 26711	"	151	15	55	80	14/0	"
" 26712	larva	78	10	33	35		clawed
" 26713	"	61	8	26	27		"
" 26715	"	69	8	31	30		"
" 16017	"	70	9.5	30.5	30		"
" 16018	"	73	9.5	28.5	34		"

NYCHODACTYLUS FISCHERI (Boulenger).

1886. *Geomolge fischeri* Boulenger, Proc. Zool. Soc. London, p. 416, pl. 39, f. 2; Nikolski 1896, Ann. Mus. Zool. St. Pétersbourg, 1, p. 77; 1905, Zap. Imp. Akad. Nauk, S. Peterburg (8), 17, p. 440; 1918, Faune de la Russe, p. 242, f. 45; Abé 1921, Tokyo Zool. Mag., 34, p. 328.

1918. *Onychodactylus fischeri* Dunn, Bull. Mus. Comp. Zool., 42, p. 454.

1913. *Onychodactylus rossicus* Nikolski, Ann. Mus. Zool. St. Pétersbourg, 18 (2), p. 260 (type locality, Ussuri); 1918, Faune de la Russe, p. 259, f. 57–58, pl. 4, f. 5.

TYPE: Brit. Mus. No. 86.5.15.11–12, collected by Doerries.

TYPE LOCALITY: Khabarovka.

RANGE: From Khabarovka to Wonsan in Korea.

DIAGNOSIS: An *Onychodactylus* with 14–15 costal grooves, generally a slight gap between vomerine series, light dorsal band rather obscure.

DESCRIPTION: Cal. Acad. Sci. No. 32170, adult male, Wonsan, Korea, collected by Victor Kuhne, June 7, 1911; 14 costal grooves; appressed toes meeting; head width 7 in length from snout to vent; head length 4 in length of body; head oval from above; eye longer than distance from tip of snout; outline of upper jaw concave as seen from side; angle of jaw back of hind angle of eye; upper and lower eyelid fitting under a fold of skin behind; a groove from hind angle of eye to gular fold; a groove from this down behind angle of jaw; limbs well developed; fingers 2 and 3 equal in length, 1 and 4 equal and shorter than 2 and 3; toes 4, 3, 5, 2, 1 in order of length, a fold of skin along hind edge of fibula and fifth toe; tail longer than body; cylindrical throughout most of its length, flattened at tip; vent a longitudinal slit joined at tip by two shorter oblique slits directed forward, and joined one fourth its length back from the tip by two other similar slits, opening of vent between tip and posterior short slits; vomerine teeth in two series of about 19 teeth each; series beginning behind middle of nares, curving forward and in to level of anterior edge of nares, thence back and in, but not so far back in middle as on sides, separated from its fellow by one-half width of naris; cream color above and below, very heavily mottled above with black, on the tail this mottling takes the form of two dorsolateral stripes; total length 160.5 mm., head 17.5, body 68, tail 75.

Variation: No adult females have been seen. Three other males, same data, all have 14 costal grooves. Boulenger (1886) says "14 or 15." In these three the third finger is longer than the second as in *O.*

japonicus. In the smallest, No. 32171, the vent is a simple longitudinal slit. In No. 32172 the tail is longer than the head and body. The vomerine teeth may be as few as 14 in a series, this being the number in No. 32172.

The larvae are similar to those of *O. japonicus*, even to the extent of having two color phases, with and without a marked dorsal light stripe. All have strong black claws and black horny beaks on the lower jaw. I give the measurements of the four adults and of the largest and smallest larvae.

32169	adult ♂	140.5	15.5	58	67	
32170	“ ♂	160.5	17.5	68	75	
32171	“ ♂	116	13.5	47.5	55	
32172	“ ♂	168	17.5	60.5	90	
32219	larva	97	9	45	43	marbled heavily
32262	“	97	11	39	47	
32199	“	95	11	40	44	an immaculate dorsal area
32993	“	27	4	13	10	
32992	“	33	4	14	15	

Abé (1921) records a female from Hokkanzan, Korea, having claws on fingers, total length 148 mm., head 14, body 51, tail 33.

The female type in the British Museum has head 14 mm. long and body 66.

The male type has total length 163, head 12, body 58, tail 93.

Apparently these Ussuri specimens have shorter heads than the ones I have seen from Korea.

REMARKS: As has been said under *O. japonicus* the two species of this genus are extremely closely related. They are closer in fact than are the *Hynobius* of the Islands in the Korean straits (*tsuensis*, *tagoi*, and *ikishimae*).

Nikolski's *Onychodactylus rossicus* was based on a claw-bearing larva 75 mm. long.

Besides the following specimens from Wonsan, Boulenger has recorded two from Khabarovka (1886), Nikolski (1896) one from the valley of the Sutchan, and (1913) one from the Ussuri, and Abé (1921) has recorded it from Hokkanzan, Korea.

Specimens seen: 228, as follows;

Korea: Wonsan, 4 adults (Cal. Acad. Sci. 32169–32172).
224 larvae (Cal. Acad. Sci. 32173, 32175–202, 32204–32209, 32211–32373, 32984–32993, 35977–35988, Amer. Mus. Nat. Hist. 14503–14506).

Genus Ranodon Kessler.

1866. *Ranodon* Kessler, Bull. Nat. Moscou, 39, p. 126 (type *Ranodon sibiricus* Kessler).
1882. *Ranidens* Boulenger, Cat. Bat. Grad. Brit. Mus. (2), p. 36 (substitute for *Ranodon* Kessler).

Range: Same as that of type species.

Diagnosis: *Hynobiidae* with no foot modification; lungs present; premaxillary fontanelle small; gills of larvae ?; vomerine teeth in two short, widely separated series; tail flat; one species.

Ranodon sibiricus Kessler.

1866. *Ranodon sibiricus* Kessler, Bull. Nat. Moscou, 39, p. 126, pl. 7, f. 1–10; Strauch 1870, Mem. Acad. Sci. St. Pétersbourg (7), 16, p. 66, pl. 11, f. 3; Severtzof 1873, Izv. Imp. Obst. Lyub. Est., 8, 2, p. 72; Wiedersheim 1876, Morph. Jahrb. 2, p. 424, pl. 29, f. 2–3 (extremities); 1877, Morph. Jahrb. 3, p. 422, pl. 23, f. 68–70 (skull); Kehrer 1886, Ber. Nat. Ges. Freiburg, I, p. 4, pl. 4, f. 3, 4, 15 (extremities); Nikolski, 1887, Tr. Spb. Obstch. Est. 19, p. 162; Schmalhausen 1917, Rev. Zool. Russe 2, p. 129 (extremities); Stejneger 1917, Proc. Biol. Soc. Washington, 30, p. 123; Nikolski, 1918, Faune de la Russe, p. 247, ff. 48, 49–55.
1882. *Ranidens sibiricus* Boulenger, Cat. Batr. Grad. Brit. Mus. (2), p. 36; Kulagin 1888, Izv. M. Obstch. Lyub. Est. 56, 2, p. 34; Boettger 1888, Ber. Offenbach Ver., p. 169; Nikolski 1899, in Fedtschenko "Reise in Turkestan," Zool. 2, pt. 7, p. 78; Kastschenko 1902, Izv. Tomsk Univ., p. 20; Nikolski 1905, Zap. Imp. Akad. Nauk. S. Peterburg (8), 17, p. 441; Elpateevski and Sabaneef 1907, Zool., Jahrb. Syst. Th. 24, p. 263; Snitnikov 1913, Ann. Mus. Zool. St. Pétersbourg 18, p. 53 (habits).
1868. *Ranodon Kessleri* Ballion, Bull. Nat. Moscou 41, p. 138 (Kopal); Severtzof, 1873, loc. cit., p. 72.
1873. *Ranodon* sp. Severtzof, loc. cit., p. 72.
1918. *Ranodon kozhevnikovi* Nikolski, Faune de la Russe, p. 251, ff. 49–55, pl. 3, f. 5 (Tashkent).

Type: None known to exist.

Type Locality: "Neighborhood of Semipalatinsk," possibly erroneous.

Range: Known certainly from Eastern Semiryechensk and Western Chinese Turkestan, and it possibly occurs in Semipalatinsk and in Syr-darya.

Diagnosis: A salamander with toes 4–5; vomerine teeth in two short widely separated series between the nares; flat tail; no labial folds; no horny covering to palms and soles.

Description: U. S. Nat. Mus. No. 14365, adult male, Kopal, Semiryechensk Gov.; 12 costal grooves; appressed toes meeting; head width 4½ in length from snout to vent; head length 3 in length of body; head oval; eye as long as its distance from tip of snout; outline of upper jaw straight as seen from side; lower eyelid fitting under upper in front and behind; angle of jaw back of hind angle of eye; a groove from eye to gular fold; a groove from this down behind angle of jaw; a fold of skin on sides between insertions of limbs; limbs well developed; fingers 2, 3, 4, 1 in order of length; toes 3, 4, 2, 5, 1 in order of length; fingers and toes flattened towards tip; tail longer than body, flattened considerably; vent a longitudinal slit; vomerine teeth beginning behind inner edge of nares, a series of 6 teeth curving forward and in to level of middle of nares, series separated from its fellow by width of nares; color (faded) light brownish gray, with a few black dots on dorsal surface; total length 104 mm., head 14, body 41, tail 49.

Variation: A nearly transformed larva, Mus. Comp. Zool. No 1964, same locality; 13 costal grooves, one costal fold between appressed toes; labial folds on upper and lower jaw; otherwise like adult; total length 97 mm., head 11, body 40, tail 46.

According to Strauch (1870) the vomerine teeth converge forward in the young, but are quite transverse in the adult.

He gives measurements of a series from Kopal and from Kuldja in Chinese Turkestan, which I here reproduce.

Locality	**Total length**	**Head**	**Body**	**Tail**	**Remarks**
Kopal	219	23	78	118	
"	208	23	74	111	
Kuldja	192	20	66	106	
Kopal	182	20	65	97	
"	151	19	54	78	
"	111	14	42	55	
"	97	12	39	46	
"	97	13	38	46	
"	100	12	40	48	Has rudiments of gills

Apparently the relative length of the tail increases with age.

Habits: Snitnikov (1913) has a paper on the habitat and habits of *Ranodon* from which I quote the following.

"This salamander is characteristic of a purely mountain fauna; an inhabitant of mountain brooks at a considerable altitude."

He was able to find none at 1500 meters or at 1850 meters. Finally at greater altitudes in small streams in the Urgan-Tas Mts. between Kopal and Djarkent, he found them in abundance.

"In a large stream which flows in this region I took a series of adults, and some of the egg-sacks. In the next days of my sojourn I tried to hunt in other similar streams and I found salamanders in all of them but no egg-sacks. All these streams are at about 1900–2000 meters of altitude and flow into a little river, the Kesken-Terek, at about 1800 meters. The streams rise from springs on the mountain-side. In these streams, which I would not really call streams, but tiny brooks, some only $\frac{1}{4}$ arshin (7 inches) wide, the animals can be found, but in those lower than 1850 meters I found none."

"The sources of these brooks are rocky and the salamanders are under the rocks in the daytime." "I only found two at the water level on wet and moss-covered rocks, whose tops were out of water. All the others were under stones in the water. Here they spend the whole day. I never saw one come out from under the stone by his own volition or swim freely in the water, or stay on the open bottom of the stream in the daytime. Only towards beginning of evening they come out from under the stones and even out of water, and you can find them along the banks, and sometimes pretty far away among Juniper bushes or in the grass."

"Dissection of some of the specimens shows that the food is largely made up of small crawfish, caddis-fly larvae and their tubes, beetles, plants, much sand which probably came from the tubes, and even pebbles as large as 6 mm. in diameter. As this food was not digested and recently eaten the salamanders eat even in the daytime.

"They lay eggs at the end of June, because I found in the first days of July, 1908, eggs in cleavage stages, probably 5–6 days old, larvae just hatched, mature eggs in which the process of hatching was observed, and eggs not yet developed and freshly laid.

"The young just before it emerges, wriggles back and forth and breaks through the thin cover of the egg, goes through the opening, and swims immediately to the bottom, swimming like a snake, with legs flat against the body.

"The egg-sacks are fastened to the under sides of flat stones in the water. The water flows under such stones, so that the egg-sacks, which are slim and spindle-shaped, and fastened by one end, are continually swept back and forth by the water.

"The eggs are protected from shock by stiff jelly. When larvae emerge the egg opens in the same direction as the opening of the bag.

"The jelly bag remains attached to the rock empty and flat after all the animals have gone.

"The bags vary a great deal in size and in number of eggs contained. Some are 2–3 vershok (4–6 inches) long, and some are even ½ arshin (14 inches) long, and the number of eggs varies accordingly. I have a bag shrunk in alcohol which measures 2½ duima (2½ inches), and contains 25 eggs.

"The egg at hatching measures 1 cm. in diameter, and the larva is 1.75 cm. long. They have well developed front legs and perceptible hind legs. The dorsal fin reaches the head. In the next stage, 2.5 cm., the dorsal fin reaches the middle of the body, and the larva has developed hind feet.

"The young are lighter in color than the adults. The background is yellow with many brown dots which later become the background.

"Many specimens have well developed gills, but otherwise do not differ from those without gills. No very large ones were found with gills, so probably those with gills are young ready to transform.

"203 and 210 mm. were the lengths of my largest adults, the medium sized examples were 150 mm. long, and the smallest were 90 and 85 mm.

"The only specimen with atrophied gills is the smallest of all and is 81 mm. long, while two with gills measure 92 and 95 mm.

"They normally live under stones, and their motion is halfway between a swim and a crawl. But in deep water they put their legs to their sides and swim with their tails. They probably feel uncomfortable for they immediately try to find a stone and go under it.

"Both Kirghiz and Cossacks believe that these animals when dried in the sun and powdered are a good medicine for broken bones. Dried salamanders can be found in every Kirghiz or Cossack village. In places they have been exterminated by catching them for this purpose."

REMARKS: *Ranodon* is an isolated species whose relationships are remotely with *Hynobius,* perhaps with its recently described compatriot *H. turkestanicus.* It certainly has nothing to do with *Batrachuperus.* The two are independent more aquatic derivatives of *Hynobius,* and the similar dentition, due to weakening of the prevomers, is probably a parallelism.

Ranodon kessleri does not seem to be a distinct form.

Strauch (1870) stated that this animal occurred in Northeast China, but as his only Chinese locality was Kuldja, which is in the extreme western part of Chinese Turkestan, "Northeast" must have been a slip for Northwest. This error has been copied many times.

Nikolski (1905) questions Kulagin's (1888) record for Tashkent. To this Elpateevski and Sabaneef (1907) reply that a specimen with this locality, and said to have been collected by Fedtschenko, is in the Moscow Museum. Nikolski (1918) bases his *Ranodon kozhevnikovi* on this specimen. The characters are minute differences in proportion from *R. sibiricus*. If *Ranodon* varies as much as *Hynobius* does the form is certainly not valid.

Kastschenko (1898, Nauk. Tomsk.) has recorded this form from north of the Altai in the Tomsk Government, but later (1902) stated that he was mistaken and really had *Molge vulgaris*. In the latter paper he says he heard from Cossacks about some salamanders near the village of Ulyabinsk, and believes these to have been *Ranodon*.

Snitnikov (1913), having found the animal living in mountain streams between 1850 and 2000 meters of altitude, doubts whether the type actually came from "the neighborhood of Semipalatinsk" because there is no country of anywhere near such elevation in that vicinity. He suggests that the type really came from Semiryechensk, which was not a separate Government when Kessler wrote, but was a part of the Semipalatinsk Government. He says also that both "Kuldja" and "Kopal" refer not to the towns but to the quite large districts of which they are the centers.

Besides Kopal, whence came the specimens I have seen, it has been recorded from the following places:

Semipalatinsk Gov.: Semipalatinsk (Kessler 1866).
Semiryechensk Gov.: River Maloi Almatinki near Vernaye, 6000 feet alt. (Severtzov 1873); river Balykty near Kopal (Nikolski 1905); river Chimbulaki near Kopal, river Kora near Kopal, headwaters of river Kesken-Terek in Urgan-Tas Mts. at alt. of 1850–2000 m. between Kopal and Djarkent (Snitnikov 1913).
Syrdarya Gov.: Tashkent (Kulagin 1888).
Chinese Turkestan: Kuldja (Strauch 1870).

Specimens seen: Two, as follows;
Semiryechensk Gov.: Kopal (U. S. N. M. 14365, M. C. Z. 1964).

Genus BATRACHUPERUS Boulenger.

1878. *Batrachuperus* Boulenger, Bull. Soc. Zool. France, p. 71 (type *Salamandrella sinensis* Sauvage).

1882. *Batrachyperus* Boulenger, Cat. Batr. Grad. Brit. Mus., (2), p. 37 (substitute for *Batrachuperus* Boulenger).

Range: Same as that of the type species.

Diagnosis: *Hynobiidae* with horny covering to palms and soles and over tips of digits; lungs present; premaxillary fontanelle small; gills of larvae without filament bearers; vomerine teeth in two short, widely separated series; tail flat; one species.

BATRACHUPERUS SINENSIS (Sauvage).

1876. *Desmodactylus Pinchoni* David, Journ. Roy. Asiat. Soc., N. China Branch, 7, p. 226 (Moupin), (nomen nudum).

1877. *Salamandrella sinensis* Sauvage, Bull. Soc. Philom., (7), 1, p. 115.

1878. *Batrachuperus sinensis* Boulenger, Bull. Soc. Zool. France, p. 71.

1882. *Batrachyperus sinensis* Boulenger, Cat. Bat. Grad. Brit. Mus., (2), p. 37, pl. 3, f. 1; Guenther, 1896, Ann. Mus. Zool. St. Pétersbourg, 1, p. 209; Barbour 1912, Mem. Mus. Comp. Zool., 40, p. 126, pl. 1, f. 1.

1898. *Salamandrella keyserlingii* Bedriaga, (part), Wiss. Result. Przewalski Zentralasiat., Zool. 3, pt. 1, p. 3.

Type: In Paris Museum. A cotype is U. S. Nat. Mus. No. 10995, adult female.

Type locality: Moupin, Sze-chuan, China.

Range: Known only from Sze-chuan and from the Thibetan province of Kham.

Diagnosis: A salamander with toes 4–4; vomerine teeth in two short widely separated series between nares; flat tail; labial folds much developed; horny covering to palms and soles and over tips of digits.

Description: U. S. Nat. Mus. No. 64420, adult female, Mt. Omei, Sze-chuan, China; 13 costal grooves; one costal fold between appressed toes; head width $4\frac{2}{3}$ in length from snout to vent; head length $2\frac{3}{5}$ in length of body; head oval from above; eye as long as its distance from tip of snout; outline of upper jaw very convex as seen from side, owing to prominent labial fold (present also on lower jaw as in larvae); angle of jaw under hind angle of eye; both eyelids fitting under a fold of skin behind; a deep crescentic groove back of eye; a groove from eye to gular fold; a groove from lower lid to angle of jaw; gular fold extends on sides of neck to dorsal surface; limbs well developed; fingers 2, 3, 4, 1 in order of length; toes 3, 2, 4, 1 in order of length; palms and soles and under surfaces and tips of fingers and toes covered with thickened horny epidermis; tail longer than body, cylindrical at base, flattened gradually to tip, last half very flat; vent more or less cross-shaped, anterior edges swollen; vomerine teeth 5 in series, beginning well in and back from the very small nares, series extending inwards in slight curve, separated from its fellow by its own length; olive, indistinctly marbled with black, lighter below, anterior lips of vent white.

Variation: A male M. C. Z. No. 2848, Liang-ho-ko, western Sze-Chuan, referred with some doubt to this species, differs in having 14 costal grooves; appressed toes overlapping; head width 5 in length from snout to vent; head length 3 in length of body; eye longer than its distance from tip of snout; labial folds less developed; no horny skin on palms and soles (but limbs in poor preservation); tail shorter than body, cylindrical at base, somewhat flattened distally, a dorsal keel on distal third; vomerine teeth 4–5 in series, beginning in from hind edge of nares, running forward and in to level of anterior edge of nares; vent opening at front end of a longitudinal groove, as in male *Hynobius*, this groove does not open into the vent, which is star-shaped, and formed by the confluence of seven grooves. Brownish gray, speckled with black above, and mottled with black below.

In two females, U. S. N. M. Nos. 64419 and 64421, from Mt. Omei the vomerine teeth are as in the described male, running forward and in instead of transversely, but reaching only to the level of the middle of the nares.

In a fourth female, U. S. N. M. No. 10995, a cotype, probably from Moupin, Sze-Chuan, the teeth are transverse as in the specimen first described, but the series is separated from its fellow by one-half its length.

In this connection, Guenther (1896) says "In very young specimens the palatine teeth are placed somewhat differently from the mature condition, and form two oblique series extending beyond the level of the choanae." Bedriaga (1898) figures this condition.

The tail is shorter than the body in four of the five adults I have seen.

The appressed toes touch in No. 10995, and in it the vent is a slit.

A larva, U. S. N. M. No. 64422 from Mt. Omei, has dorsal fin to middle of back; the foot coverings are perfectly distinct; black above, somewhat lighter below.

MEASUREMENTS.

* = costal grooves. / = costal folds between appressed toes.

Specimen	Locality	Total length	Head	Body	Tail	*	Remarks
U. S. N. M. 10995	(?) Moupin	149	18	66	65	13/0	A Cotype
“ 64419	Mt. Omei	141	19	62	60	13/1	
“ 64420	“	146	21	55	70	13/1	
“ 64421	“	130	15	57	58		
“ 64422	“ (larva)	39	6.5	15.5	17	13/0	
M. C. Z. 2848	Liang-ho-ko	136	19	59	58	14/0	Tail and color different

The tongue seems to vary considerably, but is usually small and appears retractile into a sheath.

The type shows a somewhat different color: light chocolate; irregular dark mid-dorsal and dorso-lateral streaks, the latter continuing onto tail; mottled with darker below dorso-lateral streak; middorsal area darker chocolate color.

REMARKS: U. S. N. M. No. 10995 was sent from the Paris Museum as a “Type,” with the locality “Kiang-si.” The types of *Salamandrella sinensis* having come from Moupin, Sze-Chuan, and there being no other record of this species from Kiang-si, it seems probable that this specimen actually came from Moupin.

M. C. Z. 2848 is in poor preservation, and is the only male I have seen. Hence I am not inclined to lay too much stress upon its peculiarities. But the possibility should be kept in mind that this specimen may belong to another species. It was taken “walking over a bed of damp moss among the stunted spruces and firs at the very limit of tree growth. There was no pond or stream of running water near by.” (Barbour 1912.) The appearance of the tail would lead one to suspect less aquatic proclivities than those of the other specimens with their very broad, flat tails.

I doubt very much whether this animal is more allied to *Ranodon* than to *Hynobius*. The peculiar sole modification occurs elsewhere only in *Pachypalaminus boulengeri* of Hondo. The dentition is that of *Ranodon*, but such a weakening of the prevomers might well occur twice.

Unless there are two species included in the above account, there is no close relative, and the animal stands isolated. The occasional resemblance in color to *H. keyserlingii* (seen in the described cotype), the lack of the fifth toe in both, and the fact that *keyserlingii* has slightly developed labial folds suggest a possible derivation from that species. *Batrachuperus* is an animal of high altitudes and *keyserlingii* is a northern form. There is no known geographical connection between the two at present.

Besides the specimens listed below, it has been recorded from the following places:

Thibet: Province Kham; Kuochu-chin (Guenther 1896).

China: Province Sze-chuan; Sungpan (Guenther 1896); Kzerno River, Lumbu River (Bedriaga 1898).

Specimens seen: 6, as follows;

China: Province Sze-chuan; Moupin 1 (U. S. Nat. Mus. 10995, Cotype) Mt. Omei 4 (U. S. Nat. Mus. 64419–64422) Liang-ho-ko, 12000 feet alt. 1 (M. C. Z. 2848).

NOTE: *Turanomolge mensbieri* Nikolski (1918, Faune de la Russe, p. 256, f 56, pl. 4, f. 3) described from 61 mm. long larvae collected by V. Nikolski in Russian Turkestan, does not appear to me to be a Hynobiid. It has long series of vomerine teeth posterior to each naris, and no dorsal fin. I suggest that it is the larva of a mountain brook Salamandrid, perhaps allied to *Neuergus crocatus*.

ADDENDA.

RANODON SIBIRICUS.

1913. *Ranidens sibiricus* Dorovatovski, Tr. Spb. Obstch. Est., 42, p. 25.

ONYCHODACTYLUS FISCHERI.

1912. *Onychodactylus rossicus* Nikolski, Trud. Troitsko-Savsko Kyacht. Otd. Obstch., 15, p. 28. (This may be the original description).

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AN EXAMPLE IN POTENTIAL THEORY.

By O. D. Kellogg.

AN EXAMPLE IN POTENTIAL THEORY.

By O. D. Kellogg.

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1. The First Boundary Value Problem of Potential Theory.

Given an open continuum, T, in the plane of x and y, and values, F, assigned to each of the boundary points of T, the first boundary value problem of potential theory consists in finding a function which shall have continuous second derivatives which satisfy LaPlace's equation at the points of T, and which shall approach the given values, F, on the boundary. For continuous boundary values, and for a broad class of regions T, the problem has been proven possible of solution.[1] We shall denote by $[T_1]$ the class of regions for which the problem is possible in this sense.

2. The Example.

The purpose of the present note is to give an example of a region T_1 to which previous proofs do not apply; and incidentally, to throw some light on the direction which further investigation may take.

We first recall a familiar point set obtained by the "removal of middle thirds." The set S_0 is the closed interval $\left(-\frac{1}{2}, \frac{1}{2}\right)$ of the x-axis. The set S_1 is obtained from S_0 by the removal of its open middle third, and consists, therefore, in the two closed intervals $\left(-\frac{1}{2}, -\frac{1}{6}\right)$ and

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[1] The most general results of this class appear to be due to Lebesgue, Sur le problème de Dirichlet, Rendiconti di Palermo, v. 24 (1907), pp. 371–402. His restriction on T is, that any point belongs to T if it is possible to surround the point by a curve, lying in an arbitrarily small neighborhood of the point, and consisting only of points of T. Results more general, at least in some directions, have been obtained by H. B. Phillips and Norbert Wiener, and by G. E. Raynor. Their papers have appeared, and are expected to appear, in the *Journal of Mathematics and Physics of the Massachusetts Institute of Technology*, vol. 2 (1923), pp. 105–124, and in the *Annals of Mathematics*, respectively. For simply connected regions in particular, G. C. Evans has shown that the problem is solvable when the values F are merely bounded summable functions of the values on the boundary of the conjugate to a Green function, provided a suitable understanding be agreed on as to the manner of approach to the boundary values. See Problems of Potential Theory, *Proceedings of the National Academy of Sciences*, vol. 7 (1921), pp. 89–98.

$\left(\frac{1}{6}, \frac{1}{2}\right)$. S_n is obtained from S_{n-1} by the removal of the open middle thirds of the 2^{n-1} equal intervals of which S_{n-1} consists. The set S is the points common to all the sets S_n as n runs through all positive integers.

Let K be a circle containing S in its interior, say the circle $x^2 + y^2 = 4$. The region, T^*, in question, consists of the points interior to K, excluding the points of S. The boundary points of T^* within K then form a perfect set of Borel measure 0. But in spite of the relative sparseness of the points of S and the infinite connectivity of T^*, this region belongs to the class $[T_1]$.

3. The Lebesgue Barrier Sets.

This assertion may be proved by means of the notion of barrier functions, introduced by Lebesgue.[2] Suppose that there exists, for each boundary point, B, of a given region, T, and for every positive number, ϵ, a function, $U_{B,\epsilon}(P)$, with the following properties: it is harmonic in T, and assumes at every point, P, of T, a value not less than the distance $\overline{BP}$. As P approaches B, the function approaches a number less than ϵ, or more generally, there exists a neighborhood, N, of B, such that when P is a point of T in N, $0 \leqq U_{B,\epsilon}(P) \leqq \epsilon$. A set of such functions, we call, after Lebesgue, a barrier set for the region T. *Its existence is a necessary and sufficient condition that* T *belongs to* [T₁]. While this theorem, as well as its demonstration, is implied in the report of Lebesgue, the reader may welcome the following more explicit proof.

First, suppose that the barrier set exists for T. We approach the region T by a set of regions, $t_1, t_2, t_3, \ldots$, each containing the preceding and bounded by a finite number of regular curves, and such that each point of T is interior to one of them, and hence to all later ones. Let us suppose first that the boundary values F are those of a polynomial, $F(x, y)$. Then harmonic functions exist for the regions $t_1, t_2, t_3, \ldots$, which approach the same values as F on their boundaries. These functions will at least contain a sequence, $[U_n]$, which converges uniformly in any closed subregion of T to a harmonic function U. That U will approach the required boundary value at a boundary point, B, may be seen as follows. Let L be an upper bound for the difference quotients of F in $T + T'$, T', being the boundary set of T. Then, for any n, U_n lies between $F(B) - LU_{B,\epsilon}(P)$ and $F(B) + LU_{B,\epsilon}(P)$, since its values on the boundary of t_n do, and these functions are harmonic. Hence U lies between these functions, for every positive ϵ, and hence approaches $F(B)$.

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[2] Sur le problème de Dirichlet, C. R. vol. 154 (1912), p. 335.

If F is not given as a polynomial, we first extend its definition so that it shall be continuous in the whole plane.[3] We may then approximate to F by a sequence of polynomials over a closed region containing T' in its interior, the approach being uniform. The corresponding harmonic functions of T, U_1, U_2, U_3, . . . will then approach uniformly in T a harmonic function with the required boundary values. If T' extends to infinity, we have merely to replace, in the above reasoning, the polynomials by rational functions with a single pole not in T'.

Conversely, a barrier set exists for each region T_1, one, indeed, independent of ϵ. For, to each boundary point, B, corresponds a harmonic function of T_1, $U_B(P)$, determined by the boundary values $F(P) = \overline{BP}$. To see that $U_B(P) - F(P)$ is never negative, we have only to recall Gauss' mean value theorem for harmonic functions, and to notice that the above difference, at any point, P, of T, exceeds its arithmetic mean on any circle in T_1 with center at P. Such a function can have no minimum in T_1, and as the present one vanishes on T_1' and is continuous in $T_1 + T_1'$, it can, accordingly, never be negative in this region.

4. A LOGARITHMIC SPREAD ON S.

It remains to establish the existence of a barrier set for T^*. There is no difficulty connected with the barrier functions for the points B of K, since a barrier for the open circle evidently is a barrier for T^*. As a preliminary step in the establishment of barriers for the points of S, we consider the logarithmic potential of a certain spread of attracting matter on S. It will simplify notation to shift the origin of coördinates for the present to the point $\left(-\frac{1}{2}, 0\right)$. The potential in question is then the Stieltjes integral $L(P) = \int_0^1 \log r \, d\mu(x)$, where

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[3] For the possibility of this, see Lebesgue, l. c. Rendiconti di Palermo, or, Carathéorody, Vorlesungen über reele Funktionen, Leipzig, 1918, pp. 617-620.

$\mu(x)$ is the total mass on the segment $(0, x)$, and is defined as follows: $\mu(0) = 0$, $\mu(1) = 1$, on $\frac{1}{3} \leqq x \leqq \frac{2}{3}$, $\mu(x) = \frac{1}{2}$; in general, $\mu(x)$ being defined on the gaps in S_n plus their end points (i.e. on $S_0 - S_n$ plus its derivative), it is defined on the new gaps of S_n and their end points (i.e. on $S_{n-1} - S_n$ plus its derivative) by assigning to it the arithmetic mean of its values at the two nearest points to right and left at which it has been defined. In this way it is determined at points everywhere dense, and the definition is completed by demanding that it be continuous, a thing evidently possible.

The function $L(P)$ is seen to be harmonic in the entire plane, apart from the points of S. Moreover, *it is continuous at the points of* S. To show this, we first determine a simple function dominating $\Delta\mu = \mu(x_2) - \mu(x_1)$, $(x_2 > x_1)$.

If $x_1 = 0$, then $\Delta\mu = \mu(x_2) = \frac{1}{2^n}$ when $x_2 = \frac{1}{3^n}$. If we eliminate n between the equations $x = \frac{1}{3^n}$ and $y = \frac{1}{2^n}$, we obtain the function $y = x^p$ $(p = \log 2/\log 3)$. Geometric intuition suggests that x^p dominates $\mu(x)$, and further, that $(\Delta\mu) \leqq (\Delta x)^p$, since $y = \mu(x)$ never increases more rapidly than at the origin. These inferences are correct. We first note that $y = \mu(x)$ never rises above the polygonal line through the points $\left(\frac{1}{3^n}, \frac{1}{2^n}\right)$ of $y = x^p$. For as x increases from $\frac{1}{3^n}$ to $\frac{2}{3^n}$, $\mu(x)$ remains constant at $\frac{1}{2^n}$, and $\mu(x)$ does not increase beyond half way to $\frac{1}{2^{n-1}}$ until x has progressed $\frac{2}{3}$ of the remaining distance from $\frac{2}{3^n}$ to $\frac{1}{3^{n-1}}$. Hence for $\frac{1}{3^n} \leqq x \leqq \frac{2}{3^n} + \frac{2}{3^{n+1}}$, $y = \mu(x)$ lies under the chord joining $\left(\frac{1}{3^n}, \frac{1}{2^n}\right)$ and $\left(\frac{1}{3^{n-1}}, \frac{1}{2^{n-1}}\right)$. But from $\frac{2}{3^n} + \frac{1}{3^{n+1}}$ to $\frac{2}{3^n} + \frac{2}{3^{n+1}} + \frac{2}{3^{n+2}}$, by the same reasoning, $\mu(x)$ lies under its steeper chord joining $\left(\frac{2}{3^n} + \frac{1}{3^{n+1}}, \frac{1}{2^n} + \frac{1}{2^{n+1}}\right)$ and $\left(\frac{1}{3^{n-1}}, \frac{1}{2^{n-1}}\right)$. A continuation of the argument, and the use of the continuity of $\mu(x)$ completes the proof that $\mu(x)$ lies under the polygonal line in question, and hence under the concave downward curve $y = x^p$.

Next, suppose that x_1 is the abscissa of the left end of one of the intervals of S_n. We are evidently at liberty to suppose that n is the smallest integer for which x_1 has this property. Note that all the parts of $\mu(x)$ above the intervals of S_n are congruent. Hence for $\Delta x \leqq \frac{1}{3^n}$, $\Delta\mu \leqq (\Delta x)^p$. That the same inequality holds for all Δx with the above value of x_1 follows from the congruence of the pieces of $\mu(x)$, the constancy of $\mu(x)$ between the intervals of S_n, and from the fact that if we call S_n' the set obtained by shifting S_n to the right a distance x_1, *the total length of the intervals of S_n between x_1 and x_2 is never greater than the total length of the intervals of S_n' between x_1 and x_2.* To see this, we notice first that S_n and S_n' begin at x_1 with a common interval of length $\frac{1}{3^n}$. There follows a gap of length $\frac{1}{3^n}$ between the intervals of S_n', whereas a greater gap, of length g, say, follows for S_n, since n was the smallest integer for which x_1 began an interval of S_n. The corresponding gap g in S_n' must begin at a distance g from x_1, i. e. a distance $\frac{1}{3^n}$ before the gap g in S_n terminates. Because of the symmetry of S_n to either side of a gap at least for a distance g, it follows that after the gap g in S_n', the two sets have again a common interval, and that before this point the accumulated length of intervals of S_n' has exceeded, or at least equaled that of S_n, because of the earlier occurrence of the gap g in S_n. The reasoning may now be repeated. The next gap in S_n' will be of length $\frac{1}{3^n}$, while the next gap in S_n will be longer, for otherwise we should have intervals of S_{n-1} and S'_{n-1} coinciding, which is contrary to the hypothesis that n is the smallest integer for which x_1 begins an interval of S_n.

Finally, if, instead of beginning an interval of S_n, x_1 lies in a gap between two of them, the inequality $\Delta\mu \leqq (\Delta x)^p$ holds *a fortiori*, since the diminution of x_1 will have increased Δx without changing $\Delta\mu$. Because of the continuity of $\mu(x)$, the inequality holds generally.

The continuity of $L(P)$ now follows without trouble if we separate out from the integral a part corresponding to a small interval containing the boundary point B, in whose neighborhood the continuity of $L(P)$ is to be investigated. The above inequality shows that this interval can be made so small that the integral over it may be made uniformly less than a preassigned positive ϵ in a given neighborhood of B. The rest of the integral is continuous at B, and hence so is $L(P)$.

We now know that $L(P)$ attains its negative minimum, $-m$, on S.

It is also negative at all points within a circle C with S_0 as diameter, for the integrand is never positive in this circle. On and outside any larger concentric circle, of radius R, $L(P) > \log\left(R - \frac{1}{2}\right)$.

5. THE BARRIERS FOR T^*.

We now shift our origin of coördinates back to its position of symmetry in the set S, and denote by $U_1(x, y)$ the function which $L(P) + m$ thus becomes. We then consider the subset, s, of S, namely the points common to S and one of the intervals, i, of S_n; let $(a, 0)$ be the center of this interval. Then $U_2(x, y) = U_1(a + 3^n x, 3^n y)$ is harmonic in the whole plane except at the points of s, it is never negative, and within the circle c, which has i as diameter, $U_2(x, y) \leqq m$. On and without any circle with $(a, 0)$ as center, and radius $R/3^n$, $U_2(x, y) > \log\left(R - \frac{1}{2}\right)$. Or, with $3^n R$ in place of R, we may state that on and outside a circle of radius R about $(a, 0)$, $U_2(x, y) > n\log 3 + \log\left(R - \frac{1}{2 \cdot 3^n}\right)$. The least value possible of R on K is $\frac{3}{2} + \frac{1}{2 \cdot 3^n}$. Hence, on K, $U_2(x, y) > n\log 3$.

Now let B be any point of S. We identify the above subset s with one containing B. The maximum of $\overline{BP}$ on K is $5/2$. Hence $U_3(x, y) = \frac{5U_2(x, y)}{n2\log 3}$ exceeds $\overline{BP}$ on K, is harmonic except on s, is never negative, but within c is less than $5m/n2\log 3$. Finally, $U_{B,n}(x, y) = U_3(x, y) + \frac{1}{3^n}$ exceeds $\overline{BP}$ on s and K, and hence in T^*. As P approaches B, this function approaches a value which vanishes with $1/n$, and thus has all the properties of the barrier functions, so that the existence of the barrier set, and with it the fact that T^* belongs to $[T_1]$ have been established.

6. REMARKS.

The preceding considerations give an example of the possible usefulness of the notion of barrier. As the nature of the region T which makes it a T_1 depends on the boundary "im Kleinen," a generalization of the idea of barrier may be indicated which may be useful, namely a barrier for the region bounded by a large circle and an arbitrarily small piece of the boundary set.

As to the degree of generality of the regions T_1, the following may be said. An isolated point cannot form part of the boundary (see, for instance, Osgood, Funktionentheorie, p. 565). Hence the boundary must be dense on itself, and as it is closed, it must be perfect. The above developments suggest that a criterion as to which perfect sets may enter the boundary, may be found in the possibility of logarithmic spreads on the boundary set, of only positive masses, which have potentials that are continuous on the boundary.

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THE TYPICAL SHAPE OF POLYHEDRAL CELLS IN VEGETABLE PARENCHYMA AND THE RESTORATION OF THAT SHAPE FOLLOWING CELL DIVISION.

By Frederic T. Lewis.

THE TYPICAL SHAPE OF POLYHEDRAL CELLS IN VEGETABLE PARENCHYMA AND THE RESTORATION OF THAT SHAPE FOLLOWING CELL DIVISION.

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FREE cells of plants and animals can readily be seen from all sides. Though varying greatly in form they are regarded as primarily spherules or globules, easily flattening into discoids, or elongating to form ellipsoids and rods of varying length with rounded ends. But the shape which such cells assume when surrounded and compressed by similar cells, though frequently a subject for inference, has apparently never been determined by the observation of actual cells, and on this supposition the present study was undertaken. Even though the examination of many sorts of cells for this purpose is very difficult, and in the most favorable cases is tedious, it seems impossible that such work has not already been done. But if so, it remains unknown to the writer, after some inquiry and a search of the literature at hand.

Previous publications record and reiterate that the problem of cell-shape is solved in foam. In the earliest observations, Robert Hooke found that within the shaft of a feather the cells form "a kind of solid or hardened froth, or a congeries of very small bubbles,"[1] and Grew described parenchyma as "much the same thing, as to its conformation, which the froth of beer or eggs is." Two centuries later, Errera remarks that cell-walls "must correspond with such a lamellar system as one gets in pouring soap-suds, beer, etc. from a narrow-necked bottle." Clearly these bubbles are not cubes; and although Grew ascribed cubical cells to the pith of "reed-grass," and at present epithelial cells of a certain type are commonly called "cuboidal," it appears that true cubical cells have never been shown to occur, even as rare exceptions.

Parenchymal cells which are "pentangular, sexangular, and septangular" were seen by Grew, and the prevailing hexagonal form of such cells in section is shown in some of his figures. Cut tangentially, the cells of cork were found by Hooke to be as regularly hexagonal as cells in honeycomb. If these cells should be six-sided tubes divided into little boxes by transverse diaphragms, as he described them, octahedral shapes would be produced; though if the diaphragms in adjacent cells stand at different levels, something more complicated is at once suggested.

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[1] Schwann, in his *Mikroskopische Untersuchungen*, Heft 1, 1838, p. 94, confirmed the accuracy of Hooke's comparison of the cells in the pith of a feather with those of cork, but without citing the earlier work.

In 1812, Dr. Dieterich Georg Kieser, professor of medicine at Jena and collaborator with Oken, obtained the Teylerian prize for an essay on the structure of plants. It seems that he had not then reached the conclusion published in his Phytotomie (1815) that "the higher plant consists of a mass of individual cells — cell-tissue — and the cells then assume, through mutual pressure, a form determined by mathematical laws and consequently inevitable, namely that of the *rhombic dodecahedron*."[2] He considered that this dodecahedron may have the proportions shown in Figure 1, *a*; or it may be either flattened or elongated (Figure 1, *b*). In Oken's Physiophilosophy it is declared dogmatically that the fundamental form of cells is the rhombic dodecahedron, with Kieser as authority; and its production is explained somewhat as follows. If spheres are stacked (after the manner of cannon balls) any one will rest in a depression between three which are below it; it will be surrounded by six in its equatorial region; and there will be three more in contact with it at the upper pole — twelve altogether. The model of such a cell may readily be made from balls of putty or plasticine, which by compression become rhombic dodecahedra, and it will be similar to Figure 1, *a*, but rotated so that trihedral angles are at the top and bottom respectively. Or, following the suggestion of Buffon, dry peas may be packed close in a jar and tightly secured by wire netting, after which they are made to swell until all the interstices are obliterated, by placing them in boiling water. The peas so treated are usually irregular little dodecahedra, though peas with eleven or thirteen contacts are not infrequent.[3]

An interesting feature of Kieser's work, often overlooked, is illustrated in Figure 1, *c* and *d*. He considered that truncate rhombic dodecahedra occur more often than the typical form. The upper and basal surfaces of such cells are hexagonal instead of quadrilateral, and the two quadrilaterals which previously extended from the base to the top likewise become hexagons. Such a form, with four hexagonal and eight quadrilateral surfaces, Kieser regarded as typical for the parenchyma of bark and pith, being impressed with the way in which it provided hexagonal sections when cut in several planes. The flattened form (Figure 1, *d*) will be seen to be a most interesting approximation to the shape of cells actually found by methods not at Kieser's disposal. In an excellent diagrammatic figure he showed how the truncate dodecahedra would appear when combined to make a tissue, and this perhaps led DeCandolle to warn his readers that "cells are not as regular as the published figures might lead one to believe."

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[2] Although Kieser cites no previous authorities, he may not have been the first to consider vegetable cells as dodecahedral. Allman is said to have made the suggestion, in regard to cells of dicotyledons, in a paper read before the Royal Society in 1811 and privately printed (Thompson, *Growth and Form*, p. 643). But it was through Kieser's exposition that the idea became generally and favorably known. Thus Schleiden (Müller's Archiv, 1838, p. 146) remarks that "the form of cells frequently passes into that of the rhombic dodecahedron, so beautifully determined a priori by Kieser."

[3] Buffon did not describe the shape of peas, "or better, some cylindrical seed," treated in this way further than to remark that the cylinders become hexagonal prisms. This was in connection with his study of the form of cells in honeycomb. The entire passage has been quoted by Thompson, p. 333–334.

Thus the matter rested until Plateau, in 1873, brought together his masterly researches on the statics of liquids. By means of wire frames dipped in glycerinated or soapy solutions he produced "liquid polyhedra," studying the production and arrangement of the films which make their walls. In 1886 this work in physics was applied to the problem of cell forms both by Berthold in his *Studien über Protoplasmamechanik* and by Errera, *Ueber Zellenformen und Seifenblasen.* Neither writer discloses the shape of parenchymal cells — Errera refers to the endless diversity of cell forms — but the inclination and curvature of cell walls, as seen in sections, are carefully studied, and compared with inert liquid films and "artificial cell-tissues." It is concluded that the phenomena of surface tension are of controlling importance in the shaping of cells.

In accordance with the mathematics of his time, Kieser had declared that, of all bodies which may be combined to fill space without interstices, "the rhombic dodecahedron encloses the greatest space with the least extent of surface." But in 1887, Lord Kelvin, after experimenting with Plateau's cubic skeleton frame, demonstrated that for the division of space with minimal partitional area the rhombic dodecahedron is rivaled by a more stable tetrakaidecahedron, or figure having fourteen surfaces, certain of which are slightly curved. Neglecting the curvature of such films, which he calculated with precision, as being lost in greater curves and irregularities due to shrinkage in preserved tissue, the form of this tetrakaidecahedron deserves careful attention. It has six quadrilateral and eight hexagonal surfaces, which form "thirty-six edges of intersection between faces, and twenty-four corners, in each of which three faces intersect." If the quadrilateral surfaces are equal squares, and the hexagons are equal, equilateral and equiangular hexagons, the figure becomes what Lord

Kelvin, in his Robert Boyle and Baltimore Lectures, called an "orthic tetrakaidecahedron." Such a solid has been drawn in Figure 1, *e–i*. It has already been stated that there is reason to consider the top and bottom of the cell as hexagonal, the sides and angular points of one of these hexagons being perpendicularly above or below those of the other. If this is done, Figure 1, *e*, represents either the top or basal view of the solid, seen in projection and not in perspective. The central hexagon is surrounded by alternating squares and hexagons — three of each. A lateral view, with a side and not an angle of the top and basal hexagons toward the observer, is shown in Figure 1, *f*. To pass from the top to the bottom, two surfaces must be traversed,— a square with a hexagon below it, or a hexagon with a square below it. A lateral view, with an angle of the top and basal hexagons toward the observer, is seen in Figure 1, *g*. If, instead of a hexagon, one of the squares should be regarded as the top, then the solid, seen from above, would appear as in Figure 1, *h*.[4] The lateral view with an angle of the square toward the observer has precisely the same appearance; but if a side of the square is toward us, the lateral aspect is as in Figure 1, *i*, which is the same as Figure 1, *g*, turned about. These, then, are the appearances to be sought in actual cells if they possess the tetrakaidecahedral form. The way in which such solids combine to fill space is shown in Figure 3 (Plate 1).

A further and very instructive review of the literature has been provided by D'A. W. Thompson, but it leaves him unable to decide whether the cells of vegetable parenchyma are dodecahedra or tetrakaidecahedra. He finds that these cannot be distinguished in ordinary sections, but suggests that it might be done through maceration. "Very probably," he concludes, "it is after all the rhombic dodecahedral configuration which, even under perfectly symmetrical conditions, is generally assumed."

Maceration appears to be impracticable,[5] but there is no special difficulty in reconstructing, by Born's familiar wax-plate method,

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[4] In this view the solid is seen to be a cube strongly truncated by an octahedron. Any mineral which crystallizes in the isometric system is capable of exhibiting such a combination, and it is not uncommon in gold, galena, pyrite, salt, fluorite, cuprite and diamond. But the truncating of the cube by the octahedron, or vice versa, in crystals is often slight, and the faces are usually unequal, so that the production of an orthic tetrakaidecahedron would be very rare indeed. For this information the writer is much indebted to Professor B. L. Robinson and Professor Palache. Crystals of alum, however, may be nearly orthic as seen, for example, in Figure 548, Plate 70, of Adams' *Micrographia* (4th ed., London, 1771).

[5] M. H. Dutrochet (Recherches . . . sur la structure intime des animaux et

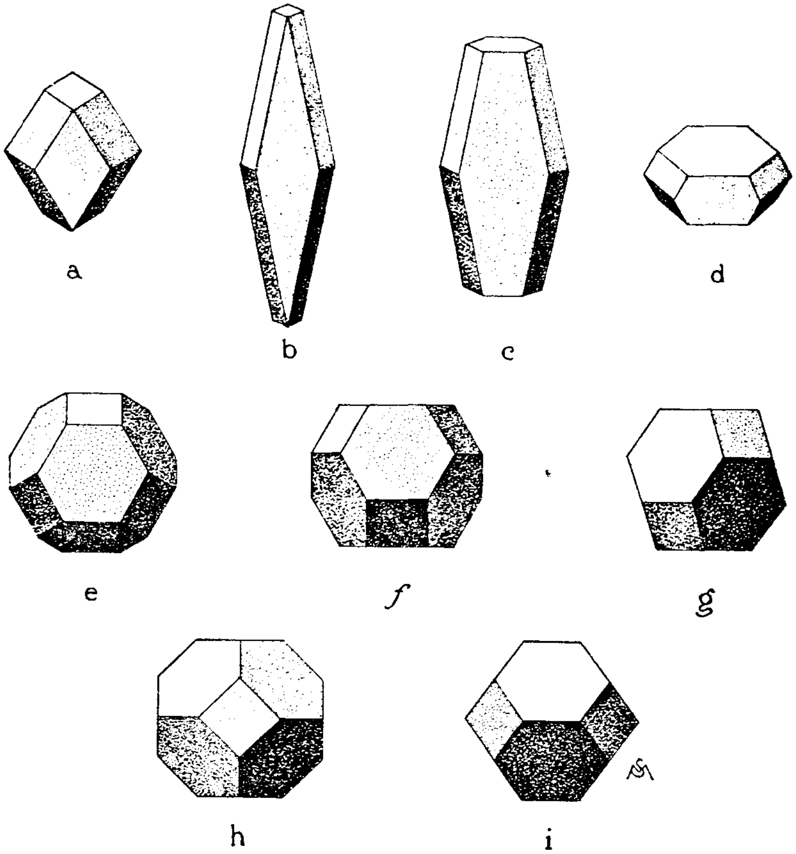

FIGURE 1. *a*, a rhomboidal dodecahedron, and *b*, an elongated form of the same. *c*, a truncate dodecahedron, and *d*, a flattened form of the same (*a–d*, after Kieser). *e–i*, an orthic tetrakaidecahedron shown in its several aspects.

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des végétaux, Paris, 1824, p. 49) succeeded in isolating the cells of pith by boiling them in nitric acid, and thus demonstrated that they are closed vesicles, independent though agglomerated. But of their shape he says merely,—"Their original form is globular; it is from the equality of the compression to which they are subjected in all directions that they often assume a symmetrical polyhedral form." It may readily be surmised why he did not describe the polyhedra in greater detail.

cells cut in serial sections, provided the cells are large enough and their walls distinct. Of human tissues, the epithelium lining the oral cavity is quite favorable, and reconstructions of eight cells have been completed; but the cells are small as compared with those of amphibians, and both small and irregular in contrast with cells of vegetable parenchyma. Hence the pith of the elder, *Sambucus canadensis*, was chosen. If a fresh moist specimen is obtained, it can be imbedded thoroughly in paraffin and cut in perfect series at a thickness of ten microns, thinner sections being undesirable. The chief difficulty will be in interpreting the cell walls which fall nearly in the plane of section and consequently appear as hazy films, but in ten-micron sections these will be dense enough to be recognized readily. Portions of pith were cut transversely and longitudinally, but the reconstructions were made exclusively from longitudinal sections. On the average a cell extended through sixteen of these sections. A small area of the tissue was drawn with the projection lantern at a magnification of 250 diameters, and selected cells in the drawing were enlarged by the pantograph to 500 diameters. After all the contacts had been studied and recorded, the models were made in wax by Ethel S. Lewis,— a task requiring patience and skill beyond that at the author's disposal. The most instructive of these models have been beautifully and accurately drawn by F. Schüyler Mathews.

The appearance of the cells in elder pith when sectioned lengthwise of the stem is shown in Figure 2. The long axis of the cells is transverse to that of the stem, contrary to the early statement of Hooke, but in accordance with a drawing by Dippel, who shows, however, that in the youngest internode in autumn the long axis may be perpendicular. Possibly after a period of rapid growth chiefly in length, the cells continue to divide transversely, thus becoming compressed and flattened, as found in the older internodes. The cells then have an accordion-like arrangement which is demonstrated by children in removing the pith by pushing it out of the stem. A long stretch may thus be cramped into a short space, but it regains almost fully its original length when released from the stem. It may well be that a similar but moderate contraction in the paraffin has rendered the cells modeled somewhat flatter than is normal, but this would not alter the number or shape of their contacts. The intercellular spaces, essential for the life of the cells, serve to blunt the edges and angles of the polyhedra, the form of which it is our purpose to make clear. Consequently these relatively small spaces (shown in Figure 2) have been disregarded. Further, the waviness of some walls, when evidently

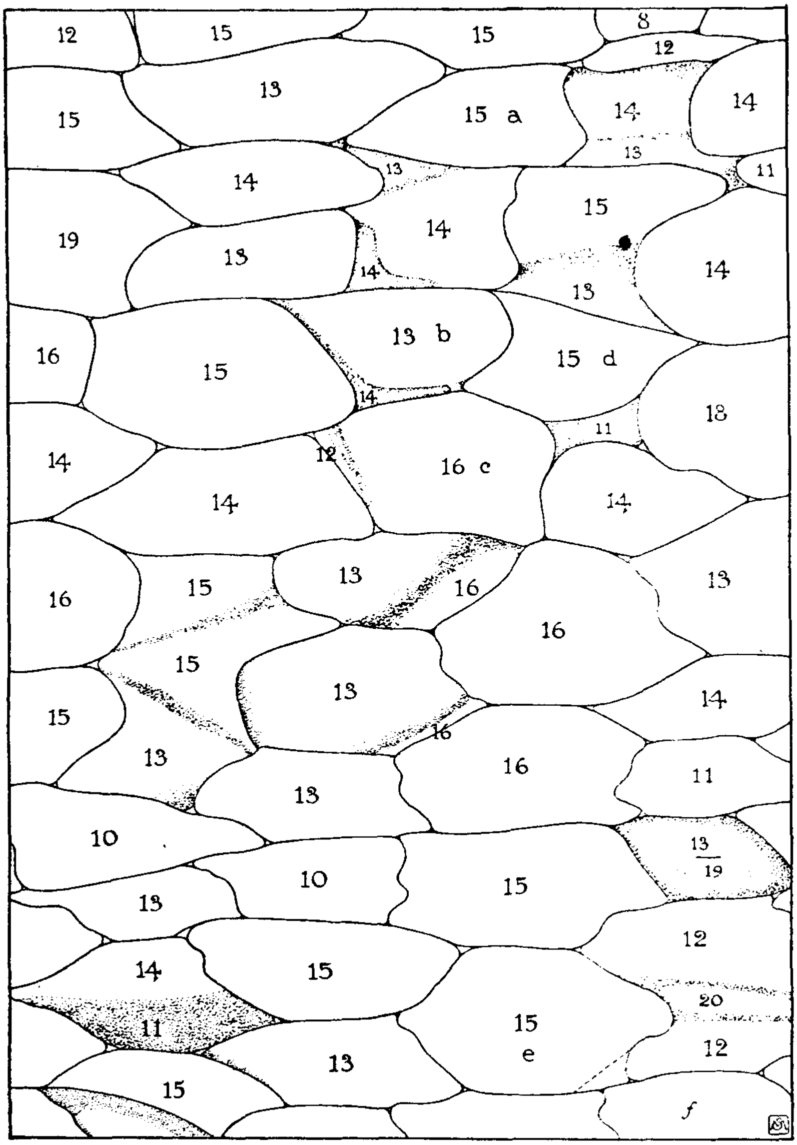
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FIGURE 2. Longitudinal section of pith of the elder, *Sambucus canadensis.* × 220 diam. The numeral in each cell indicates the number of contacts which that cell has with other cells. Two numerals on a septum indicate the contacts of the cells above and below the septum. The letters are explained in the text.

due to shrinkage, has been deliberately smoothed out, altogether giving the models somewhat flatter surfaces and sharper edges than occurs in the preserved tissue. But this suppression of accidental irregularities has been permitted only when necessary to reveal the true pattern of the cell as indicated by its contacts with others.

The preliminary count of contacts which the cells have with those about them is partly recorded in Figure 2, by numerals placed within the cell outlines. Of the sixty-three cells there shown, the average number of contacts is 13.96. A contact is always a potential facet, and usually, as seen in the figure, is actually such. The contacts of thirty-seven more cells were counted, but, as it happened, the average number per cell remained unchanged — 13.96. In Table I is shown the number of cells, in the one-hundred counted, having the number of contacts indicated at the top of the several columns. The cells range from hexahedra to icosahedra, with no special tendency to form dodecahedra. More than half of them are 13-, 14- or 15-hedra. This is consistent with a typical tetrakaidecahedral form.

TABLE I.

No. of Surfaces	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
No. of Cells	1	1	2	0	2	8	8	21	16	19	10	2	3	6	1

Irregularities are due in part to the great difference in the volume of the cells. Calculated roughly from the displacement of the wax models, these cells ordinarily range in size from .0004 to .0014 c. mm., with an average volume very close to one-thousandth of a cubic millimeter. Since such a cavity if filled with human blood would contain between four and five thousand corpuscles, it is evidently a cell of considerable size. The largest cell modeled measured .00224 c. mm., and the smallest was crowded and flattened to a mere .00008 c. mm. Cells with an excessive number of contacts are usually very large, and of such form as to suggest that an expected division in some particular plane has failed to occur. Thus they may be octagonal above and below, with two tiers of lateral surfaces, so that a vertical division would reduce them nearly to normal size and shape. Or they may be hexagonal above and below, with three tiers of lateral surfaces around most of their circumference; these would be reduced by a horizontal division. Certain instances of small cells with few contacts have resulted from an atypical division of cells approximately tetrakaidecahedral, with subsequent arrest of development. The irregularities of the cells, due to their manner of growth, should not be underestimated. Among the forty-two cells modeled, there is not one with fourteen surfaces, eight of which are hexagons and six quadrilaterals, arranged as in the tetrakaidecahedron of Figure 1, and yet they seem to show convincingly that such is their typical shape, or the form from which they may all have been derived.

The typical shape is almost realized in a cell with fifteen contacts shown near the base of Figure 2 and labeled *c*. If its lower lateral wall on the right had taken the course of the dotted line in the figure, its fifteenth contact (with the cell *f*) would have been eliminated. This disturbing contact is shown as a small triangular area, *3*, in the basal view of the model, Figure 4 (Plate 1). It is there shown that the central hexagon is not surrounded by alternating quadrilaterals and hexagons, inasmuch as there are two adjacent pentagons on the left, and a third pentagon, due to the intrusion of the triangular surface *3*, on the right. The same cell, in lateral view, looking directly at the quadrilateral surface shown in Figure 4, is seen in Figure 5. Below the quadrilateral of the upper tier is a hexagon, and below the adjacent hexagon of the upper tier is a quadrilateral, which with the hexagon on the under surface are reproducing exactly the complex pattern of the type. Turning to a lateral view of this cell facing the adjacent pentagons of the upper tier, it is seen, in Figure 6, that there are also pentagons below them. If this cell were absolutely typical, its lower tier should present a succession of surfaces with the following numbers of sides — 4, 6, 4, 6 — whereas it does show 4, 5, 5, 6; and above them, in place of 6, 5, 5, there should be 6, 4, 6. But all these deviations would be rectified if the boundary between the pentagons in the upper tier were swung like a pendulum to the left, passing the boundary between the pentagons in the lower tier. In other words, differences in the relative volume of adjacent cells may convert two quadrilaterals and two hexagons into four pentagons. Thus it is seen that with the shifting of a single boundary and the elimination of a small triangular fifteenth contact, the cell shown in Figure 4 would be typical throughout.

It may be noted, however, that this cell is not at all an *orthic* tetrakaidecahedron, but, as if from flattening, the upper and lower tiers of lateral surfaces tend to meet at an equatorial ridge. In the orthic cells in the diagram, Figure 3, the squares make one third and the hexagons two thirds, of the distance from the top of the cell to the bottom; but in actual cells with an equatorial ridge, the squares almost equal the hexagons in vertical measurement. A cell (Figure 2, *a*) with less of an equatorial ridge than that which has just been described, is shown in Figure 7. As seen by comparison with Figure 3, six of the seven surfaces which appear in the drawing are exactly according to the type.

A prolific cause of deviations from the tetrakaidecahedral form is found in the process of cell-division; and that this must be so, is evident upon reflecting that the division of a typical cell, whether vertical or transverse, reduces the number of surfaces from fourteen to eleven. Figure 11 is the pattern of vertical division through angles of the top and bottom hexagons, but the models afford no evidence that division in this plane actually takes place; Figure 12 shows vertical division through the sides of these hexagons; and Figure 9 illustrates transverse or horizontal division. With the completion of the process, a constriction occurs along the plane of division, whereby the daughter cells become more globular. In all cases the resulting cells have but eleven surfaces. Restoration of those which are lacking will depend upon the division of adjacent cells.

In elder-pith growth is chiefly in length, and division is predominantly transverse. The orientation of the cells, as will be seen, is in accordance with this. If the growth were chiefly in thickness, as in cork, it would be expected that the cells would be so turned that their inner and outer surfaces, instead of their upper and lower surfaces, would be hexagonal. In a typical cell oriented as in pith the transverse plane passes through hexagonal surfaces only, being above or below the quadrilateral surfaces in every case (Figure 9). If the cells in contact with the one which has thus divided, likewise divide transversely in the middle, the planes of their division will encounter the surfaces of the central cell as shown by the dotted lines in Figure 9. The position of these lines may perhaps be better understood by dividing several cells in the group shown in Figure 3 transversely through the middle. A line will pass across every lateral hexagonal surface, from square to square, bisecting the sides of the squares; and they cut off from each hexagon a small quadrilateral area, leaving the remaining portion still hexagonal. The contraction which follows the division of the adjacent cells produces ridges along the lines passing from square to square, and the tension or pull upon these squares converts them into hexagons. The diagram, Figure 8, shows, then, the restoration of the tetrakaidecahedral form, following the transverse division of the cell shown in Figure 9 and of those in contact with it, the corresponding surfaces in the two drawings being marked with the same letters. Every square has become a hexagon, and is above or below a quadrilateral portion cut off from an original hexagon. Every hexagon of the original cell has produced a quadrilateral above and below, and its middle part remains hexagonal.

Can any such geometrical process be assumed to occur in actual cells? In Figure 2, at *b* and *c*, there are two cells which are shown inverted in Figure 10. Except that one of these has three more surfaces than the other (16 and 13 respectively) their configuration is suggestive of twins. On the side hidden in the figure each cell presents four surfaces meeting at a salient point, such tetrahedral angles being very exceptional and contrary to the arrangement of the experimental soap-films. There is also a striking duplication of surfaces shown in Figure 10, and they are so oriented as to indicate that this pair of cells resulted from a transverse division. The surfaces *a–e* would then correspond with those of the same letters in Figures 8 and 9. There is an additional fifth contact of *a* at *x:* typically this surface should be quadrilateral and its upper border horizontal. The surfaces *b* and *c* are in contact with one and the same cell and are typical, as are all four surfaces on the right. There a single cell is in contact with the quadrilateral and the hexagon below it. On the left, the upper hexagon is typical, but below it there are two surfaces instead of three, and these are in contact with a single cell. Evidently the cell corresponding with that adjoining *g*, *h*, and *i* in Figure 9 failed to divide with the rest. Consequently the surface *d* has five instead of six sides, and since the quadrilateral, *i*, is lacking, the entire cell has thirteen instead of fourteen contacts. The general arrangement of these faces is convincingly like the theoretical pattern.

Vertical division of the type shown in Figure 12 should produce a characteristic cell, having four pentagonal surfaces (two of which can be seen in the diagram, Figure 13) and a hexagonal face extending uninterruptedly from the top pentagon to the basal pentagon. Figure 14 represents an actual cell, which as seen from one side exhibits precisely these features. From other points of view it is not perfectly typical; yet it has exactly eleven surfaces, and clearly represents a cell which, after vertical division, failed to regain the tetrakaidecahedral form. Vertical division occurring directly over a cell dividing transversely would subdivide the hexagon beneath, producing on the top of the underlying cell an additional surface like that marked *x* in Figure 10.

Cells more difficult to interpret are shown in Figure 16, which represents a pair evidently derived from the transverse division of a single predecessor. In one of their lateral aspects they present, as seen in the figure, four superimposed irregular pentagons, with angles jutting alternately to the left and right. The regularly superimposed quadrilaterals and hexagons on either side of them can not, by any change in the pentagons, be brought into harmonious relations with each other. It happens, however, that the cell shown in Figure 15 may explain the situation. This cell presents an atypical feature which is not very unusual, namely three, instead of two, superimposed lateral surfaces. A hexagon has a quadrilateral both above and below it. This would happen if the cell shown in Figure 9 failed to divide at the time when division took place in an adjoining cell at the plane indicated by the dotted line between *a* and *b*. This would convert the quadrilaterals *f* and *j* into pentagons; they would become hexagons through the transverse division of certain cells on the hidden side of the model. Thus the atypical features of the cell shown in Figure 15 may be satisfactorily accounted for. Now if this cell should divide transversely along the line of dashes, and adjacent cells against the faces *a* and *d* should divide along the dotted lines, a pair of cells almost identical with those in Figure 16 would result. The conditions in twelve of the thirteen surfaces there shown would be exactly reproduced. Division of a cell on the back of the model in Figure 15 would change the pentagonal surface *c* to a hexagon and make the analogy perfect.

There is sometimes evidence of an unequal cell division, as with the pair of cells shown in Figure 17. The cell from which they came was under the average size, and was pentagonal above and below, having twelve surfaces altogether. It exhibited a pronounced equatorial ridge. Transverse division took place below the ridge in a plane passing through a quadrilateral surface and separating scarcely more than one third of the cell from the rest. The volumes of the models are 55 c.c. and 30 c.c. respectively. In the drawing a dotted line indicates approximately the plane of normal division. As a result of the unequal partition, the upper cell retained a portion of all the original surfaces except that which formed the base, and with a new pentagonal basal surface, it continued to have twelve facets. The lower cell, however, has only eight facets, of which the three lateral ones shown in the figure extend continuously from the top to the bottom surfaces. The latter, as before mentioned, are pentagonal. On the side away from the observer, similar conditions obtain except that a small triangular fragment of one of the surfaces of the upper tier has been included in the lower cell.

The smallest cell observed, having a volume in the model of 10 c.c., was so very small and flattened that it required special examination to be sure it was not a distended intercellular space. That possibility could be definitely excluded. The model seems a formless body (Figure 19) yet one which may be tentatively explained as due to an unequal vertical division of a tetrakaidecahedron. The fragment of the pattern-model shown in Figure 18 has seven surfaces, but the actual cell in Figure 19 has but six. Four of these can be seen in the figure. The others are a very small triangular facet at the top of the hidden side, and a broad pentagonal area covering the remainder of that side. If the contact x in the pattern should be eliminated, there would remain six surfaces, each of which would have the number of sides actually found in the hexahedral cell.

Eleven of the forty-two cells modeled have now been considered, and it has been shown that great variations in the number, shape and arrangement of their facets are entirely consistent with a typical tetrakaidecahedral form. An equal division of a tetrakaidecahedron produces cells with only eleven surfaces; an unequal division may reduce the number still further, to eight or even seven, as has been shown by examples. On the other hand, if all the cells surrounding a particular tetrakaidecahedral cell should divide transversely, the undivided central cell would have three tiers of lateral surfaces and twenty contacts, the top and base remaining hexagonal. A cell among those modeled which closely approximates this form was evidently produced by a vertical division at a time when the cells around it divided transversely. If all the cells surrounding a particular cell, except the one immediately above it and the one immediately below it, should divide in halves vertically, and all in one plane passing through the sides and not the angles of the top and basal surfaces, then the central cell would be octagonal above and below; it would have sixteen lateral surfaces arranged in two tiers,— eighteen surfaces altogether; and there would be two points, on opposite sides of the cell, where four surfaces would meet, producing unstable tetrahedral angles. Such tetrahedral angles, made by the vertical bisection of a quadrilateral and of the hexagon above or below it as the case may be, would be the meeting place of two quadrilaterals side by side and two pentagons side by side, and precisely this grouping is seen in one of the models, in which the four surfaces meet at a salient point. Another model, being that of the largest cell studied, illustrates the entire configuration under discussion. Its upper surface is octagonal, and the two added borders are clearly the result of the bisection of two opposite sides of the original hexagon, owing to the vertical division of the adjoining cells. The two tetrahedral angles produced by these vertical divisions have slipped into pairs of closely adjacent trihedrals. Among other irregularities this cell has an additional surface — nineteen instead of eighteen — but on the whole it is remarkably close to the theoretical pattern.

Another instance of this sort is seen in Figure 21 in connection with the large cell at the left of the group. Its two facets *a* and *b* result from the subdivision of a quadrilateral surface, caused by the vertical division of an adjoining cell. The constriction following this division drew out the angle between *a* and *b* on the top surface, and contributed one of the two extra sides possessed by the octagon. It may be noted that the plane of the ridge between *a* and *b* is continued downward by the plane of a vertical division which produced the surface *c*, but in such a way that the formation of a tetrahedral angle is avoided, as the surface *b* is pentagonal instead of quadrilateral. The large cell with the octagonal top in Figure 21, is in relation with similar large flat cells both above and below, thus forming a column which could be split, by vertical division, into two columns of cells of average size, with hexagonal instead of octagonal bases and tops. But the restoration of the tetrakaidecahedral form after vertical divisions is a complex process of readjustment, which cannot be explained by any simple scheme.

Thus far cells have been considered individually or in pairs. As a concluding observation, the mutual relations of cells in a group of seven (Figure 21) may be compared with those of a cluster of orthic tetrakaidecahedra shown in Figure 20. The pattern-group needs no explanation other than to note that the lowest cell in the midline has been cut in halves transversely. It could not, in wax, be made to constrict at its plane of division, so that it presents in the figure relatively a much larger upper surface than would occur in actual cells. The relative heights of the top surfaces of the cells in this pattern should be carefully observed. That of the central cell is at the lowest level. The distance from the central cell to the top of the half-cell in the midline below may be described as a half-step. Thence it is a half-step further up to the top of the cells on either side of it, which rise above the central cell by the width of a square. The distance from these to the tops of the cells seen above them in the figure, is a full-step, for these cells rise above the central cell by the width of a hexagon. They mark the highest plane in the drawing. Finally it is a step down from them to the top of the cell in the midline above.

The relative levels of the upper surfaces of all the actual cells shown in Figure 21 are like those in the pattern, Figure 20. Eighteen of the surfaces appearing in Figure 21 have been marked with figures indicating the number of their sides, and they correspond, both in respect to these numbers and in their relations to one another, with the eighteen surfaces similarly marked in the group of orthic tetrakaidecahedra.

The problem which it was undertaken to solve, by presenting actual examples for inspection, is that of the shape of cells when surrounded on all sides and compressed by similar cells. Elder-pith was selected as presumably typical of this condition, and so far as elder-pith is concerned, the cells are shown to be tetrakaidecahedra, modified, especially through cell-division, in rather definite ways. The surprisingly simple manner of restoring the tetrakaidecahedral form after the transverse division of an entire group of cells, was brought to light by the direct observation of the models. It is a pleasure to acknowledge the way in which the mathematicians and physicists have anticipated our principal conclusion; yet in such shapes as we have found, they may not recognize the verification of their calculations and experiments.

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EXPLANATION OF PLATES.

PLATE I.

FIGURE 3.	A group of orthic tetrakaidecahedra.
FIGURES 4, 5, and 6.	Three views of a model of cell *e* in Figure 2, having fifteen surfaces. × 275 diam.
FIGURE 7.	Model of cell *a* in Figure 2, having fifteen surfaces. × 333 diam.

PLATE II.

FIGURES 8 and 9.	Orthic tetrakaidecahedra utilized to show the restoration of the original form following transverse division.
FIGURE 10.	Model of a pair of cells, evidently resulting from a transverse division, lettered to correspond with Figures 8 and 9. The upper cell has 16 contacts, the lower, 13, being cells *b* and *c* of Figure 2, inverted. × 266 diam.
FIGURES 11 and 12.	Orthic tetrakaidecahedra divided vertically through the angles, and through the sides, respectively, of their top and basal surfaces.
FIGURE 13.	The right half of Figure 12, for comparison with Figure 14.
FIGURE 14.	Model of a cell with 11 surfaces, produced from a tetrakaidecahedron by vertical division. × 333 diam.
FIGURE 15.	Model of a cell with 18 surfaces, which through transverse division, accompanied by that of certain adjoining cells, would produce conditions similar to those in Figure 16. × 250 diam.
FIGURE 16.	Model of a pair of cells, evidently resulting from a transverse division, showing atypical features explained in the text. × 225 diam.
FIGURE 17.	Model of a pair of cells resulting from an unequal transverse division. × 333 diam.
FIGURE 18.	Portion of an orthic tetrakaidecahedron for comparison with Figure 19.
FIGURE 19.	Model of a small flattened cell having six surfaces. × 270 diam.

PLATE III.

FIGURE 20.	A group of orthic tetrakaidecahedra for comparison with Figure 21.
FIGURE 21.	A group of seven cells (the central cell being *d* of Figure 2) showing 18 surfaces which correspond in position and the number of their sides with those bearing numerals in Figure 20. × 260 diam.

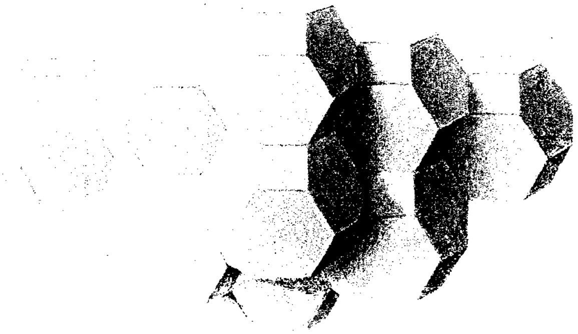


Fig. 3

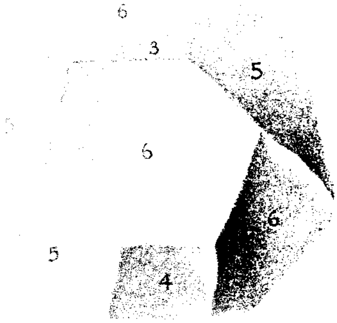
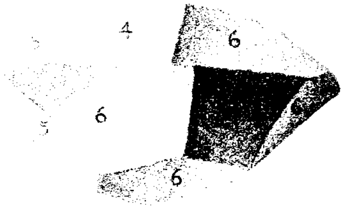


Fig. 4



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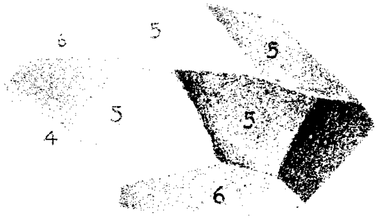

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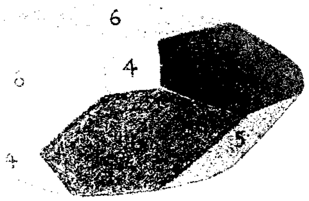

Fig. 7

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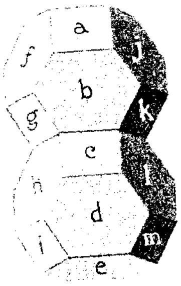
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Fig. 8

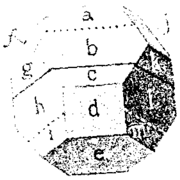
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Fig. 9

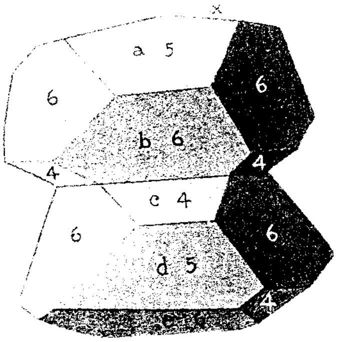
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Fig. 10

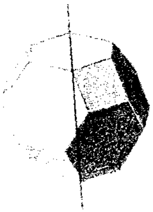

Fig. 11

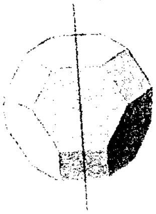

Fig. 12

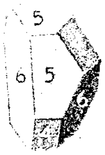

Fig. 13

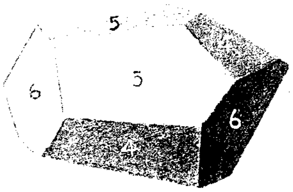

Fig. 14

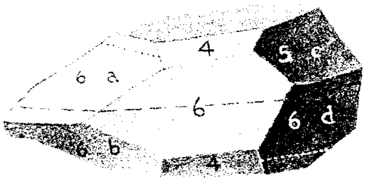
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Fig. 15

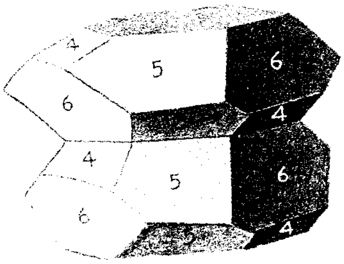
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Fig. 16

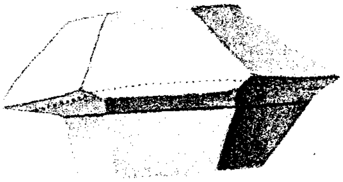

Fig. 17

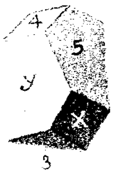

Fig. 18

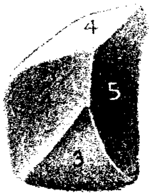

Fig. 19

F. S. M. Del.

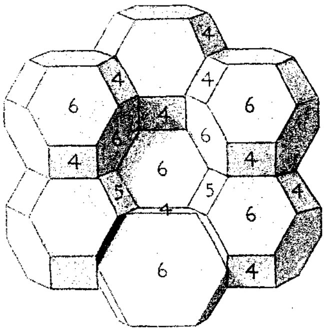
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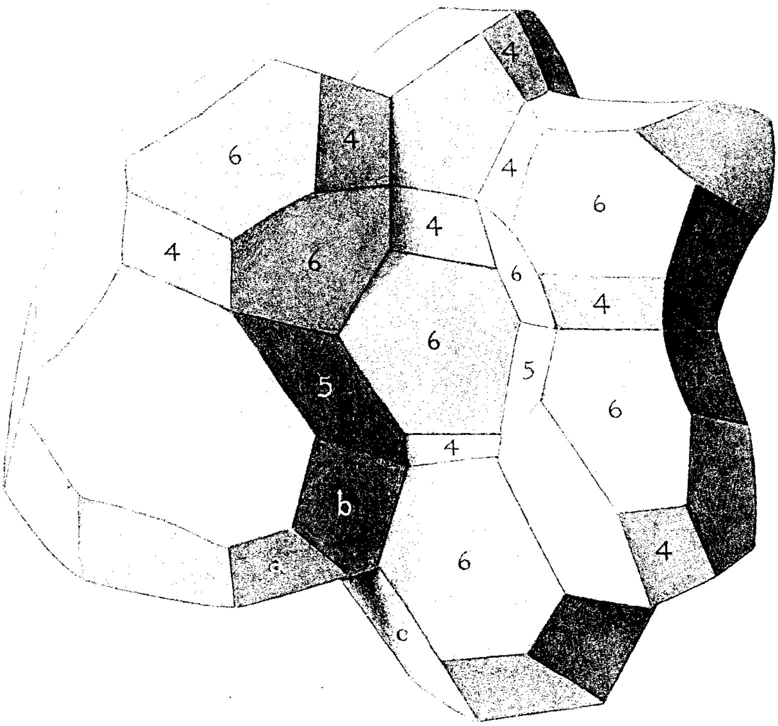
Fig 20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c

Fig. 21

F. S. M. DEL.

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THE EFFECT OF PRESSURE UPON OPTICAL ABSORPTION.

By Frances G. Wick.

Investigations on Light and Heat, made and published with aid
from the Rumford Fund.

THE EFFECT OF PRESSURE UPON OPTICAL ABSORPTION.

By Frances G. Wick.

Presented by P. W. Bridgman. Received March 22, 1923.

The effect of pressure upon various optical properties of matter has been the subject of considerable research, a very good account of which is given in a paper by Wahl.[1] Experimental difficulties have greatly limited the results obtained in high pressure optical work since the pursuit of problems along this line involves the construction of apparatus having transparent windows which will stand high pressure without fracture and which are mounted in such a way as to prevent leaks. Little work has been done at pressures above 1000 atmospheres although Wahl occasionally succeeded in reaching a limit of 4000 atmospheres.

The work described in this paper was undertaken at the suggestion of P. W. Bridgman and it was done with pressure apparatus designed by him and manipulated under his supervision in the Jefferson Physical Laboratory, through the courtesy of Harvard University. Experiments were made upon the absorption of light in solutions and in solids up to pressures of 3500 atmospheres. The difficulties encountered were many and the results are principally qualitative but they appear to be of sufficient interest to warrant publication at this time since this is a field which has been left practically unexplored.

Apparatus.

The part of the apparatus used for this work which has to do with the production, transmission, and measurement of pressure is similar to that commonly used by Bridgman[2] in different kinds of pressure experiments. The only parts of the outfit peculiar to this particular work were the observation chamber, in which the specimen under consideration was subjected to pressure, and the optical system used for the study of the absorption of light. The general arrangement of the apparatus is shown in Figure 1. The specimen, S, was mounted at the bottom of an observation chamber, O, made by drilling a hole $\frac{11}{16}$ of an inch in diameter and 7 inches deep along the axis of a steel cylinder 8 inches in diameter and $8\frac{1}{2}$ inches high. Light from a small automatic carbon arc, A, made parallel by a lens, L, was transmitted through transverse openings in the cylinder in which were mounted glass windows at G, G. A Hilger spectrograph of the constant deviation type was so placed that the collimator, C, was in line with the light transmitted through the specimen, S. A cadmium spark, N, was used as a comparison source, and a movable mirror, M, was used to reflect this light into the collimator.

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[1] Wahl, Phil. Trans. Royal Soc. London, Vol. 212A, p. 117.

[2] Bridgman, Proc. Am. Acad., 47, p. 321, 1911–12; 49, p. 627, 1913–14.

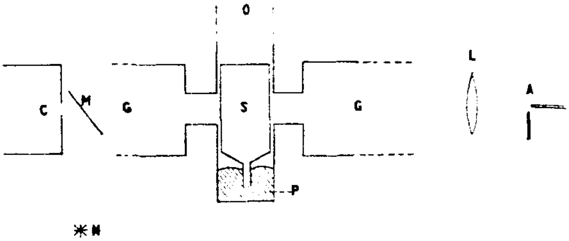


FIGURE 1.

For the study of liquids, the specimen, S, was contained in an inverted cylindrical glass bottle, as shown in Figure 1, with a narrow neck which dipped into mercury, P, to prevent the mixing of the oil through which the pressure was transmitted, with the solution under observation. When the apparatus was used for the study of solids, the specimen was placed in a brass holder mounted in position S.

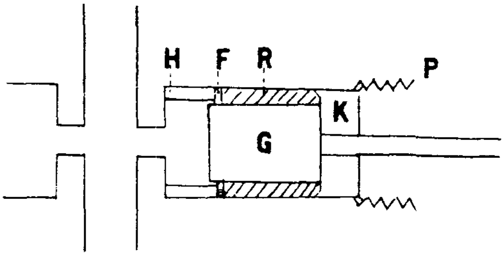


FIGURE 2.

The details of mounting the glass windows are shown in Figure 2. The cylindrical windows were cut from a piece of plate glass $1\frac{1}{4}$ inches thick and were so mounted that a rubber washer, R, a short piece of pure rubber tubing, filled the space between the glass and the steel. The washer was wrapped with fine thread after it was put on the windows in order to make it possible to insert the window and the washer into the opening. The thread breaks upon tightening the packing when the plug is screwed in at P. A thin fibre washer, F, placed at the end of the steel bearing, H, served as a guide for the window. A piece of thin paper was placed between the end of the window and the steel bearing, K, which was held in place by the plug screwed in at P. A hole $\frac{3}{16}$ of an inch in diameter through the center of the bearing and the plug allowed for the transmission of light.

Pressure was supplied by a hand pump of 1000 kg per sq. cm. capacity connected to an intensifier having a piston ratio of 4 to 1. The intensifier served to separate the pump liquid from the clear oil in the optical parts. A manganin pressure gauge was inserted between the intensifier and the plug at the top of O, Figure 1, through which connection was made with the observation chamber.

PROCEDURE.

The observation chamber, O, Figure 1, was filled with neutral white oil, the medium used for the transmission of pressure, and air bubbles were removed by means of a vacuum pump. A steel container holding the mounted specimen was lowered into position S, and the plug through which connection was made to the intensifier was screwed into the top of O. The source of light, A, the lens, L, and the collimator of the spectroscope, C, were then placed accurately in line with the windows, G G and the specimen, S.

The spectrum of the light transmitted through the specimen was photographed, first, at normal pressure, then, after the pressure was raised. By use of a divided slit, the two exposures were made adjacent to each other and the comparison spectrum was photographed on the edge of each without any change in the adjustment of the apparatus. For most of the work Wratten and Wainwright panchromatic plates were used but for the study of absorption bands of greater wave-length than 7000 Ångstrom units, Seeds 26 plates dyed with Eastman red-sensitive dye were prepared. Simultaneously with the taking of the picture of the absorption spectrum, the corresponding pressure was determined by measuring the change in resistance of the manganin gauge.

DIFFICULTIES.

Glass is a very unreliable substance when subjected to high pressure and the breaking of the windows of the apparatus caused considerable trouble. It was difficult to mount the windows in such a way that they should not become cramped against some part of the steel when the pressure was raised. This was especially likely to happen if the ends were not accurately perpendicular to the axis, which was doubtless the case with some of the windows used since they were made by grinding cylindrical pieces from thick plate glass such as is used in portholes of ocean liners. The curved surfaces were finished off in the lathe and made true but no attempt was made to grind and polish the faces of the windows. The final mounting, shown in Figure 2, seemed reasonably safe. The fibre washer, F, slightly wider than the shoulder, H, was inserted to prevent the crowding of the glass against this shoulder in case the outer face of the window was not exactly perpendicular to the axis.

A second difficulty arose from the fact that, at high pressures, the rubber of the washers dissolved in the neutral white oil used in the observation chamber. This caused a diminution in the intensity of the transmitted light which took place gradually after the application of pressure and caused considerable trouble at pressures above 2000 atmospheres. When the apparatus was allowed to stand at pressures of this order, the absorption of light continued to increase, a difficulty which proved very serious since the work was done photographically and red-sensitive plates which required long exposures were used. Upon relieving the high pressure after it had been maintained for some time, the light transmitted through the specimen came back to almost its original brightness at a pressure of approximately 2000 atmospheres. After the pressure had been run up and down several times, the intensity of the light transmitted at normal pressures was somewhat diminished and, upon removal of the specimen from the observation chamber, the oil next to the windows was found to be cloudy, due to the gradual disintegration of the rubber washers which had to be replaced from time to time to prevent leaks. Different varieties of oil were tried and a heavy variety called "Nujol" was found to cause the least trouble. By changing the oil in the observation chamber frequently the difficulties due to lack of transparency in "Nujol" were largely eliminated.

A comparison of the relative intensities of light transmitted by a given specimen at different pressures is a matter of interest but no such comparison can be made with the apparatus so constructed as to bring rubber washers in contact with oil at high pressures. If this source of trouble could be removed, it would then be desirable to substitute for the arc a more constant source of light. Nitrogen filled tungsten lamps, the "pointolite" and a tungsten arc kindly furnished by the General Electric Company were tried for this work but were found unsatisfactory because of the increase in the time of exposure required with a less intense source of light. This made the difficulty due to the gradual dissolving of the washers more marked.

Selection of Material.

The work was begun upon substances having well-marked absorption bands, a number of which were known to show some change in the bands upon variation in temperature or concentration, since it seemed desirable to make a comparison of any possible pressure effect with variation in absorption already known to be due to other agencies. After a preliminary study of various absorbing materials, the following were selected for study:

Aqueous solutions of salts of the rare earths neodymium, praseodymium, erbium and "didymium."

Aqueous solutions of the uranyl salts.

Aqueous solutions of cobalt chloride.

A synthetic ruby.

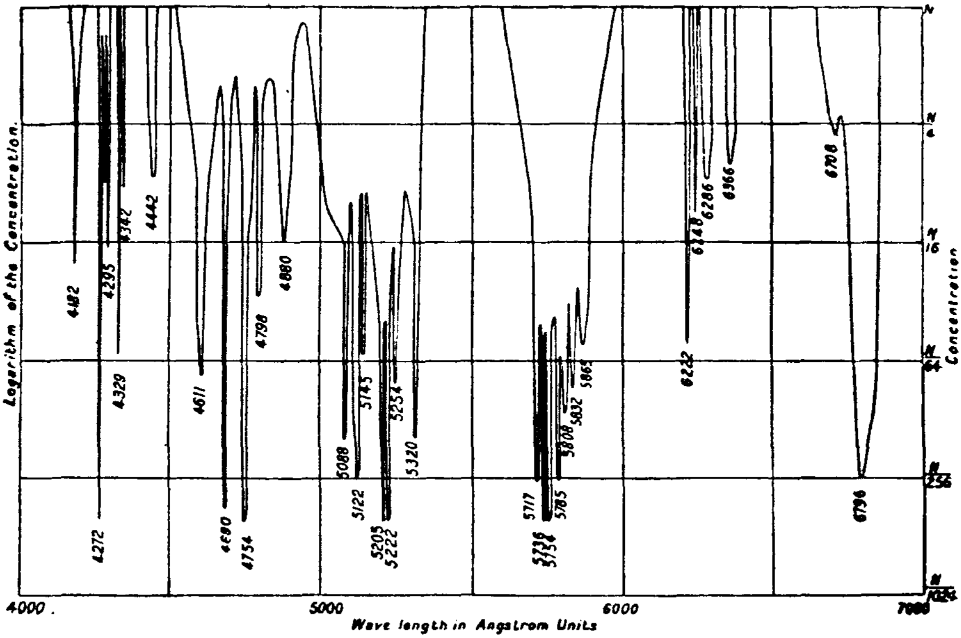
Samples of colored glass.

Neodymium Solutions.

The absorption spectra of aqueous solutions of neodymium salts are known to be rich in bands of unusual sharpness which are much alike for dilute solutions of different salts. Neodymium ammonium nitrate, obtained through the kindness of the Welsbach Company, was the salt used for most of this work but a number of observations were also made upon the nitrate, chloride, and acetate formed from pure neodymium oxalate obtained from Professor James of New Hampshire University.

As is well known, the bands are not all visible under the same conditions but different ones may be brought out by varying the thickness and concentration of the absorbing layer. In these experiments the thickness of the absorbing specimen was limited by the size of the observation chamber to about 1.4 cm. and it was possible to vary only the concentration of the solution in order to bring out different bands. The exact concentration of solution used was not known except in the case of neodymium ammonium nitrate solutions which varied from .0698 to .693 normal.[3]

The position and width of the absorption bands in dilute solutions of neodymium salts for different concentrations are shown in Figure 3 taken from the work of Rech[4] and copied directly from Little's[5] book. Along the horizontal axis, the width of the various absorption bands in Ångstrom units is indicated for different concentrations which are



Absorption spectrum of aqueous neodymium chloride (Rech; thickness of absorbing layer = 20 cms.).

FIGURE 3.

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[3] The term "normal solution" is used to indicate one that contains the same number of grams of solute per liter of solvent as the number which represents the molecular weight of the salt dissolved.

[4] Rech, Zeits. Wiss. Photochem., 1905, 3, 411.

[5] Little, Text-book of Inorganic Chemistry, Vol. IV, p. 290.

shown on the vertical axis. The figure shows nothing with regard to relative intensities of bands but it gives a good idea of the absorption bands present in different concentrations. For most of the high pressure work, a .2 normal solution was found to be most satisfactory which, with the thickness of about 1.4 cm. of cell used in the apparatus, gave an absorption spectrum comparable with that of an N/64 solution 20 cm. thick as shown in Figure 3.

An increase of pressure was found to produce changes in the appearance of the absorption spectra of neodymium solutions which were observed both visually and photographically but which, on account of the low dispersion of the spectrograph used, are not of sufficient magnitude to reproduce satisfactorily. In general, the following changes in the absorption spectra accompany an increase of pressure:—

1.— Some bands show a marked change with pressure increase while others show little or no effect. The bands most affected are the group between 5717 and 5865, the doublet 5205 and 5222, and the sharp blue band 4272. (The first group of these bands will be referred to hereafter as band 5800, and the doublet will be referred to as band 5200.)

2.— The effect of increase in pressure is to narrow and intensify the bands and to cause the component bands of a group to become more distinctly resolved. The narrowing of the bands is usually unsymmetrical, most of the reduction in width taking place on the red side.

3.— Some bands which are weak and diffuse at normal pressure are brought out at high pressures.

4.— An increase in pressure causes a slight shift in the position of certain bands in the red end of the spectrum toward the violet and a shift of the sharp blue band, 4272, toward the red.

Neodymium Ammonium Nitrate.

A very sharp absorption band at 7324,[6] not shown in Figure 3, which came out only on plates dyed to make them red-sensitive, was shifted toward the shorter wave-lengths upon application of 3500 atmospheres pressure. The width of this band is about 20 [7] Ångstroms and the shift amounts to from 3 to 6 Ångstroms. This band is sharply defined at normal pressure and its appearance is not perceptibly changed by pressure.

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[6] The wave-length of this band is taken from Little's Text-book of Inorganic Chemistry, p. 288.

[7] The values given for the width, narrowing, and shift of absorption bands are only approximate since the poor definition of some of the bands and the

The most conspicuous effect of pressure is that shown upon the prominent group of bands at 5800. In the photographs taken at normal pressure, this group appears as a single band showing very little resolution, which terminates sharply on the violet side but shades off gradually toward the red. The effect of pressure is to narrow and intensify the components of the group so that they appear more sharply defined and distinctly separated in the central and short wave-length end of the band, and to narrow the whole band by cutting off or weakening the long wave-length end. In some concentrations, the component of longest wave-length is practically suppressed by pressure. There is also a slight shift of the violet side of this group toward the shorter wave-lengths. In a .2 normal solution a pressure of 1700 atmospheres causes a shift of approximately 5 Ångstroms on the violet side of this band and a narrowing of about 20 Ångstroms on the red side.

The double band at wave-length 5200 having a width of about 25 Ångstroms in a .2 normal solution has its components decidedly sharpened and intensified by the application of a pressure of 1700 atmospheres. There also appears to be a very slight shift of the violet side of this band toward the shorter wave-lengths, amounting to 1 or 2 Ångstroms, together with a narrowing on the red side of 3 or 4 Ångstroms.

The fine blue band at 4272 was made narrower and sharper by pressure and it was also intensified and shifted toward the red, the opposite direction from the shift observed in the case of the red bands. Jones and Strong[8] have described this band as varying in appearance in different solutions, sometimes appearing double and sometimes single. In the more concentrated solutions observed in this work, this band appears double with the dimmer component on the side toward the longer wave-lengths. This component is weakened or caused to disappear by an increase in pressure while the stronger component is sharpened and intensified. The total width of the band in a solution .287 normal at ordinary pressure is about 15 Ångstroms. The effect of a pressure of 1500 atmospheres is to narrow the red side

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uncertainties arising from differences in exposure of the plates make exact measurements impossible.

[8] Jones and Strong, Carnegie Publication 130, p. 84.

about 8 Ångstroms, to shift the violet edge about 2 units toward the longer wave-lengths, and to reduce the width of the whole band to about 5 units.

Bands 5088 and 5122 are somewhat narrowed by pressure and the separation between them is made more distinct. 4690, 4611 and 4329, are slightly narrowed and intensified by pressure but the effect is very small. The group of red bands, 6730,[9] 6790, and 6890, appearing only in the more concentrated solutions, is made more distinct by pressure.

In the neodymium ammonium nitrate solutions, band 4754, shown in Figure 3, did not appear. Two broad, hazy bands not shown in this figure, at wave-lengths 4410 to 4465 and at 4813, were observed which were evidently due to an impurity of praseodymium. No decided effect upon either of these bands could be noticed when the pressure was raised.

Neodymium Nitrate.

Solutions of neodymium nitrate, prepared from pure neodymium oxalate obtained from James, show a pressure effect which is similar to that observed in the ammonium nitrate. An excess of free acid seems to increase the effect. The band at 5800 which, at normal pressure shows itself to be made up of several poorly defined components, is resolved and intensified by a pressure of 2500 atmospheres. It is made narrower on the red side and the components of the group which are of longest wave-length disappear at this high pressure. The band at 5200 is sharpened and intensified and the band at 4272 which is not so sharp as in the ammonium nitrate, is made narrower and slightly shifted toward the red.

Neodymium Chloride.

The effect of pressure upon neodymium chloride is similar to that observed in the nitrate but much smaller. The components of the groups at 5800 and 5200 were more distinct at normal pressures than in the nitrate but the effect of pressures from 1500 to 1800 atmospheres was exceedingly small. Concentrations were used which were greater than those of the other salts and bands appeared between 6222 and 6366 which did not show in the nitrate solutions. These were slightly intensified by pressure and the red bands, 6730, 6790, and 6890 were also intensified and slightly shifted toward the violet.

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[9] Wave-lengths taken from Little,— Text-book of Inorganic Chemistry, Vol. IV, p. 288.

Neodymium Acetate.

The absorption of neodymium acetate[10] differs from that of the other salts in that the bands are shifted toward the longer wavelengths and are much more diffuse. Single, unresolved bands appear instead of the groups of narrower bands seen in the other salts at the corresponding concentrations. A pressure of 1440 atmospheres produces a slight intensification of the band at about 5800 and separates it more distinctly into its two components.

"Didymium" Nitrate.

Didymium is a mixture of praseodymium and neodymium and the spectrum of the nitrate is similar to that of the neodymium ammonium nitrate and shows the same pressure effects. In the concentrations used, aqueous solutions of didymium nitrate show very distinctly the two components of band 4272 at normal pressure. At a pressure of 1700 atmospheres, the dimmer component practically disappears and the stronger one becomes sharper and more intense.

EFFECTS OF CHANGES IN TEMPERATURE AND CONCENTRATION UPON NEODYMIUM ABSORPTION SPECTRA.

Effects very similar to those due to increase of pressure have been observed as a result of the lowering of the temperature of neodymium solutions.[11] Jones and his colleagues, in an extensive series of investigations of aqueous solutions between 0° and 190°, found that all absorption bands have a tendency to become more diffuse at high temperatures but that no marked change in the appearance of the bands takes place except in the cases of 4272 and 5800, both of which remain fixed on the violet edge and widen on the red edge diffusely with rise in temperature. These two bands are the ones most changed by an increase in pressure and it is observed that the effect of raising the pressure is to produce an unsymmetrical narrowing similar to that produced by a reduction of temperature. It is known from the work of Becquerel,[12] Du Bois and Elias[13] and others, that the absorption bands of solutions and of crystals of the rare earth salts become finer and increase in intensity with lowering of temperature and are remarkably narrow at the temperature of liquid air. Becquerel states that the width of the absorption bands in some substances varies proportionally with the square root of the absolute temperature and that the total amount of energy absorbed at low temperatures is greater than the total amount absorbed at high temperatures.

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[10] Jones and Strong, Carnegie Publication, 130, p. 79.

[11] Jones and Strong, Carnegie Publication 130, pp. 72, 77, 83. Jones and Guy, Carnegie Publication 190, p. 9.

[12] Becquerel, Phil. Mag., Vol. 16, p. 153, 1908.

[13] Du Bois and Elias, Ann. der Phys., Vol. 27, p. 233.

Marked changes are also produced in the absorption spectra of neodymium solutions by changes in concentration as may be observed from Figure 3. Bands 5800, 5200, and 4272, the ones most affected by variations in pressure or temperature, are also most influenced by changes in concentration. These bands are resolved and narrowed unsymmetrically by dilution much as they are by either increasing the pressure or lowering the temperature. The resemblance between the effects of pressure and dilution upon band 4272 is especially marked as may be observed by comparing the pressure effect described in this paper with the effect produced by dilution as described by Jones and Anderson.[14]

The changes produced in the neodymium absorption spectra by increasing the pressure, lowering the temperatures or by dilution are similar to those which have been observed by Nichols and Howes [15] in the uranyl salts upon lowering the temperature to that of liquid air. The absorption bands of these salts occur in groups which are poorly resolved at ordinary temperatures. The component bands of the groups are sharpened and intensified by reduction in temperature and the envelope of the individual groups, as well as the envelope of the groups taken as a whole, is pinched off more on the red side than on the violet side. Some bands which appear on the red side of the group at 20° are weakened or caused to disappear entirely by a reduction of temperature to −180°. In general, the effect of cooling is that the absorption spectrum, considered as a unit, suffers a narrowing on cooling which is more marked toward the red than toward the violet end.

PRASEODYMIUM SALTS.

The absorption bands of praseodymium salts are wide and poorly defined and the effect of pressure upon the solutions observed in this work, the sulphate and the nitrate, was found to be very small. A solution of praseodymium sulphate, .083 normal concentration, made from material furnished by the Welsbach Company, showed little change when subjected to a pressure of 1700 atmospheres. A nitrate solution made from pure praseodymium oxalate furnished by James, showed a narrowing and intensifying of the components of the broad band with crests at 5882 and 5964. Bands at 4813 and 4688 appeared to have their red sides shifted slightly toward the red but these bands are not sharp and the effect is doubtful.

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[14] Jones and Anderson, Carnegie Publication 110, p. 88.

[15] Nichols and Howes, Carnegie Publication 298, Ch. V, pp. 61, 79.

From observations made by other observers, it is known that the absorption bands of praseodymium solutions change very little with variations in temperature and concentration in comparison with the changes which occur in solutions of the neodymium salts.

Erbium Nitrate with Yttrium.

Erbium nitrate with traces of yttrium in aqueous solution made from material furnished by James, shows an absorption spectrum in which some of the bands are very narrow and sharp. The effect of a pressure of 1500 atmospheres was found to be so small that it appeared uncertain as to whether or not any change took place. Bands 6535 and 6490 and the very narrow crest at 5231 appeared to be narrowed and intensified by pressure.

Uranyl Salts.

Solutions of the uranyl salts were suggested by E. L. Nichols as substances upon which to try the effect of pressure. The absorption bands in aqueous solutions of uranyl potassium sulphate and uranyl ammonium chloride appeared to be made somewhat more distinct by a pressure of 1500 atmospheres but the effect was not pronounced.

Cobalt Chloride.

Aqueous solutions of cobalt chloride are well known to be sensitive to variation in temperature, changing from a purplish-red color at room temperature to a decided blue at higher temperatures in the more concentrated solutions. As a result of repeated observations of the absorption spectra of solutions varying from .2 to 1.6 normal, at pressures up to 1500 atmospheres, the conclusion may be drawn that the effect of increasing the pressure is similar to that of lowering the temperature. There is a broad absorption band in the middle part of the visible spectrum with transmission bands at both ends, the width of this absorption band varying with the concentration and thickness of the solution used. Some of the photographs of the spectra of these solutions showed that, upon increase of pressure, the transmission in the red end of the spectrum was increased and that in the blue end was decreased causing the color of the transmitted light to become more red at higher pressures as it does at lower temperatures, the greatest change taking place in concentrated solutions. The wide, diffuse absorption band having its center at about 5200 was observed to change its limits with increase of pressure but no quantitative measurements could be made on account of difficulties already mentioned in connection with the apparatus.

GLASS.

Samples of glass made at the Corning Glass Works, some of which were known to change color with variation in temperature, were obtained through the kindness of Dr. H. P. Gage. The effect of pressure upon these specimens was such as to produce a change in color in some but not in others.

No observable effect was produced upon the absorption spectra of canary glass, a blue-green variety marked 4102[16] or upon "Didymium" glass by a pressure of 1400 atmospheres. In the last mentioned, the didymium bands were similar to those observed in concentrated solutions of the salts of this substance and such solutions did not show the marked change with pressure which was observed in the more dilute solutions.

Orange glass, G 34, was observed, at a pressure of 3700 atmospheres, to become more yellowish in color due to a shifting of the transmission band which is in the red end of the spectrum, toward the shorter wave-lengths. Orange glass, G 36, has a similar color change, becoming more yellowish under 1300 atmospheres pressure The single transmission band is cut off more on the red side than on the violet side by an increase of pressure. A red glass, G 20, has a much stronger transmission band at high pressures than at low and the band is extended toward the shorter wave-lengths. That the intensity of the transmitted light is increased at high pressure can easily be observed both visually and photographically.

A comparison of the changes in absorption due to pressure with those due to variation in temperature shows that an increase of pressure upon at least three specimens of the glass produces an effect which is similar to that produced by lowering of temperature. Gibbs,[17] in his work upon colored glass at high temperatures, found that the effect of heating is, in most cases, to increase absorption and to shift the absorption bands toward the longer wave-lengths. Gibson,[18] in a study of colored glass at low temperatures, found that the transmission becomes greater at low temperatures and that absorption bands are generally shifted toward the violet. A specimen of orange glass was found by him to become more yellow at −180° and a brilliant red at 430°. A similar increase of transmission and shift of absorption bands toward the red with increase of pressure has been described in the case of glasses of red and orange color, G 20, G 34, and G 36.

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[16] The numbers given are those sent by Dr. Gage with the different specimens of glass.

SYNTHETIC RUBY.

A synthetic ruby 2.8 mm. thick kindly furnished by E. L. Nichols, showed a small change in absorption at pressures up to 1600 atmospheres. Absorption bands with centers at 6688, 6593, 4768, and 4687 were slightly intensified by pressure and the characteristic close doublet which, according to Gibson,[19] has its components at 6943 and 6928 at room temperature and looks practically like a single band at room temperature and normal pressure, is slightly sharpened by pressure so that the components are more distinct and the band appears to be shifted by a very small amount toward the violet.

DuBois and Elias,[20] Mendenhall and Wood,[21] and Gibson[22] have investigated the effect of decrease of temperature upon absorption in the ruby and they all found the narrow red doublet to be sharpened and more distinctly resolved. It is also shifted toward the violet about 9 Ångstroms with lowering of the temperature to −180 degrees. Numerous absorption bands of wave-length between 6950 and 7100 were observed by DuBois and Elias and by Mendenhall and Wood at liquid air temperatures. Gibson did not observe these bands but his results show two bands at 5905 and 5965 breaking off from the broad absorption band at these low temperatures and also a change in the position of the maximum of this absorption band.

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[17] Gibbs, Phys. Rev., Vol. 31, p. 463, 1910.
[18] Gibson, Phys. Rev., Vol. 7, p. 194, 1916.
[19] Gibson, Phys. Rev., Vol. 8, p. 38, 1916.
[20] Du Bois and Elias, Ann der Phys., Vol. 27, p. 233; Vol. 35, p. 617.
[21] Mendenhall and Wood, Phil. Mag., Vol. 30, p. 316.
[22] Gibson, Phys. Rev., Vol. 8, p. 38.

The effect of pressure upon the ruby is similar to that of a lowering of temperature in that the characteristic red doublet is sharpened and shifted toward the violet by both agencies. An increase in pressure did not, however, bring out the bands which are referred to by the investigators mentioned as coming out at low temperatures.

SUMMARY.

A comparison of the pressure effects with those due to other agencies shows that, in general, the changes which take place in the absorption spectra upon increase of pressure are similar to those which take place upon lowering the temperature or decreasing the concentration. The similarity of these three effects, especially in the case of neodymium solutions, is very marked in that the same absorption bands are affected in the same way, becoming narrower and sharper unsymmetrically. There are, however, some differences in the effects due to these different sources. There is a slight shift in the position of the neodymium bands upon increase in pressure which does not take place upon lowering of temperature or decrease of concentration. This shift is, as has been mentioned, toward the red in most cases but toward the violet in the case of band 4272. The only neodymium solutions which show appreciable shift with rise in temperature are those [23] to which some dehydrating agent such as calcium chloride has been added. It is also observed that bands barely visible under normal conditions are brought out more strongly by either increase of pressure or lowering of temperature but this effect is not produced by dilution. In order to bring more bands into prominence by change in concentration, solutions of neodymium salts must be made more concentrated rather than less as may be observed from Figure 3.

It is not possible, from the observations made in this work, to make any exact quantitative comparison of the effects upon the absorption spectra due to different agencies but a study of the photographs of the absorption spectra of neodymium ammonium nitrate taken at different pressures, temperatures, and concentrations makes possible a rough estimate of relations. The lowering to the temperature of liquid air of a thin layer of concentrated solution of neodymium nitrate giving an absorption spectrum at room temperature similar to that observed in some of the solutions used for pressure work, produced a much greater effect in sharpening and intensifying the bands than that which was observed in the highest pressures used in these experiments. The effect of a pressure of 3500 atmospheres was about equal to the effect due to the lowering of temperature which took place when the cooled specimen was supported in a Dewar flask above liquid air and attached to a metal strip, the lower end of which dipped into the liquid air. The temperature of this specimen was not measured but a rough estimate indicates that it was at least 60 below zero centigrade. A comparison of the pressure and concentration effects shows that an increase of 1700 atmospheres pressure upon a .2 normal solution of a neodymium nitrate is approximately equivalent to a reduction of the concentration of this solution to one half its value, that is, to making the concentration of the solution .1 normal.

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[23] Jones and Strong, Am. Chem. Jour., Vol. 43, p. 130.

The changes observed in the absorption spectra upon increase of pressure cannot be explained by any actual temperature or concentration change which might take place as a result of pressure since any direct effect due to either of these sources is known to be usually in the opposite direction. Upon application of pressure, the specimen becomes warmer but the heating effect amounts to only a few degrees with the highest pressures used and this slight rise in temperature rapidly disappears in the heavy steel cylinder. If correction were to be made for it, the corrected value of the pressure effect would be larger than the observed effect. Upon application of the highest pressures used, the actual volume of the aqueous solutions was decreased about 10 per cent. If this decrease of volume under pressure is equivalent to an increase in concentration, the effect of it should be to broaden rather than narrow the bands, and correction for it should make the actual pressure effect larger than the observed effect.

The fact that such agencies as variations in pressure, temperature, and concentration affect some of the absorption bands of solutions but not others and the fact that the bands affected undergo changes which are not the same for different bands, seem to indicate that there must be more than one simple type of mechanism involved in the production of these absorption bands.

From a comparison of the absorption bands of neodymium crystals with those of the same salts in solution, it is observed that the bands in the crystals are sharper than those in the solutions under normal conditions. Since absorption bands in solutions are made sharper by pressure, it seems possible that solutions under pressure may have something approaching structure tending to make the bands more as they are in the solid state. However, until more is known about the mechanism of the absorption of light, an explanation of the change in absorption due to increase in pressure and the relation of this effect to the effects produced by lowering of temperature and decrease in concentration, confront the experimenter as difficult problems to be solved eventually by the theoretical physicist.

The work described in this paper was done during a part of the year in which the writer held the Sarah Berliner Research Fellowship. In conclusion, she wishes to acknowledge her indebtedness to Professor Lyman, Director of the Jefferson Physical Laboratory, for generously extending to her the privileges of a guest in the laboratory, and to Professor Saunders for suggestions with regard to the optical part of the work, particularly in connection with the dyeing of plates used in the deep red. She wishes especially to express her sincere thanks to Professor Bridgman for his constant interest and valuable suggestions and help during the whole course of the work.

Vassar College,
Poughkeepsie, N. Y.

Proceedings of the American Academy of Arts and Sciences.

Vol. 58. No. 17.—September, 1923.

RECORDS OF MEETINGS, 1922–23.

BIOGRAPHICAL NOTICES.

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LIST OF THE FELLOWS AND FOREIGN HONORARY MEMBERS.

STATUTES AND STANDING VOTES.

RUMFORD PREMIUM.

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(Title Page and Table of Contents.)

RECORDS OF MEETINGS.

One thousand one hundred and sixteenth Meeting.

OCTOBER 11, 1922.— STATED MEETING.

The Academy met at its House at 8.15 P.M.

The PRESIDENT in the Chair.

There were thirty-five Fellows present.

The Minutes of the Annual Meeting, of May 10, 1922, were read and approved.

Letters were presented by the Corresponding Secretary:— from W. S. Adams, Nathan Banks, T. M. Carpenter, Edward Channing, Stanley Cobb, Gano Dunn, J. L. Goodale, E. C. Kemble, George LaPiana, R. W. Lovett, William McDougall, A. K. Porter, A. C. Redfield, A. F. Rogers, P. J. Sachs, R. C. Tolman, W. H. Weston, Jr. and C. H. C. Wright, accepting Fellowship; from T. C. Allbutt, A. S. Eddington, Emmanuel de Margerie and Henri Pirenne, accepting Foreign Honorary Membership; from Thomas A. Edison declining election.

The President announced the deaths of the following Fellows:— Alexander Graham Bell, Class I., Section 2; Harold Clarence Ernst, Class II., Section 4; Henry Marion Howe, Class I., Section 4; Edward Hall Nichols, Class II., Section 4; Alexander Smith, Class I., Section 3; Williston Walker, Class III., Section 1.

The Corresponding Secretary presented the following biographical notices:— Julius von Hann, by R. deC. Ward; William Pfeffer, by D. H. Campbell; Seth Carlo Chandler, by E. S. King; Edward Hall Nichols, by E. H. Bradford.

On the recommendation of the Council, it was

Voted, To make a supplementary appropriation of $300 from the income of the General Fund, for the expenses of the Library.

The following communications were presented:—

Worthington C. Ford, "A Map of Virginia," with lantern slide illustrations.

Arthur Gordon Webster, "Real and Pseudo-Isophelematic Lines and their Use and Misuse in Geography."

The following papers were presented by title:—

"The Effect of Pressure on the Electrical Resistance of Cobalt, Aluminum, Nickel, Uranium, and Caesium," by P. W. Bridgman.

"The Compressibility of 30 Metals as a Function of Pressure and Temperature," by P. W. Bridgman.

"Note on Two Remarkable Ascomycetes," by Roland Thaxter.

Brief reports of delegates who represented the Academy at Padua, Brussels, and Paris, were then presented:—

C. H. Moore, The 700th Anniversary of the Founding of the University of Padua.

A. E. Kennelly, The 150th Anniversary of the Brussels Academy of Sciences and Letters.

J. R. Jewett, The 100th Anniversary of the Société Asiatique de Paris.

The meeting was dissolved at 10.30 P.M.

One thousand one hundred and seventeenth Meeting.

November 8, 1922.— Stated Meeting.

The Academy met at its House at 8.15 P.M.

The President in the Chair.

There were forty-two Fellows and two guests present.

The Minutes of the meeting of October 11 were read and approved.

The Corresponding Secretary announced that the Council had elected Harold Murdock, Treasurer, to fill the vacancy caused by the death of Henry H. Edes.

The President announced the deaths of the following Fellows:— John Casper Branner, Class II., Section 1; Henry Herbert Edes, Class III., Section 4; William Stewart Halsted, Class II., Section 4; Thomas Nelson Page, Class III., Section 4; and of the Foreign Honorary Member, Hermann Diels, Class III, Section 2.

A biographical notice of Harmon Northrop Morse, by Ira Remsen, was presented by the Corresponding Secretary.

The President reported that the Committee appointed to revise

Chapter III of the Statutes concerning procedure in the election of new members had made a report which would be submitted for approval by the Academy.

The Librarian made a statement regarding the Library and urged that Fellows make more frequent use of its valuable collections.

The following communications were presented:—

William McDougall, "Psychic Research: a Neglected Responsibility of Men of Science."

George F. Moore, "A Word about the Astronomical Theories of the Greeks."

The following paper was presented by title:—

"The Minimum Audible Intensity of Sound," by Clifford M. Swan, presented by P. W. Bridgman.

The meeting was dissolved at 10.15 P.M.

One thousand one hundred and eighteenth Meeting.

DECEMBER 13, 1922.—STATED MEETING.

The Academy met at its House at 8.20 P.M.

The PRESIDENT in the Chair.

There were twenty-five Fellows present.

The Minutes of the November 8th meeting were read and approved.

The Corresponding Secretary reported the gift, by Professor Charles Sprague Sargent, of copies of Volumes I and II of the Memoirs of the Academy, 1785 to 1804, in the original bindings, with the bookplate of Winthrop Sargent, elected a Fellow in 1788.

The President announced the death of Robert Wheeler Willson, Class I., Section 1.

The Council recommended to the Academy the following standing vote in regard to Resident Associates:—

Standing Vote, no. 4. The Council, under such rules respecting nominations as it may prescribe, may elect as Associates of the Academy a limited number of men of mark in affairs or of distinguished service in the community.

Associates shall be entitled to the same privileges as Fellows, but shall not have the right to vote.

The admission fee and annual dues of Associates shall be the same as those of Fellows residing within fifty miles of Boston.

It was

Voted, To lay this recommendation on the table until the next meeting, notice to be sent out.

The following communication was presented:—

S. Burt Wolbach, "Tumors: A Résumé of Facts and Theories," with lantern slide illustrations.

After the Communication, ballots were cast on the question of revising Chapter III of the Statutes, on the Election of Fellows and Foreign Honorary Members, as submitted in print.

The revision was adopted and Chapter III of the Statutes now reads as follows:

CHAPTER III

Election of Fellows and Foreign Honorary Members

Article I. The procedure in the election of Fellows and Foreign Honorary Members shall be as follows:

Nominations to Fellowship or Foreign Honorary Membership in any Section must be signed by two Fellows of that Section or by three Fellows of any Sections, and sent to the Corresponding Secretary accompanied by a statement of the qualifications of the nominee and brief biographical data.

Notice shall be sent to every Fellow not later than the fifteenth of January in each year, reminding him that all nominations must be in the hands of the Corresponding Secretary before the fifteenth of February following.

A list of the nominees, giving a brief account of each, with the names of the nominators, shall be sent to every Fellow with a request that he return the list with such confidential comments and indications of preference as he may choose to make.

All the nominations, with any comments thereon and with expressions of preference on the part of the Fellows, shall be referred to the appropriate Class Committees, which shall canvass them, and report their recommendations in writing to the Council before the Stated Meeting of the Academy in April.

Elections of Fellows and Foreign Honorary Members shall be made by the Council before the Annual Meeting in May, and announced at that meeting.

Persons nominated in any year, but not elected, may be carried over to the list of nominees for the next year at the discretion of the Council, but shall not be further continued unless renominated.

The meeting was dissolved at 10.15 P.M.

One thousand one hundred and nineteenth Meeting.

JANUARY 10, 1923.— STATED MEETING.

The Academy met at its House at 8.20 P.M.

The PRESIDENT in the Chair.

There were twenty-three Fellows and two guests present.

The Minutes of the December 13th meeting were read and approved.

On recommendation of the Council, it was

Voted, To appropriate the sum of $400 from the General Fund to be placed at the disposal of the House Committee.

The President announced the deaths of the following Foreign Honorary Members:— Friedrich Delitzsch, Class III., Section 2; Albert Venn Dicey, Class III., Section 1; Henry Jackson, Class III., Section 2; George Walter Prothero, Class III., Section 3; Eduard Seler, Class III., Section 2.

The following communication was presented:—

William J. Cunningham, "The Problem of Railroad Consolidation in New England."

Two papers were presented by title:—

"Lichenes in Insula Trinidad a Professore R. Thaxter Collecti," by Edward A. Vainio, presented by Roland Thaxter.

"Some New Fossil Parasitic Hymenoptera from Baltic Amber," by C. T. Brues.

The meeting dissolved at 9.50 P.M.

One thousand one hundred and twentieth Meeting.

FEBRUARY 14, 1923.— STATED MEETING.

The Academy met at its House at 8.25 P.M.

The PRESIDENT in the Chair.

There were twenty-three Fellows present.

The Minutes of the meeting of January 10, 1923, were read and approved.

The Corresponding Secretary reported the receipt of a biographical notice of William Stewart Halsted, by Harvey Cushing.

On recommendation of the Council, it was

Voted, To make a supplementary appropriation of $250 for the current year from the General Fund, in accordance with the action appointing Mrs. Holden, Assistant Librarian Emerita.

It was unanimously

Voted, To amend Paragraph 4 of the Standing Votes, so that it now reads:—

4. The Council, under such rules respecting nominations as it may prescribe, may elect as Associates of the Academy a limited number of men of mark in affairs or of distinguished service in the community.

Associates shall be entitled to the same privileges as Fellows, but shall not have the right to vote.

The admission fee and annual dues of Associates shall be the same as those of Fellows residing within fifty miles of Boston.

It was suggested that biographical notices in the cases of such Associates would not be required.

The President announced the death of Edward Emerson Barnard, Fellow in Class I., Section 1.

The following communications were presented:—

George LaPiana, "The Form of the Earth in the Middle Ages."

Waldemar Lindgren, "Geological Fieldwork in Arizona, Mexico, and South America," with lantern slide illustrations.

One paper was presented by title:—

"A Revision of the Atomic Weight of Silicon. The Analysis of Silicon Tetrachloride and Tetrabromide," by Gregory P. Baxter, Philip F. Weatherill and Edward W. Scripture, Jr., presented by G. P. Baxter.

The meeting dissolved at 10.25 P.M.

One thousand one hundred and twenty-first Meeting.

MARCH 14, 1923.— STATED MEETING.

The Academy met at its House at 8 P.M.

The PRESIDENT in the Chair.

There were forty Fellows and several guests present.

In the absence of the Corresponding Secretary, the Recording Secretary was requested to assume his duties.

The Transactions of the last meeting were read and approved.

The President announced the deaths of the following Fellows:— Edward Williams Morley, Class I., Section 3; John Trowbridge, Class I., Section 2; and of one Foreign Honorary Member, Heinrich Brunner, Class III., Section 1.

On recommendation of the Council, the following appropriations were made for the ensuing year:—

From the income of the General Fund, $9,500, to be used as follows:—

for General and Meeting expenses	$1,200.00
for Library expenses	2,800.00
for Books, periodicals and binding	1,200.00
for Assistant Librarian Emerita	500.00
for House expenses	2,500.00
for Treasurer's expenses (including five insurance premiums due this year)	1,200.00
at the disposition of the President	100.00

From the income of the Publication Fund, $3,368.62, to be used for publication.

From the income of the Rumford Fund, $3,832.49, to be used as follows:—

for Research	$1,000.00
for Books, periodicals and binding	200.00
for Publication	600.00
for use at the discretion of the Committee	2,032.49

On recommendation of the Council, it was

Voted, To make a supplementary appropriation from the General Fund of $300, to be used for the purchase of Books and Periodicals and for Binding, for the present year, 1922–23.

An amendment to the Statutes was proposed to bring them into conformity with the changes recently made in the Statutes and Standing Votes, striking out all references to Resident Associates. The amendment was adopted, twenty-three votes in the affirmative, none in the negative.

The President appointed the Nominating Committee as follows:

Desmond FitzGerald, of Class I.

Edward M. East, of Class II.

Kirsopp Lake, of Class III.

The following communications were presented:—

George W. Pierce, "Demonstration of Radio-telephony."

Charles T. Brues, "Insects Preserved in Baltic Amber, with remarks on other Tertiary Insects," with lantern slide illustrations.

Three papers were presented by title:—

"The Chilean Species of Metzgeria," by Alexander W. Evans.

"The Typical Shape of Polyhedral Cells in Vegetable Parenchyma and the Restoration of that Shape Following Cell Division," by Frederic T. Lewis.

"The Effect of Pressure on Optical Absorption," by Frances G. Wick, presented by P. W. Bridgman.

The meeting dissolved at 10 o'clock, and was followed by a concert by Radio-telephony.

One thousand one hundred and twenty-second Meeting.

APRIL 6, 1923.— OPEN MEETING.

A Reception was held at the House of the Academy from four to six o'clock, in honor of Sir Joseph John Thomson, of Cambridge, England, who was in this country as the guest of the Franklin Institute of the State of Pennsylvania, and was accompanied by Vice-President W. C. L. Eglin and Secretary R. B. Owens of the Institute.

There were about two hundred Fellows and guests, including ladies, present.

Tea was served in the Reception Room on the third floor.

One thousand one hundred and twenty-third Meeting.

APRIL 11, 1923.— STATED MEETING.

The Academy met at its House at 8 P.M.

The PRESIDENT in the Chair.

There were twenty-three Fellows and one Foreign Honorary Member present.

In the absence of the Recording Secretary, the Corresponding Secretary was requested to assume his duties.

The Transactions of the last meeting were read and approved.

The following communications were presented:—

Henry H. Clayton, 1. "Variations in Solar Radiation and the Weather," with lantern slide illustrations. 2. "Evidences of Tidal Strains in the Solar Atmosphere possibly due to Planetary Attraction."

William M. Davis, "The Marginal Belts of the Coral Seas."

The meeting was dissolved at 10 P.M.

One thousand one hundred and twenty-fourth Meeting.

MAY 9, 1923.— ANNUAL MEETING.

The Academy met at its House at 8 P.M.

The PRESIDENT in the Chair.

There were forty-two Fellows and several guests present.

The Transactions of the last meeting were read and approved.

The President announced the death of George Lincoln Goodale, Fellow in Class II., Section 2.

The Corresponding Secretary reported the receipt of a biographical notice of M. P. Knowlton, by Wm. C. Wait.

On recommendation of the Council, it was

Voted, That the President be authorized to appoint delegates to represent the Academy at the July meeting in London of the Royal Asiatic Society.

On recommendation of the Council, the following appropriation was made for 1923–24:

From the income of the C. M. Warren Fund, $1,072.00, for use at the discretion of the Committee.

It was

Voted, To refer to a committee for report in the autumn the following proposed amendment to the Statutes:— Addition to Chapter II, Art. 5,

"Any Fellow shall also be exempt from annual dues who has paid such dues for forty years, or, having attained the age of seventy-five, has paid dues for twenty-five years."

The following report of the Council was presented:—

Since the last report of the Council, there have been reported the deaths of fifteen Fellows: — Edward Emerson Barnard, Alexander Graham Bell, John Casper Branner, Henry Herbert Edes, Harold Clarence Ernst, George Lincoln Goodale, William Stewart Halsted, Henry Marion Howe, Edward Williams Morley, Edward Hall Nichols, Thomas Nelson Page, Alexander Smith, John Trowbridge, Williston Walker, Robert Wheeler Willson; and seven Foreign Honorary Members: — Heinrich Brunner, Friedrich Delitzsch, Albert Venn Dicey, Hermann Diels, Henry Jackson, George Walter Prothero, Eduard Seler.

Nineteen Fellows and four Foreign Honorary Members were elected by the Council and announced to the Academy in May 1922. One Fellow has declined election.

The roll now includes 572 Fellows and 64 Foreign Honorary Members (not including those elected in April 1923).

The annual report of the Treasurer, Harold Murdock, was read, of which the following is an abstract:—

GENERAL FUND.

Receipts.

Balance, April 1, 1922	$7,037.54	
Investments	4,536.34	
Assessments	3,400.00	
Admissions	130.00	
Rumford Fund for payment in 1922 in error	27.56	
Sundries	248.40	$15,379.84

Expenditures.

Assistant Librarian Emerita	$250.00
Expenses of Library	4,627.57

Assistant Treasurer	$250.00	
Treasurer's Expenses	456.89	
Expense of House	2,819.70	
General Expenses	1,099.55	
President's Expenses	80.00	
Biographical Notices	64.24	
Transferred to Publication Fund	1,500.00	
Interest on Bonds	146.54	
Income transferred to principal	367.23	$11,661.72
Balance, April 1, 1923		3,718.12
		$15,379.84

RUMFORD FUND.

Receipts.

Balance, April 1, 1922	$5,929.22	
Investments	4,131.51	
Sale of Publications	39.06	
Sale of Spectrometer	50.00	$10,149.79

Expenditures.

Research	$1,775.00	
Books, periodicals and binding	204.10	
Publications	1,317.03	
Table of Constants	200.00	
Interest on Mortgage	10.39	
Income transferred to principal	189.55	$3,696.07
Balance, April 1, 1923		6,453.72
		$10,149.79

C. M. WARREN FUND.

Receipts.

Balance, April 1, 1922	$2,402.06	
Investments	1,105.42	$3,507.48

Expenditures.

Research	$1,250.00	
Vault rent, part	3.00	
Interest on Bonds and Stock bought	70.42	
Charged to cancel premium on Bonds	150.00	
Income transferred to Principal	51.86	$1,525.28
Balance, April 1, 1923		1,982.20
		$3,507.48

PUBLICATION FUND.

Receipts.

Balance, April 1, 1922	$5,120.13	
Appleton Fund Investments	1,618.13	
Centennial Fund Investments	2,428.85	
Authors' Reprints	159.55	
Sale of Publications	600.35	
General Fund Income	1,500.00	$11,427.01

Expenditures.

Publications	$4,579.13	
Vault Rent, part	10.00	
Interest on Bonds bought	80.94	
Charged to cancel premium on Bonds	175.00	
Income transferred to Principal	207.99	$5,043.06
Balance, April 1, 1923		6,383.95
		$11,427.01

FRANCIS AMORY FUND.

Receipts.

Investments	$1,646.17

Expenditures.

Publishing statements	$62.95
Interest on Bonds bought	23.38

Charged to cancel premium on Bonds . . .	$213.75	
Income transferred to Principal	1,346.09	$1,646.17

The following Reports were also presented:—

Report of the Library Committee.

The Librarian begs to report for the year 1922–1923, as follows:—

During the year 75 volumes and 18 unbound numbers of periodicals have been borrowed by 33 persons, including 27 Fellows and 6 libraries. Many books have been consulted and used at the library. All books taken out have been returned or satisfactorily accounted for.

The number of books on the shelves at the time of the last report was 38,411. During the year, 672 volumes have been added, making the number now on the shelves 39,083. This includes 94 purchased from the General Fund, 27 from the Rumford Fund, and 551 received by gift or exchange. The number of pamphlets added was 439.

The expenses charged to the library during the financial year are:

Salaries	$3,221.25
Binding:—	
General Fund	1,049.70
Rumford Fund	54.10
Purchase of periodicals and books:—	
General Fund	327.08
Rumford Fund	122.44
Miscellaneous	29.54

The library has suffered a severe loss in the death of Miss Wyman, the new assistant librarian, who was taken ill in April 1922, and died in little over two months. The librarian desires to put on record his appreciation of the excellence of her services. After some searching a very desirable candidate was found in Mrs. W. G. Ball, who was appointed assistant librarian, from October first, and has begun her work most auspiciously. Mrs. Smith carried on during June and July, and has been placed in the status of part-time assistant. Mrs. Holden, who served us so long and faithfully, has been by vote of the Academy provided with a moderate pension.

A. G. Webster, *Librarian.*

May 9, 1923.

REPORT OF THE RUMFORD COMMITTEE.

The Committee organized on October 11, 1922, by electing Theodore Lyman to be Chairman and Arthur G. Webster, Secretary.

The following grants in aid of researches in light and heat have been made during the academic year 1922–1923:

October 11, 1922. To Professor W. W. Campbell of the Lick Observatory for the purpose of installing a coarse grating to cover the upper end of the Crossley reflecting telescope at the Lick Observatory	$475
To Professor A. G. Webster of Clark University in further aid of his researches in the Thermodynamic Problems of Interior Ballistics	500
To Professor Norton A. Kent of Boston University for the continuation of his researches on the Constitution of Spectral Lines	150
December 13, 1922. To Professor John R. Roebuck of the University of Wisconsin for the procuring of porous plugs for the Joule-Kelvin Experiment	125
February 14, 1923. To Professor Wm. Duane an additional grant (to No. 239) for the continuation of his researches on the Heat Energy of Electrons	200

Reports of Progress in their respective researches have been received from the following persons:— R. T. Birge (research finished), P. W. Bridgman, A. L. Clark, F. Daniels, A. L. Foley, P. F. Gaehr, R. C. Gibbs, H. P. Hollnagel, H. L. Howes, L. R. Ingersoll, N. A. Kent, F. E. Kester, E. Kremers, C. L. Norton, F. Palmer, Jr., J. A. Parkhurst, H. M. Randall, T. W. Richards, F. K. Richtmyer, F. A. Saunders, A. W. Smith, B. Spence, L. S. E. Thompson (research finished), O. Tugman (research finished), F. W. Very, A. G. Webster, F. G. Wick, R. W. Wood.

The following papers in the Proceedings have been published with aid from the Rumford Fund since the presentation of the last Report of this Committee:

Atmospheric Attenuation of Ultra-Violet Light, E. R. Schaeffer, Vol. 57, No. 14.

The Ratio of the Calorie at 73° to that at 20°, Arnold Romberg, Vol. 57, No. 15.

Vision and the Technique of Art, A. Ames, Jr., C. A. Proctor and Blanche Ames, Vol. 58, No. 1.

The Compressibility of Thirty Metals as a Function of Pressure and Temperature, P. W. Bridgman, Vol. 58, No. 5.

The following paper has been approved for publication: The Effect of Pressure on Optical Absorption, Frances G. Wick.

A sum not to exceed $478 was granted toward the expense of publication of a paper by A. Ames, Jr., C. A. Proctor and Blanche Ames, on Vision and the Technique of Art.

THEODORE LYMAN, *Chairman.*

May 9, 1923.

REPORT OF THE C. M. WARREN COMMITTEE.

The Committee had at its disposal at the beginning of the fiscal year 1922-23, $3,368.47. During the year ending March 31, 1923, grants to the amount of $1,250.00 were made. The balance on that date was $2,118.47.

At a meeting of the Committee held April 14, 1923, it was voted to request the Treasurer to invest $1,000.00 of the accumulated income and to add this amount to the principal of the fund.

Grants have been made during the year ending April 1, 1923, as follows:

May 4, 1922.— To Prof. Charles James, New Hampshire College, $500.00 to be applied toward an investigation of the Ytterbium earths.

May 4, 1922.— To Prof. Charles A. Kraus, Clark University, $500.00 to continue his work on the constitution of metallic substances.

Dec. 8, 1922.— To Prof. James B. Conant, Harvard University, $250.00 to be used in a research on the electrochemical study of the reversible reduction of organic compounds.

At a meeting of the Committee held April 9, 1923, the following grants were made:

To Prof. David E. Worrall of Tufts College, $250.00 to aid a research on the chemistry of mustard oils.

To Prof. R. R. Renshaw, New York University, $200.00 to be used in the study of the physiological activity of choline and its derivatives.

Reports of investigations aided by grants made by the Committee have been recently published by Professors Conant, Hibbert, James, Kraus and Willard.

A potentiometer bought with money granted by the Committee to Prof. H. H. Willard has recently been returned to the Chairman and lent to Prof. R. R. Renshaw. It is planned to have all apparatus of a general nature returned so that it can be used in a number of researches.

James F. Norris, *Chairman.*

May 9, 1923.

Report of the Publication Committee.

During the twelve months since the publication of the last report there have been printed Nos. 11–18, inclusive, of Vol. 57 of the Proceedings, and Nos. 1–8 of Vol. 58. Vol. 57 contains the largest number of papers, though not the largest number of pages printed, since the issuance of Vol. 48 (1912–13), and the current volume will be nearly as large.

Sales of publications remain at a very satisfactory figure.

The financial statement follows:

Receipts.

Balance, April 1, 1922	$6,003.69	
Appropriation	3,951.95	
Sales of publications	600.35	
Received for authors' reprints	159.55	$10,715.54

Expenses.

Engraving, printing and binding	$4,439.64	
Cartage and mailing	113.87	
Committee's expenses	15.62	$4,569.13
Balance, April 1, 1923		$6,146.41

The above figures do not include the sum of $1,317.03 received from the Rumford Committee for publication of Rumford papers.

Respectfully submitted,

LOUIS DERR, *Chairman.*

May 9, 1923.

REPORT OF THE HOUSE COMMITTEE.

The House Committee submits the following report for 1922–23:—

With the balance of $125.44 left from last year, an appropriation of $2,700, and $110 received from other societies for the use of the rooms, the Committee has had at its disposal the sum of $2,935.44. The total expenditure has been $2,929.70, leaving an unexpended balance on April 1, 1923, of $5.74. The expenditure has been as follows:—

Janitor		$925.00
Electricity	A. Light	180.98
	B. Power	69.00
Coal		1,015.48
Care of Elevator		51.50
Gas		54.93
Water		18.48
Telephone		74.02
Upkeep		39.36
Ash Tickets		31.68
Furnishings		352.45
Janitor's Materials and Sundries		116.82
Total Expenditure		$2,929.70

The amount of $110 contributed by other societies for the use of the building leaves the net expense of the House $2,819.70.

Meetings have been held as follows:—

The Academy	
Stated Meetings	8
Open Meetings	1
American Antiquarian Society	1

Archæological Institute	1
Colonial Dames	1
Colonial Society	4
Geological Club of Boston	3
Harvard-Technology Chemical Club	5
New England Botanical Club	3
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The rooms on the first floor have been used for Academy Council and Committee meetings, and also by the Colonial Society, the Trustees of the Children's Museum, and the Thursday Evening Club.

During Convocation Week of the American Association for the Advancement of Science, December 26–30, the Academy kept open house for the members of the Association, tea being served each afternoon.

Respectfully submitted,

John Osborne Sumner, *Chairman.*

May 9, 1923.

On recommendation of the Treasurer, it was

Voted, That the Annual Assessment be $10.00.

The annual election resulted in the choice of the following officers and committees:

George F. Moore, *President.*
Arthur E. Kennelly, *Vice-President for Class I.*
William M. Wheeler, *Vice-President for Class II.*
Arthur P. Rugg, *Vice-President for Class III.*
Harry W. Tyler, *Corresponding Secretary.*
Charles B. Gulick, *Recording Secretary.*
Harold Murdock, *Treasurer.*
Arthur G. Webster, *Librarian.*

Councillors for Four Years.

George F. Swain, *of Class I.*
Irving W. Bailey, *of Class II.*
Jeremiah D. M. Ford, *of Class III.*

Finance Committee.

ARTHUR LORD, THOMAS BARBOUR, PAUL J. SACHS.

Rumford Committee.

THEODORE LYMAN,
ARTHUR G. WEBSTER,
ELIHU THOMSON,
LOUIS BELL,
PERCY W. BRIDGMAN,
HARRY M. GOODWIN,
CHARLES L. NORTON.

C. M. Warren Committee.

JAMES F. NORRIS,
HENRY P. TALBOT,
GREGORY P. BAXTER,
WALTER L. JENNINGS,
ARTHUR D. LITTLE,
LAWRENCE J. HENDERSON,
FREDERICK G. KEYES.

Publication Committee.

LOUIS DERR, *of Class I.*
HERBERT V. NEAL, *of Class II.*
ALBERT A. HOWARD, *of Class III.*

Library Committee.

HARRY M. GOODWIN, *of Class I.*
THOMAS BARBOUR, *of Class II.*
WILLIAM C. LANE, *of Class III.*

House Committee.

JOHN O. SUMNER,
WM. STURGIS BIGELOW,
ROBERT P. BIGELOW.

Committee on Meetings.

THE PRESIDENT,
THE RECORDING SECRETARY,
GEORGE H. PARKER,
EDWIN B. WILSON,
EDWARD K. RAND.

Auditing Committee.

GEORGE R. AGASSIZ, JOHN E. THAYER.

The Council reported that the following gentlemen were elected members of the Academy:—

Class I., Section 1 (Mathematics and Astronomy):
  Samuel Alfred Mitchell, of University, Virginia, as Fellow.
  Oswald Veblen, of Princeton, New Jersey, as Fellow.
Class I., Section 3 (Chemistry):
  Tenney Lombard Davis, of Cambridge, as Fellow.
  Duncan Arthur MacInnes, of Cambridge, as Fellow.
Class I., Section 4 (Technology and Engineering):
  William Hovgaard, of Cambridge, as Fellow.
  James Robertson Jack, of Cambridge, as Fellow.
  William Henry Lawrence, of Boston, as Fellow.
  Samuel Wesley Stratton, of Cambridge, as Fellow.
  Edward Pearson Warner, of Cambridge, as Fellow.
Class II., Section 2 (Botany):
  Joseph Charles Arthur, of Lafayette, Indiana, as Fellow.
  Elvin Charles Stakman, of St. Paul, Minnesota, as Fellow.
Class II., Section 3 (Zoölogy and Physiology):
  Manton Copeland, of Brunswick, Maine, as Fellow.
  George Albert Boulenger, Brussels, as Foreign Honorary Member.
Class II., Section 4 (Medicine and Surgery):
  Roger Irving Lee, of Cambridge, as Fellow.
  Hans Zinsser, of Boston, as Fellow.
Class III., Section 1 (Philosophy and Jurisprudence):
  Leslie Colby Cornish, of Augusta, Maine, as Fellow.
Class III., Section 2 (Philology and Archæology):
  Henri Guy, of Grenoble, as Foreign Honorary Member.
Class III., Section 3 (Political Economy and History):
  Carl Lotus Becker, of Ithaca, New York, as Fellow.
  Michael I. Rostovtzeff, of Madison, Wisconsin, as Fellow.

The following communication was presented:—

Edwin B. Wilson, "Statistics."

The following papers were presented by title:—

"An Example in Potential Theory," by O. D. Kellogg.

"Mitochondrial Bodies in the Spermatogenesis of Chorthippus curtipennis (Scudd.)," by E. L. Mark and L. C. Wyman.

"Trajectory Surfaces and a Generalization of the Principal Directions in any Space," by Joseph Lipka.

The Meeting was dissolved at 10.10 P.M.

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JOEL ASAPH ALLEN (1838–1921).

Fellow in Class II, Section 3, 1871.

The American Academy of Arts and Sciences was the first learned society to recognize the scientific worth of Joel Asaph Allen.

The young man had come to Cambridge to study with Agassiz. He had been a member of the famous expedition to Brazil; he had made collecting trips into the Middle West, and into Florida, and had published some thirty or more scientific papers. This was fifty years ago. On the year of his election he was made "Assistant" in the Museum of Comparative Zoölogy, and assigned to work on the Plains, and in the Rocky Mountains. He did pioneer work in these regions, where the buffalo were "so numerous on one occasion that they darkened the plains as far as the eye could reach," and where he witnessed sanguinary engagements between Custer and the Sioux.

Having collected an enormous amount of zoölogical material he became absorbed in "working up" scientific collections, either those which he himself had made or which came to him because of his genial and sympathetic nature from other collectors.

Although of frail physique, his powers of concentration were most exceptional. While at work he seemed to live in another world, unconscious of those about him and absolutely preoccupied with the task he had undertaken. He labored daily to the point of exhaustion, and looked upon rest only as a means of obtaining the necessary energy for an additional day of devotion to his dominating interest.

In 1885 he left the neighborhood of Boston and of Cambridge to assume the duties of Curator of the Department of Ornithology and Mammalogy at the American Museum of Natural History. Here he rounded out and completed his life work. During his incumbency exploring expeditions were sent into all parts of America, into Europe, Asia, Africa, the Islands of the Pacific, Arctica and Antarctica. The collections of mammals and birds increased from a paltry fifteen hundred to upwards of a quarter of a million specimens. He described between five and six hundred new species of mammals, and his published scientific papers aggregated nearly fifteen hundred titles.

Conscious of his own "timidity" in the presence of strangers, embarrassed before audiences, men were drawn to him because of his sterling worth, and honors were generously awarded him because of the value of his scientific papers. His election to Fellowship in this Academy was followed by membership in no less than thirty scientific and learned societies.

In his autobiography which appeared only a few years before his death, referring to his youth, he wrote; "All I aspired to was opportunity for scientific research, believing that diligence, singleness of purpose, and honest work would bring its own reward." His aspirations were amply fulfilled and diligence, singleness of purpose, and honest work certainly brought their own reward.

Hermon C. Bumpus.

WILLIAM STEWART HALSTED (1852-1922).

Fellow in Class II, Section 4, 1901.

Professor Halsted, certainly one of the most cultivated, and regarded by many as the most eminent surgeon of his time, in view of the character of his contributions, died at noon on Thursday, the seventh of September, in the Johns Hopkins Hospital, of which he had been surgeon-in-chief since soon after its opening. At that time, in 1889, neither he nor his clinical colleagues, Osler and Kelly, had as yet turned forty.

A man of unique personality, shy, something of a recluse, fastidious in his tastes and in his friendships, an aristocrat in his breeding, scholarly in his habits, the victim for many years of indifferent health, he nevertheless was one of the few American surgeons who may be considered to have established a school of surgery, comparable, in a sense, to the school of Billroth in Vienna. He had few of the qualities supposed to accompany what the world regards as a successful surgeon. Over-modest about his work, indifferent to matters of priority, caring little for the gregarious gatherings of medical men, unassuming, having little interest in private practice, he spent his medical life avoiding patients — even students, when this was possible — and, when health permitted, working in clinic and laboratory at the solution of a succession of problems which aroused his interest. He had that rare form of imagination which sees problems, and the technical ability combined with persistence which enabled him to attack them with promise of a successful issue. Many of his contributions, not only to his craft but to the science of medicine in general, were fundamental in character and of enduring importance.

As a schoolboy at Phillips-Andover and as an undergraduate at Yale, he was prominent in sports rather than in the class-room, and in his senior year was captain of one of the early university football teams. Like many other young men his ambition was not fired until his entrance into a professional school, and when, after his graduation in 1874, he entered the College of Physicians and Surgeons (Columbia) in New York, he settled down to prove his mettle with the result that three years later, on getting his degree, he was awarded a prize for leading his class in scholarship. After serving as interne at Bellevue he was appointed house physician to the newly erected New York Hospital. Subsequently, two years were passed in Europe where he devoted himself more especially to the subjects of anatomy and embryology. He studied at Vienna, Leipzig and Würtzburg, and his later surgical trend and investigative proclivities were distinctly colored by the German and Austrian surgery of the day.

On his return from abroad in 1880, he was made assistant demonstrator and subsequently demonstrator of anatomy at the College of Physicians and Surgeons. He also held a number of hospital positions, first at the Charity Hospital where from 1881 to 1887 he was an attending surgeon and director of the out-patient department. For three years he was also surgeon-in-chief to the Emigrant Hospital, Ward's Island; and later, from 1885 to 1887, an attending surgeon to both the Bellevue and Presbyterian Hospitals. During this period in New York, following his return from abroad, he supported himself mainly by teaching, and with Dr. George E. Munroe he organized a famous extramural course for students, consisting of practical exercises in the laboratory and at the bedside, to take the place of the time-honored quizzes which it was long the fashion for the New York students with hospital aspirations to attend.

During his last few years in New York he undertook an anatomico-surgical investigation on the anæsthetizing effect of the then little-known and newly introduced drug, cocaine. In this research, which had been begun in 1885, he was the first to utilize for surgical purposes the principle of nerve blocking, and was accustomed to demonstrate to dentists how painless extractions or even more extensive operations on the jaws might thus be carried out. He was the first, also, at this time, to demonstrate spinal anæsthesia by introducing the drug into the lumbar meninges. In the course of these studies he used himself as a subject, injecting his own peripheral nerves in order to map out the areas of anæthesia, and, unaware of the danger he was running, contracted an habituation to the drug, from which, with the help of a devoted professional friend, he effectually broke himself.

It was natural enough that cocaine was subsequently abhorred by him, and after Schleich's solution came to be generally employed as a local anæsthetic, he usually preferred to infiltrate with salt solution alone, which has certain anæsthetizing properties, rather than use even the diluted drug. Fifteen years later when the writer of this note, as Dr. Halsted's resident surgeon, stumbled anew upon the principle of nerve blocking for operations on hernia and published a paper on the subject, he was utterly unaware that his chief had ever made studies with cocaine of any sort, so reticent was he about this particular matter and so little did questions of priority interest him. It has remained for the dentists to call attention to his original work on regional anæsthesia, and a few months before his death they made due public acknowledgment of what Dr. Halsted himself had never laid claim to, and the knowledge of which he had even withheld, at least until recent years, from his house officers.

Before this tragic episode interrupted what would doubtless have been a brilliant career in New York, he had published a number of papers which showed promise of his technical gifts and abilities as an investigator, but it was not until he was brought to Baltimore in the late eighties by William H. Welch and got to work in the original pathological building there with Franklin P. Mall, Councilman, Flexner and others, that his unusual capacity for research was shown at its full worth.

The studies of compensatory thyroid hypertrophy, one of his early researches, published in the first volume of the Johns Hopkins Hospital Reports, remained for twenty years the basis of our views regarding exophthalmic goitre as an expression of functional overactivity. The correctness of his observations and interpretation of them, indeed, remained unquestioned until he himself repeated the experiments and, failing to corroborate his original results, promptly reported the fact before one of the biological societies. It was a striking example of his scientific honesty, and it remained for someone else to point out, on the basis of new facts relating to iodine, how it was that his original interpretation had been nearer the truth than his later one.

Another of his early studies was on intestinal resection and suture, and he introduced a method of anastomosis of the bowel, based on the distribution of the blood supply and on the correct placement of the sutures, far superior to that of any of his predecessors. These two subjects, the surgery of the thyroid and intestine, continued to engage his attention to the end, and among his last publications was a monograph entitled "The Operative Story of Goitre" published two years before his death; and another on the bulkhead principle of intestinal anatomosis.

His interest lay not in the number of cases he might operate upon but in working at certain principles of surgery, and in the course of his experiments upon the thyroid and parathyroid bodies, he hit upon what is known as Halsted's Law, namely, that "a transplant of a portion of a ductless gland will survive only when a physiological deficit has been produced."

On the opening of the hospital in 1889 he turned his attention to questions of technique, and was among the first American surgeons to grasp fully the principle of the new aseptic surgery. The introduction of silver as suture material and as a covering for wounds because of its bactericidal qualities was due to him. He studied the healing of an aseptic blood-clot in closed wounds. He introduced gutta-percha in the form of "protective" as a dressing for open wounds. He showed how silk could be safely buried in the tissues, an important principle many surgeons are incapable of learning. He was among the first to insist upon absolute blood-stilling in the course of operations in days when operations were bloody affairs, and he introduced the form of delicate pointed forceps for hæmostosis now universally in use. He also introduced rubber gloves into surgery in the early nineties, and, being himself a painstaking rather than a brilliant or spectacular operator, it was long before gloves came into use in other clinics — indeed, for years they were very much scoffed at as clumsy impediments to manipulation.

His operation for cancer of the breast revolutionized the treatment of these cases, and the same might be said of his hernia operation, though in this he shared the honors with Bassini, an Italian, who introduced a high inguinal operation with repair of the canal at about the same time. In the late nineties his attention was chiefly centered upon the diseases of the gall-bladder and its ducts, and the early radical operation on the common duct emanated from his clinic. Possibly few men in the country knew more than did he about the condition from which he was destined to succumb — a stone in the ampulla of Vater.

In later years he devoted himself chiefly to studies relating to the blood-vessels and evolved a method whereby in cases of aneurysm the major trunks could be slowly constricted, and in this as in all other subjects which his studies illuminated, his inventive genius was displayed, as well as his thorough knowledge of anatomy and pathology. He was the first successfully to ligate the left subclavian artery in its first portion for aneurysm and the only surgeon who is recorded to have performed this rare procedure twice.

Halsted's honors were many. In 1900 at the centennary of the Royal College of Surgeons of England he, with J. C. Warren of Boston, W. W. Keen of Philadelphia, and Robert F. Weir of New York, were the four Americans chosen to receive an honorary fellowship. A few years later he was made an F. R. C. S. of Edinburgh, and also an LL.D. both of Edinburgh and of his alma mater, Yale. Columbia gave him a D.Sc. and he was a member of the National Academy of Sciences as well as of many other foreign and American scientific bodies. Though his publications were comparatively few — rarely more than one or two a year — he wrote well and painstakingly, and many of his papers will remain among our surgical classics. The one surgeon he perhaps admired more than any other was the late Theodore Kocher of Berne, Switzerland, and the two men, in manner and methods surgical, in imagination and ideals, had very much in common. Both of them held their professorships for an unusual number of years — Kocher for forty-five years, Halsted for thirty-three.

Halsted was a man who taught by example rather than precept. He was a safe, fastidious and finished surgeon, by no means a brilliant and showy operator after the style cultivated by many of his contemporaries. He cared nothing for administration, and up to ten years ago at least, his staff never met as a body. He was not a successful teacher of undergraduates. A bed-to-bed ward visit was almost an impossibility for him. If he was interested he would spend an interminable time over a single patient, reviewing the history, taking notes, having sketches made, carrying the problem to the laboratory and perhaps working on it for weeks. Meanwhile his associates and assistants would run his clinic as best they could. In this way his school developed — none of his pupils after his own fashion, to be sure — it would have been impossible to imitate him — all of them, nevertheless, influenced enormously by his attitude toward surgery, and by his operative methods.

His loss to the Johns Hopkins Hospital which he served so faithfully and long, and to which he bequeathed his property, will be irreparable. It will be equally so to his many and devoted disciples. One of his long series of resident-surgeons, who, as others have done, came to know him better after leaving his service, just as many sons learn to know their fathers not until after they have grown up, has in all respect and affection written this inadequate note of appreciation.

"Who knows whether the best of men be known, whether there be not more remarkable persons forgot than any that stand remembered in the known account of time?"

HARVEY CUSHING.

MARCUS PERRIN KNOWLTON (1839-1918).

Fellow in Class III, Section 1, 1911.

Marcus Perrin Knowlton, son of Merrick and Fatima (Perrin) Knowlton, was born in Wilbraham, Massachusetts, February 3, 1839. Little is known of his boyhood. He is remembered as an earnest student at Munson Academy where he received his early instruction. He took high rank in his class of 1860 at Yale. He depended largely upon his own exertions to meet the expenses of his education. He taught at Westfield and in the Hopkins Grammar School at New Haven, and, for a short time after his graduation from Yale, he was principal of the Union School at Norwalk, Connecticut. The destruction of this school by fire and the consequent interruption of his occupation, gave the immediate impetus to his study of the law as a profession.

He began at the office of James G. Allen, at Palmer, but soon removed to the offices of Wells and Soule in Springfield, where, under these distinguished lawyers — both later to be Justices of the great court upon which he also was to sit — he rapidly mastered the principles of the English law.

On September 24, 1862, he was admitted to the bar, after the oral examination customary at that time — in his case largely a perfunctory ceremony so well was his capacity recognized. He immediately began practice by himself at Springfield, but within two years formed the partnership of Knowlton and Greene, soon to be superseded by the distinguished law firm of Stearns & Knowlton. George L. Stearns, the senior partner, was a leading lawyer of western Massachusetts, a great jury advocate, differing in numberless ways from his staid junior. The two complemented each other remarkably, and Stearns & Knowlton prospered. It became the leading law firm of the state west of Worcester.

His devotion to the law and the interests of his clients was not allowed to stand in the way of service to the community. Mr. Knowlton served his city in its city council, and his district in the great and General Court of Massachusetts, where he became a member of the House in 1878 and later, in 1880 and 1881, of the Senate. A republican of earnest conviction, wide information, and sound judgment, he imbued his fellow legislators with firm belief in his great capacity. Governor Long, who had served with him in the legislature and recognized his ability as man and lawyer, appointed him an Associate Justice of the Superior Court, and he was commissioned on August 31, 1881, to succeed Judge William Allen.

Judge Knowlton in the six years of his service in the Superior Court impressed the bar, the jurors in attendance when he presided, and the litigants whose causes came before him, with his courtesy, his patience, his comprehensive grasp alike of the theoretical law and of the practical common sense involved in a sound determination, his ability to state clearly the issues of fact and of law, and, above all, his absorbing desire to secure their respective rights to all parties. His appointment to the Supreme Judicial Court, which came from Governor Ames, September 14, 1887, to fill the vacancy caused by the resignation of Judge Gardner, was the natural recognition by the Commonwealth of his desert, and of its desire to profit still further from the service of this gifted son. He first sat with the court at Greenfield on September 20, 1887; and his first opinion came down on October 20, 1887, in the case of *O'Keefe v. Northampton*, reported in 145 Mass. 115. Other opinions followed rapidly. Fifteen opinions from his pen occur in the first volume of reports issued after his appointment. They are full, clear, simply stated, broad-minded, and sensible, and they cover a wide range of subjects. His decisions grew in force with the passing years; but, throughout, they are characterized by wide knowledge of law, sound common sense, wise consideration of the ultimate consequences of the principles laid down, and keen sense of justice, expressed in clear, simple and forceful words, logical, coherent, persuasive and convincing.

On December 8, 1902, Chief Justice Holmes vacated his office by becoming a Justice of the Supreme Court of the United States, and on December 17 Governor Crane appointed Judge Knowlton to be Chief Justice. Massachusetts has had many able Chief Justices, but only Chief Justice Shaw would be ranked higher by the consensus of lawyers than Chief Justice Knowlton.

His industry was enormous. He wrote the opinion of the court in eight hundred thirty-seven cases as an Associate Justice and in seven hundred thirty-three as Chief Justice. He wrote also twenty-nine dissenting opinions, of which four were written while Chief Justice. It is worth remarking that the law as contended for by him in most of these dissenting opinions was eventually established as the law of the Commonwealth. He was fair and open-minded in the consideration of a legal problem. His conclusion was the result of sound, logical processes of thought; and, once reached, was not easily abandoned. He had the courage of his convictions, the rock-ribbed determination which is the result, not of prejudice, but of conscientious, painstaking, laborious thought.

Only Chief Justice Shaw has exceeded him in the number of his written opinions; twenty-one hundred sixty-one in thirty years, as against fifteen hundred ninety-nine in twenty-four years.

His last opinion was *Ryan v. Boston Elevated Railway*, 209 Mass. 292. Thus his writings extend through sixty-five volumes of Massachusetts reports.

He resigned from his office September 7, 1911, owing to failing eyesight.

Fortunately this trouble with his eyes was overcome, and he was called upon for still further service to Massachusetts as Chairman of the trustees appointed to preserve and rehabilitate the Boston and Maine Railroad.

For many years he was one of the trustees of Munson Academy and his early school was an object of his affectionate thought and care.

He was a faithful attendant at the services of the Church of the Unity at Springfield, interested in the religious work of the society.

He received the degree of LL.D. from Yale, from Harvard and from Williams.

He died May 7, 1918, at Springfield.

His school, his college, his city and his state delighted to honor him; and to them all he gave the loyal service of a devoted and exceptionally able son.

WILLIAM C. WAIT.

HARMON NORTHROP MORSE (1848–1920).

Fellow in Class I, Section 3, 1914.

Professor Morse was born at Cambridge, Vermont, October 15, 1848, and died September 8, 1920.

His earliest paternal American ancestor was John Morse who came from England in 1639 and settled in New Haven.

His father, Harmon Morse, was a believer in hard work, few holidays, and little schooling. He looked upon all forms of recreation as objectionable. The death of the boy's mother, when he was too young to remember her, removed a much needed gracious and loving influence. His brother, Anson, and his young sister, Delia, were comrades and comforters in his life, which for the most part lacked the elements of love and geniality.

The courage and ambition of the boy overcame all obstacles. His maternal grandfather left a legacy whereby each of the three children was enabled to prepare for higher education, and thus Harmon was led to Amherst College entering in 1869. He was graduated from Amherst in 1873, then went to Göttingen, and received the degree of

Ph.D. from that university in 1875. While there he gave his time to chemistry as the principal subject and mineralogy as the subordinate. Wöhler was still living but had retired from active service. Nevertheless he spent part of his time in the laboratory and a few favored students, generally Americans, were given the privilege of working with him. Morse was one of these, and throughout his life he took delight in recalling his personal experiences with Wöhler. The active head of the laboratory was Hübner and it was with him that most of Morse's advanced chemical work was done. It is interesting to note that Hübner was distinctly an organic chemist and those who worked with him naturally tended to follow in that field. For some reason, however, very little of Morse's later work was in that field. His best work, that which gained him distinction, was really in physics, though, in reference to present custom, probably should be called physical chemistry or perhaps chemical physics.

Returning to America in 1875, Morse was given a minor assistantship at Amherst and spent the academic year there under Harris and Emerson. These two men have had a marked influence on the lives of many men. The number of chemists and geologists who had their early training under them I do not know, but I do know that among them were many of note. In Johns Hopkins alone there were two Amherst men, students of Emerson, who became successively professors of geology. I refer to George Huntington Williams and William Bullock Clark. Here also Morse came. I have elsewhere (see Science, November 26, 1920) told that story in brief and I may here repeat what I there said.

In 1875 it was announced that the Johns Hopkins University would begin its work in the year 1876. Shortly after it became known that the writer of this notice was to be the professor of chemistry in the new university he received a call from Morse who brought a letter of introduction from Emerson. This letter led me to take more than ordinary interest in the bearer. Whatever we were to do in Baltimore, it seemed clear that I should need an assistant, and I told him I would in due time arrange for his appointment. Hearing a little later of the fellowships that were to be awarded I secured one of these for Morse and so his connection with the Johns Hopkins University began. Before the doors were opened, however, he was designated associate, and we began our work together for better or for worse. We had no laboratory. We had less than a handful of students. What was to come of it? I need not go into the story thus suggested except to say that we were absolutely untrammeled and left to work out our own salvation. Morse and I were of one mind as to the object to be attained and there were no discussions in regard to the methods to be adopted. They were not original, but they had never been tried in this country. There had never been an opportunity. The opportunity that many of us had hoped for, had dreamed of, was furnished by the bounty of Johns Hopkins and the wisdom of his trustees and of President Gilman.

Morse remained an associate until 1883, when he became an associate professor. In 1892 he was promoted to be professor of inorganic and analytical chemistry, and in 1908 he became director of the chemical laboratory. In 1916 he withdrew from active service and became professor emeritus.

From the beginning of our work in the new university the importance of research was emphasized. That was indeed its most characteristic feature. Morse was as anxious as any of us to take part in this work. For one reason and another it was some time before he got going. To be sure he did show his hand in some small and rather unpromising pieces of work and I think he became discouraged, but he was faithful to his teaching. Gradually, however, his researches opened up new fields and he began their exploration. This is not the place for a full review of his contributions, and those of his last years so overshadow all that preceded that a reference to these alone will do substantial justice to his memory.

In the early nineties he turned his attention seriously to the question of the stability of solutions of potassium permanganate, and in 1896 he published an article on "The production of permanganic acid by manganese superoxide," A. J. Hopkins and M. S. Walker appearing as joint authors. Pursuing this subject Morse and H. G. Byers in 1900 published an article "On the cause of the evolution of oxygen when oxidisable gases are absorbed by permanganic acid." The results were such that it became desirable to obtain an aqueous solution of pure permanganic acid. It was decided to prepare this by dissolving the heptoxide in water. In an article by Morse and J. C. Olsen that appeared in 1900 occurs the following passage:

> (We) accordingly prepared a quantity of the anhydride by mixing potassium permanganate and concentrated sulphuric acid in vessels cooled by ice and salt. We soon learned, however, that something more than a low temperature is essential to safety in handling the product; for a minute quantity of the anhydride — certainly less than half a drop — which had been separated from the sulphuric acid, exploded with great violence and with disastrous results to one of us.[1] Some idea of the force of the explosion may be gained from the fact that one of the flying fragments of glass passed entirely through a burette which was mounted in the vicinity, leaving holes over half the diameter of the burette, edges of which were entirely free from cracks. After this experience, we decided to abandon the anhydride as a source of the acid, and to work out, if practicable, an electrolytic method of separating it from its salts.

The electrolytic method worked very satisfactorily, and led to the further use of this method in the preparation of osmotic membranes. The first results of this investigation are given in an article by Morse and D. W. Horn that appeared in 1901. They say:

> In this connection, it occurred to the authors that if a solution of a copper salt and one of potassium ferrocyanide are separated by a porous wall which is filled with water, and a current is passed from an electrode in the former to another in the latter solution the copper and the ferrocyanogen ions must meet in the interior of the wall and separate as copper ferrocyanide at all points of meeting, so that in the end there should be built up a continuous membrane well supported on either side by the material of the wall. The results of our experiments in this direction appear to have justified the expectation and to be worthy of a brief preliminary notice.

This marks the real beginning of the work on osmotic pressure with which the name of Morse will always be associated. But before the cells were available and therefore before any reliable measurements could be made, years of patient, skilful work were still necessary. Difficulties that seemed insurmountable frequently arose and necessitated new efforts. It must be said that some of us in the laboratory, including myself, at times lost faith in the ultimate success of the work and were perhaps inclined to advise the use of cells that were not perfect. But Morse went steadily on. He had in mind a practically perfect cell that could be used for high pressures as well as low. He tried all sorts and conditions of clay and after many, many discouragements he succeeded in finding what he wanted and in making a satisfactory glaze quite different from any available, and he achieved success.

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[1] To make this story complete it should be added that Morse was the "one of us" here referred to. A piece of glass passed through the tissues of his neck in close proximity to the jugular vein. His escape from death was almost miraculous.

In 1902 he and J. C. W. Frazer described "The preparation of cells for the measurement of high osmotic pressures." A careful reading of this article will give some idea of the tremendous difficulties that were met and overcome. The closing paragraph may be advantageously quoted in this connection:

> The difficulties of construction are by no means completely overcome, and we have in view a number of changes which we hope will prove of advantage. That these difficulties are of great magnitude will be realized if one considers that in our last experiment the pressure which was measured and which was still below what we were called upon to control would suffice to raise a column of water at 20° to a point 15 meters higher than the top of the Eiffel tower, or which would raise from its base a marble shaft whose height is 120 meters. These comparisons will perhaps make it clear that the most painstaking attention to every detail of construction is absolutely essential to success when an apparatus like ours is to be made up of several parts, consisting of different materials, and which must be united without the usual mechanical means of securing strong joints.

Soon after this the Carnegie Institution of Washington lent its powerful aid to the large investigation thus begun. In 1914 the institution published a memoir entitled "The Osmotic Pressure of Aqueous Solutions: Report on Investigations made in the Chemical Laboratory of the Johns Hopkins University during the years 1899–1913. By H. N. Morse." In it is given a detailed account of this remarkable piece of experimental work. Any one who reads it understandingly will recognize that no one but a master of experiment could have done this. The work required the highest degree of resourcefulness and skill, of patience and persistence. Any one of ordinary caliber would have stopped short of the accomplishment. Morse was not satisfied with anything but perfection as nearly as this could be reached, and as it never can be reached he worried about the residual no matter how small it might be. In the concluding chapter of the Carnegie Memoir occur these words:

The work reported upon in the preceding chapters is only a fraction of the task which the author hopes to accomplish, or to see accomplished by others. The investigation — already fifteen years old — was undertaken, in the first instance, with a view to developing a practicable and fairly precise method for the direct measurement of the osmotic pressure of aqueous solutions. The need of such a method for the investigation of solutions seemed to the author very great and very urgent.

Honors came to him rather late but they came, the chief among these was the award of the Avogadro Medal of the Turin Academy of Sciences, in 1916.

In 1911 an international congress of scientists assembled at Turin, Italy, to celebrate the centennial of the announcement of the hypothesis of Avogadro. Those in attendance decided to award a medal to be known as the Avogadro Medal. This medal was to be awarded to the investigator who should, in the judgment of the awarding committee, make the most valuable contribution to the subject of molecular physics during the years 1912, 1913 and 1914.

A few words in regard to Morse, the man. He was quiet and uneffusive. He did not care for the ordinary intercourse with his fellowmen. He lived, when not in the laboratory, for his family and a few kindred spirits. He married twice and had four children — a daughter and three sons. His second wife, who was Miss Elizabeth Dennis Clark, of Portland, Maine, his daughter and two sons survive him. In his later years his wife was of great assistance to him in preparing his articles for publication and was a true helpmate in every way.

For many years he spent his summers at Chebeague in the beautiful Casco Bay. Here he had a simple comfortable cottage and a garden. He delighted to work, both in and out of the house, and this gave him his exercise. He was rather stout and he knew that he needed exercise to keep his weight down. He therefore indulged in walking, bicycling, and finally in motoring, and he managed to keep fairly well. But after his retirement in 1916, his health failed. His strength gave out and his courage also. He did not dare to take his car out of the garage, and his walks were very short. I saw him in May, just before he went to Maine, and thought he seemed more like his old self. He even talked of taking up his work again. It was not to be. I heard nothing from him after that. And then came the dispatch announcing his rather sudden and entirely unexpected death. He was buried at Amherst, a place that meant so much to him — where he had spent his college years and for some time had had a summer home.

IRA REMSEN.

EDWARD HALL NICHOLS (1864-1922).

Fellow in Class II, Section 4, 1913.

Edward Hall Nichols was born in Reading, Mass., January 6, 1864. His death occurred in Boston on June 12, 1922. The end may be considered a happy one, for it came in the active performance of his beneficent professional duty, suddenly and without prolonged anxiety or suffering. His career was a noteworthy one, not only for its useful activity, but also for its distinction in various diverse accomplishments. He was a proficient surgeon, an excellent teacher, both in surgical pathology and in clinical surgery, and a leading esteemed surgical practitioner. He was independent and thorough in his pathological research and his contributions were of importance.

That of the work of the Cancer Research Commission was characterized by marked charity, independent and sound statement; it cleared the way of much confusion of opinion and was of much assistance to his successors on the Harvard Cancer Commission. Perhaps his most important research was on the microscopic pathology of Arthritis Deformans which in thoroughness of investigation, definiteness of objective and in its conclusions could be regarded as the most important work on the subject as yet published.

For several years Professor Nichols served on the Committee to regulate the athletics at Harvard University and his advice and influence in promoting sound training and in improving true sportsmanship were highly appreciated by all interested in the improvement of college sport. He was selected by President Roosevelt as a member of a committee to draw up needed rules for the improved regulation of college foot ball, which saved the game from possible brutality and increased its spectacular popularity. In his college days, an unusual ball player, he never lost his enthusiasm in all proficiency in college athletics and his advice as a coach and medical adviser was of great assistance. His most important contribution to the domain of applied surgery was in the Surgery of Diseases of Bones and Joints. Many important contributions to surgical literature have been made by Dr. Nichols, and he left unfinished portions of a work on Surgical pathology in coöperation with Professor Mallory. It is expected that the work will be finished.

The culminating effort in Dr. Nichols' career may be considered his directorship of two Hospital Units in France during the late war. The first was somewhat unusual and was organized and supported under the authority of Harvard University before the United States went into the war. The command was enrolled and served under British control and commission, though still retaining American citizenship, an arrangement especially made by the British government and involved at the outset tact and firmness on the part of the director in bringing a raw medical group, well trained for professional work in civic life, but needing the regulations under military routine.

After America went into the great war, Dr. Nichols headed another Hospital Unit, organized from the draft of the Boston City Hospital. This service continued until the end of the war.

Like many others, some time was needed before Dr. Nichols recovered from the war strain. He was, however, at the time of his death, again in full promise of his unusual abilities with every expectation of many years of useful activity. His death deprived the profession of one of its valued leaders.

Dr. Nichols is survived by his widow, Mrs. Elizabeth J. Nichols, and by a son and daughter.

EDWARD H. BRADFORD.

EDWARD JAMES YOUNG (1829–1906).

Fellow in Class III, Section 2, 1870.

Edward James Young was born in Boston April 1, 1829, and died in Waltham, June 23, 1906. His father was Dr. Alexander Young, minister of the New South church in Boston, but more widely known for his very useful books "Chronicles of the Pilgrim Fathers" and "Chronicles of the First Planters of the Colony of Massachusetts Bay." In 1835 he entered the Chauncy-Hall School, under the principalship of its founder, Gideon F. Thayer, and passed in 1839 to the Boston Latin School of which Epes Sargent Dixwell was then Head Master. He graduated from Harvard College in the class of 1848. For the next two years he was a teacher in the Brimmer School and the Boston Latin School. In 1850 he entered the Harvard Divinity School which was then at a low ebb academically, having but two resident professors. Young seems to have exhausted the resources of the School in two years, for in 1852 he went to Germany where he spent four years in theological study, principally at Göttingen and Halle. At that time in Germany theology was passing through a period of reaction from the powerful influence of Schleiermacher, coinciding with the political reaction after the troubles of 1848. Neo-Lutherans were sharply attacking both the Reformed Church and the Party of Union inspired not more by zeal for orthodoxy than by ecclesiastical and political chauvinism. Of this conflict, Young gave an excellent account in three articles published in the Christian Examiner of May and November, 1855 and January 1856, in one of which he made a vigorous plea for the enlargement of theological education at Harvard. These articles had been preceded by an elaborate survey of recent theological publications in Germany (Christian Examiner, May 1854) and a translation of the introductory article of the new "Protestant Church Gazette" by H. Krause (Monthly Religious Magazine, March 1854). He also translated into English the first volume of Erdmann's History of Philosophy which, however, failed to find a publisher. Returning to this country, he was ordained pastor of the Channing Religious Society in Newton, (June 18, 1857), where he remained until 1869 when he was appointed Hancock Professor of Hebrew and other Oriental Languages and Dexter Lecturer on Biblical Literature in Harvard University. For this position he had served an apprenticeship for one year as teacher of the Bible in the Boston School for the Ministry. But he came to Harvard at a peculiarly unpropitious time for a teacher of Hebrew. The language was still required but students had lost interest in it on account of changing opinions as to the inspiration and authority of the Bible coincident with broadening theories concerning the function of a minister in the life of the community. It was often argued that if the teacher were more interesting, the students would be more interested whereas the real reason lay deeper than the merits or demerits of any particular instructor. In addition, the science of the Higher Criticism of the Old Testament was becoming better known and theological students were keenly interested in its results which, however, required for their effective presentation a broader training, greater imaginative power and a more consistently held point of view than could be justly expected of a teacher trained in earlier conditions. Accordingly Professor Young's position at Harvard soon became exceptionally difficult and in 1880 he resigned his chair and returned to the life of a parish minister. On November 14, 1880, he was installed pastor of the Unitarian Church in Waltham which he served until October 30, 1892, but continued to reside in Waltham until his death in 1906.

Dr. Young (the degree of S.T.D. was conferred upon him by Tufts in 1887) was an active member of many learned and religious societies to most of which he also rendered valuable official service. He was recording secretary of the Massachusetts Historical Society from 1883 to 1906; secretary of the Ministers' Club from 1874 to 1899; president of the Society for the Promotion of Theological Education from 1894 to 1906. As a teacher, he is reported to have been didactic rather than stimulating, more apt to catalogue the opinions of others than to state his own conclusions. His publications consist of Memoirs prepared for the Massachusetts Historical Society and published in its Proceedings (February 1881, J. L. Diman; December 1895, O. B. Frothingham; Jan. 1896, A. P. Peabody), essays and occasional sermons and addresses, the most important of which were: Oration at the Two Hundredth Anniversary of the battle of Green Hill, Sudbury, Boston, 1876. The Value of the study of Hebrew for a Minister (Unitarian Review, May 1879); Subjects for Master's degree in Harvard 1655–1791 (Proceedings of Massachusetts Historical Society, June 1880); Early Religious Customs of New England, Cambridge, 1882. He also wrote a memoir of the life of Samuel Abbot Smith contained in a volume of the latter's sermons, Christian Lessons and a Christian Life, Boston, 1866.

William W. Fenn.

American Academy of Arts and Sciences

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[1] Died May 15, 1923. [2] Died June 14, 1923. [3] Died May 11, 1923.

LIST

OF THE

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(Corrected to July 1, 1923.)

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CLASS I., SECTION II.— *Physics*.— 52.

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Henry Lefavour	Boston
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CLASS II.— *Natural and Physiological Sciences.*— 188.

SECTION I.— *Geology, Mineralogy, and Physics of the Globe.*— 54.

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William Otis Crosby	Jamaica Plain
Reginald Aldworth Daly	Cambridge
Edward Salisbury Dana	New Haven, Conn.
William Morris Davis	Cambridge
Benjamin Kendall Emerson	Amherst
William Ebenezer Ford	New Haven, Conn.
James Walter Goldthwait	Hanover, N. H.
Louis Caryl Graton	Cambridge
Herbert Ernest Gregory	New Haven, Conn.
William Jackson Humphreys	Washington, D. C.
Ellsworth Huntington	New Haven, Conn.
Oliver Whipple Huntington	Newport, R. I.
Robert Tracy Jackson	Peterborough, N. H.
Thomas Augustus Jaggar	Honolulu, H. I.
Douglas Wilson Johnson	New York, N. Y.
Arthur Keith	Washington, D. C.
James Furman Kemp	New York, N. Y.
Alfred Church Lane	Cambridge
Andrew Cowper Lawson	Berkeley, Cal.
Charles Kenneth Leith	Madison, Wis.
Waldemar Lindgren	Brookline
Frederic Brewster Loomis	Amherst
Alexander George McAdie	Readville
John Campbell Merriam	Washington, D. C.
William John Miller	Northampton
Charles Palache	Cambridge
Raphael Pumpelly	Newport, R. I.
Percy Edward Raymond	Lexington
William North Rice	Middletown, Conn.
Austin Flint Rogers	Palo Alto, Cal.
Robert Wilcox Sayles	Chestnut Hill
Waldemar Theodore Schaller	Washington, D. C.
Charles Schuchert	New Haven, Conn.

William Berryman Scott	Princeton, N. J.
Hervey Woodburn Shimer	Hingham
Thomas Wayland Vaughan	Washington, D. C.
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Robert DeCourcy Ward	Cambridge
Charles Hyde Warren	New Haven, Conn.
David White	Washington, D. C.
Herbert Percy Whitlock	New York, N. Y.
Bailey Willis	Palo Alto, Cal.
Arthur Winslow	Boston
John Eliot Wolff	Chestnut Hill
Jay Backus Woodworth	Cambridge
Frederick Eugene Wright	Washington, D. C.

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John Merle Coulter	Chicago, Ill.
Bradley Moore Davis	Ann Arbor, Mich.
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Rollins Adams Emerson	Ithaca, N. Y.
Alexander William Evans	New Haven, Conn.
Merritt Lyndon Fernald	Cambridge
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Elmer Drew Merrill	Manila, P. I.
Winthrop John Vanleuven Osterhout	Cambridge

Charles Vancouver Piper	Washington, D. C.
Alfred Rehder	Jamaica Plain
Benjamin Lincoln Robinson	Cambridge
Charles Sprague Sargent	Brookline
William Albert Setchell	Berkeley, Cal.
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Thorne Martin Carpenter	Boston
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Manton Copeland	Brunswick, Me.
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William Healey Dall	Washington, D. C.
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Gilman Arthur Drew	Woods Hole
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Richard Swann Lull	New Haven, Conn.
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Albert Davis Mead	Providence, R. I.
Gerrit Smith Miller	Washington, D. C.
Edward Sylvester Morse	Salem
Herbert Vincent Neal	Tufts College
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George Howard Parker	Cambridge
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William Emerson Ritter	La Jolla, Cal.
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William Lyman Underwood	Belmont
Addison Emory Verrill	Westville, Conn.
John Broadus Watson	New York, N. Y.
Arthur Wisswald Weysse	Boston
William Morton Wheeler	Jamaica Plain
Harris Hawthorne Wilder	Northampton
Edmund Beecher Wilson	New York, N. Y.
Frederick Adams Woods	New York, N. Y.
Robert Mearns Yerkes	Washington, D. C.

Class II., Section IV.— *Medicine and Surgery.*— 41.

Edward Hickling Bradford	Boston
Charles Macfie Campbell	Cambridge
Alexis Carrel	New York, N. Y.
Henry Asbury Christian	Boston
Stanley Cobb	Ponkapoag
Rufus Cole	New York, N. Y.
Harvey Cushing	Boston
David Linn Edsall	Cambridge
Simon Flexner	New York, N. Y.
Joseph Lincoln Goodale	Boston
Robert Battey Greenough	Boston
Ross Granville Harrison	New Haven, Conn.
William Henry Howell	Baltimore, Md.
Reid Hunt	Boston
Henry Jackson	Boston
Elliott Proctor Joslin	Boston
William Williams Keen	Philadelphia, Pa.
Roger Irving Lee	Cambridge
Robert Williamson Lovett	Boston
Frank Burr Mallory	Brookline
William James Mayo	Rochester, Minn.
Samuel Jason Mixter	Boston
Francis Weld Peabody	Boston
Theophil Mitchell Prudden	New York, N. Y.
William Lambert Richardson	Boston
Milton Joseph Rosenau	Brookline
Frederick Cheever Shattuck	Boston
Theobald Smith	Princeton, N. J.
Charles Wardell Stiles	Washington, D. C.
Richard Pearson Strong	Boston
William Sydney Thayer	Baltimore, Md.
Ernest Edward Tyzzer	Wakefield
Frederick Herman Verhoeff	Boston
Henry Pickering Walcott	Cambridge
John Warren	Boston
John Collins Warren	Boston

William Henry Welch	Baltimore, Md.
Francis Henry Williams	Boston
Simeon Burt Wolbach	Boston
Horatio Curtis Wood	Philadelphia, Pa.
Hans Zinsser	Boston

Class III.— *Moral and Political Sciences.*— 200.

Section I.— *Theology, Philosophy and Jurisprudence.*— 53.

Thomas Willing Balch	Philadelphia, Pa.
Simeon Eben Baldwin	New Haven, Conn.
Willard Bartlett	Brooklyn, N. Y.
Joseph Henry Beale	Cambridge
Charles Henry Brent	Buffalo, N. Y.
Howard Nicholson Brown	Boston
Charles Warren Clifford	New Bedford
Leslie Colby Cornish	Augusta, Me.
Edmund Burke Delabarre	Providence, R. I.
James De Normandie	Lincoln
Frederic Dodge	Belmont
Edward Staples Drown	Cambridge
William Harrison Dunbar	Cambridge
William Herbert Perry Faunce	Providence, R. I.
William Wallace Fenn	Cambridge
Frederick Perry Fish	Brookline
Paul Revere Frothingham	Boston
George Angier Gordon	Boston
Alfred Hemenway	Boston
William Ernest Hocking	Cambridge
Charles Evans Hughes	Washington, D. C.
Frederick John Foakes Jackson	New York, N. Y.
Charles Francis Jenney	Boston
William Lawrence	Boston
Frederick Lawton	Boston
Arthur Lord	Plymouth
William Caleb Loring	Boston
Nathan Matthews	Boston

William McDougall	Cambridge
Samuel Walker McCall	Winchester
Edward Caldwell Moore	Cambridge
John Bassett Moore	New York, N. Y.
James Madison Morton	Fall River
George Herbert Palmer	Cambridge
Charles Edwards Park	Boston
Leighton Parks	New York, N. Y.
Francis Greenwood Peabody	Cambridge
George Wharton Pepper	Philadelphia, Pa.
Roscoe Pound	Belmont
Elihu Root	New York, N. Y.
James Hardy Ropes	Cambridge
Arthur Prentice Rugg	Worcester
Austin Wakeman Scott	Cambridge
Henry Newton Sheldon	Boston
Moorfield Storey	Boston
William Howard Taft	Washington, D. C.
William Jewett Tucker	Hanover, N. H.
William Cushing Wait	Medford
Eugene Wambaugh	Cambridge
Edward Henry Warren	Boston
Winslow Warren	Dedham
Samuel Williston	Belmont
Woodrow Wilson	Washington, D. C.

CLASS III., SECTION II.— *Philology and Archæology.*— 56.

Francis Greenleaf Allinson	Providence, R. I.
William Rosenzweig Arnold	Cambridge
Maurice Bloomfield	Baltimore, Md.
Franz Boas	New York, N. Y.
Carl Darling Buck	Chicago, Ill.
Eugene Watson Burlingame	New Haven, Conn.
Edward Capps	Princeton, N. J.
George Henry Chase	Cambridge
Walter Eugene Clark	Chicago, Ill.
Roland Burrage Dixon	Cambridge

Franklin Edgerton	Philadelphia, Pa.
William Curtis Farabee	Philadelphia, Pa.
Jesse Walter Fewkes	Washington, D. C.
Jeremiah Denis Mathias Ford	Cambridge
Basil Lanneau Gildersleeve	Baltimore, Md.
Pliny Earle Goddard	New York, N. Y.
Charles Hall Grandgent	Cambridge
Louis Herbert Gray	Lincoln, Neb.
Charles Burton Gulick	Cambridge
Roy Kenneth Hack	Cambridge
William Arthur Heidel	Middletown, Conn.
George Lincoln Hendrickson	New Haven, Conn.
Bert Hodge Hill	Athens, Greece
Elijah Clarence Hills	Berkeley, Cal.
William Henry Holmes	Washington, D. C.
Edward Washburn Hopkins	New Haven, Conn.
Joseph Clark Hoppin	Pomfret, Conn.
Albert Andrew Howard	Cambridge
William Guild Howard	Cambridge
Aleš Hrdlička	Washington, D. C.
Eugene Xavier Louis Henry Hyvernat	Washington, D. C.
Carl Newell Jackson	Cambridge
Hans Carl Gunther von Jagemann	Cambridge
James Richard Jewett	Cambridge
Alfred Louis Kroeber	Berkeley, Cal.
Kirsopp Lake	Cambridge
Henry Roseman Lang	New Haven, Conn.
Charles Rockwell Lanman	Cambridge
John Livingston Lowes	Cambridge
David Gordon Lyon	Cambridge
Clifford Herschel Moore	Cambridge
George Foot Moore	Cambridge
Hanns Oertel	Munich, Germany
Chandler Rathfon Post	Cambridge
Edward Kennard Rand	Cambridge
George Andrew Reisner	Cambridge
Edward Robinson	New York, N. Y.
Fred Norris Robinson	Cambridge

Rudolph Schevill	Berkeley, Cal.
Edward Stevens Sheldon	Cambridge
Herbert Weir Smyth	Cambridge
Franklin Bache Stephenson	Washington, D. C.
Charles Cutler Torrey	New Haven, Conn.
Alfred Marston Tozzer	Cambridge
Clark Wissler	New York, N. Y.
James Haughton Woods	Cambridge

Class III., Section III.— *Political Economy and History.*— 46.

Wilbur Cortez Abbott	Cambridge
Brooks Adams	Quincy
George Burton Adams	New Haven, Conn.
Charles McLean Andrews	New Haven, Conn.
John Spencer Bassett	Northampton
Carl Lotus Becker	Ithaca, N. Y.
Charles Jesse Bullock	Cambridge
Thomas Nixon Carver	Cambridge
Edward Channing	Cambridge
John Bates Clark	New York, N. Y.
Archibald Cary Coolidge	Boston
Richard Henry Dana	Cambridge
Clive Day	New Haven, Conn.
Davis Rich Dewey	Cambridge
Ephraim Emerton	Cambridge
Henry Walcott Farnam	New Haven, Conn.
Max Farrand	New Haven, Conn.
William Scott Ferguson	Cambridge
Irving Fisher	New Haven, Conn.
Worthington Chauncey Ford	Cambridge
Edwin Francis Gay	New York, N. Y.
Frank Johnson Goodnow	Baltimore, Md.
Evarts Boutell Greene	Champaign, Ill.
Arthur Twining Hadley	New Haven, Conn.
Albert Bushnell Hart	Cambridge
Charles Homer Haskins	Cambridge
Isaac Minis Hays	Philadelphia, Pa.

Charles Downer Hazen	New York, N. Y.
George La Piana	Cambridge
Henry Cabot Lodge	Nahant
Abbott Lawrence Lowell	Cambridge
William MacDonald	New York, N. Y.
Charles Howard McIlwain	Cambridge
Roger Bigelow Merriman	Cambridge
Samuel Eliot Morison	Concord
William Bennett Munro	Boston
Charles Lemuel Nichols	Worcester
James Ford Rhodes	Boston
William Milligan Sloane	New York, N. Y.
John Osborne Sumner	Boston
Frank William Taussig	Cambridge
William Roscoe Thayer	Cambridge
Frederick Jackson Turner	Cambridge
George Grafton Wilson	Cambridge
George Parker Winship	Charles River
Allyn Abbott Young	Cambridge

Class III., Section IV.—*Literature and the Fine Arts.*—45.

Irving Babbitt	Cambridge
George Pierce Baker	Cambridge
William Sturgis Bigelow	Boston
Le Baron Russell Briggs	Cambridge
Charles Allerton Coolidge	Boston
Frederick Shepherd Converse	Boston
Samuel McChord Crothers	Cambridge
Wilberforce Eames	New York, N. Y.
Edward Waldo Emerson	Concord
William Emerson	Cambridge
Arthur Fairbanks	Cambridge
Frank Edgar Farley	Middletown, Conn.
Arthur Foote	Brookline
Edward Waldo Forbes	Cambridge
Kuno Francke	Gilbertsville, N. Y.
Daniel Chester French	New York, N. Y.

Horace Howard Furness	Philadelphia, Pa.
Robert Grant	Boston
Morris Gray	Boston
Chester Noyes Greenough	Cambridge
James Kendall Hosmer	Minneapolis, Minn.
Mark Antony DeWolfe Howe	Boston
Archer Milton Huntington	New York, N. Y.
George Lyman Kittredge	Cambridge
William Coolidge Lane	Cambridge
John Ellerton Lodge	Boston
Charles Martin Tornov Lœffler	Medfield
Charles Donagh Maginnis	Brookline
Allan Marquand	Princeton, N. J.
Albert Matthews	Boston
Harold Murdock	Brookline
William Allan Neilson	Northampton
William Lyon Phelps	New Haven, Conn.
Arthur Kingsley Porter	Cambridge
Herbert Putnam	Washington, D. C.
Denman Waldo Ross	Cambridge
Paul Joseph Sachs	Cambridge
John Singer Sargent	London, Eng.
Ellery Sedgwick	Boston
Henry Dwight Sedgwick	Cambridge
Richard Clipston Sturgis	Boston
Charles Howard Walker	Boston
Owen Wister	Philadelphia, Pa.
George Edward Woodberry	Beverly
Charles Henry Conrad Wright	Cambridge

FOREIGN HONORARY MEMBERS.—65.

(Number limited to seventy-five.)

Class I.— *Mathematical and Physical Sciences.*— 24.

Section I.— *Mathematics and Astronomy.*— 8.

Johann Oskar Backlund	Petrograd
Arthur Stanley Eddington	Cambridge
Jacques Salomon Hadamard	Paris
Godfrey Harold Hardy	Oxford
Felix Klein	Göttingen
Tullio Levi-Civita	Rome
Charles Emile Picard	Paris
Charles Jean de la Vallée Poussin	Louvain

Class I., Section II.— *Physics.*— 7.

Svante August Arrhenius	Stockholm
Oliver Heaviside	Torquay
Sir Joseph Larmor	Cambridge
Hendrik Antoon Lorentz	Haarlem
Max Planck	Berlin
Sir Ernest Rutherford	Cambridge
Sir Joseph John Thomson	Cambridge

Class I., Section III.— *Chemistry.*— 4.

Fritz Haber	Berlin
Henri Louis Le Chatelier	Paris
Wilhelm Ostwald	Leipsic
William Henry Perkin	Oxford

Class I., Section IV.— *Technology and Engineering.*— 5.

Heinrich Müller Breslau	Berlin
Ferdinand Foch	Paris

Joseph Jacques Césaire Joffre	Paris
Vsevolod Evgenievich Timonoff	Petrograd
William Cawthorne Unwin	London

CLASS II.— *Natural and Physiological Sciences.*— 22.

SECTION I.— *Geology, Mineralogy, and Physics of the Globe.*— 10.

Frank Dawson Adams	Montreal
Charles Barrois	Lille
Waldemar Christofer Brögger	Christiania
Sir Archibald Geikie	Haslemere, Surrey
Viktor Goldschmidt	Heidelberg
Albert Heim	Zürich
Emmanuel de Margerie	Paris
Gustaf Adolf Frederik Molengraaff	Delft
Sir William Napier Shaw	London
Johan Herman Lie Vogt	Trondhjem

CLASS II, SECTION II.— *Botany.*— 5.

John Briquet	Geneva
Adolf Engler	Berlin
Ignatz Urban	Berlin
Hugo de Vries	Lunteren
Eugene Warming	Copenhagen

CLASS II.— SECTION III.— *Zoölogy and Physiology.*— 3.

Maurice Caullery	Paris
Sir Edwin Ray Lankester	London
George Henry Falkiner Nuttall	Cambridge

CLASS II., SECTION IV.— *Medicine and Surgery.*— 4.

Rt. Hon. Sir Thomas Clifford Allbutt	Cambridge
Sir Thomas Barlow, Bart.	London
Francis John Shepherd	Montreal
Sir Charles Scott Sherrington	Oxford

CLASS III.— *Moral and Political Sciences.*— 19.

SECTION I.— *Theology, Philosophy and Jurisprudence.*— 3.

Rt. Hon Arthur James Balfour, Earl of Balfour	Prestonkirk
Raymond Poincaré	Paris
Rt. Hon. Sir Frederick Pollock, Bart.	London

SECTION II.— *Philology and Archæology.*— 6.

Wilhelm Dörpfeld	Athens
Henri Guy	Grenoble
Hermann Georg Jacobi	Bonn
Arthur Anthony Macdonell	Oxford
Alfred Percival Maudslay	Hereford
Ramon Menendez Pidal	Madrid

SECTION III.— *Political Economy and History.*— 5.

Adolf Harnack	Berlin
Alfred Marshall	Cambridge
Rt. Hon. John Morley, Viscount Morley of Blackburn	London
Henri Pirenne	Ghent
Rt. Hon. Sir George Otto Trevelyan, Bart.	London

SECTION IV.— *Literature and the Fine Arts.*— 5.

Georg Brandes	Copenhagen
Thomas Hardy	Dorchester
Jean Adrien Antoine Jules Jusserand	Washington, D. C.
Rudyard Kipling	Burwash
Sir Sidney Lee	London

STATUTES AND STANDING VOTES

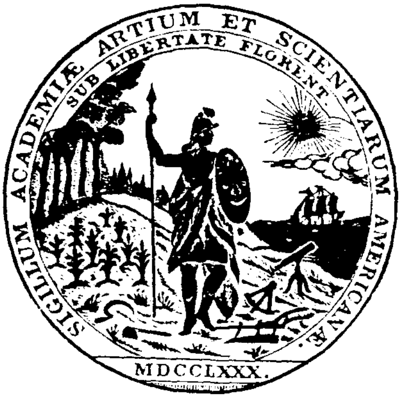
STATUTES

Adopted November 8, 1911: *amended May* 8, 1912, *January* 8, *and May* 14, 1913, *April* 14, 1915, *April* 12, 1916, *April* 10, 1918, *May* 14, 1919, *February* 8, *April* 12, *and December* 13, 1922, *February* 14, *and March* 14, 1923.

CHAPTER I

THE CORPORATE SEAL

ARTICLE 1. The Corporate Seal of the Academy shall be as here depicted:


SIGILLUM ACADEMIÆ ARTIUM ET SCIENTIARUM AMERICANÆ.
SUB LIBERTATE FLORENT.
MDCCLXXX.

ARTICLE 2. The Recording Secretary shall have the custody of the Corporate Seal.

See Chap. v. art. 3; chap. vi. art. 2.

CHAPTER II

Fellows and Foreign Honorary Members and Dues

Article 1. The Academy consists of Fellows, who are either citizens or residents of the United States of America, and Foreign Honorary Members. They are arranged in three Classes, according to the Arts and Sciences in which they are severally proficient, and each Class is divided into four Sections, namely:

Class I. *The Mathematical and Physical Sciences*
  Section 1. Mathematics and Astronomy
  Section 2. Physics
  Section 3. Chemistry
  Section 4. Technology and Engineering

Class II. *The Natural and Physiological Sciences*
  Section 1. Geology, Mineralogy, and Physics of the Globe
  Section 2. Botany
  Section 3. Zoölogy and Physiology
  Section 4. Medicine and Surgery

Class III. *The Moral and Political Sciences*
  Section 1. Theology, Philosophy, and Jurisprudence
  Section 2. Philology and Archaeology
  Section 3. Political Economy and History
  Section 4. Literature and the Fine Arts

Article 2. The number of Fellows shall not exceed Six hundred, of whom not more than Four hundred shall be residents of Massachusetts, nor shall there be more that Two hundred and ten in any one Class.

Article 3. The number of Foreign Honorary Members shall not exceed Seventy-five. They shall be chosen from among citizens of foreign countries most eminent for their discoveries and attainments in any of the Classes above enumerated. There shall not be more than Twenty-five in any one Class.

Article 4. If any person, after being notified of his election as Fellow, shall neglect for six months to accept in writing, or, if a Fellow resident within fifty miles of Boston shall neglect to pay his Admission Fee, his election shall be void; and if any Fellow resident within fifty miles of Boston shall neglect to pay his Annual Dues for six months after they are due, provided his attention shall have been called to this Article of the Statutes in the meantime, he shall cease to be a Fellow; but the Council may suspend the provisions of this Article for a reasonable time.

With the previous consent of the Council, the Treasurer may dispense (*sub silentio*) with the payment of the Admission Fee or of the Annual Dues or both whenever he shall deem it advisable. In the case of officers of the Army or Navy who are out of the Commonwealth on duty, payment of the Annual Dues may be waived during such absence if continued during the whole financial year and if notification of such expected absence be sent to the Treasurer. Upon similar notification to the Treasurer, similar exemption may be accorded to Fellows subject to Annual Dues, who may temporarily remove their residence for at least two years to a place more than fifty miles from Boston.

If any person elected a Foreign Honorary Member shall neglect for six months after being notified of his election to accept in writing, his election shall be void.

See Chap. vii. art. 2.

ARTICLE 5. Every Fellow resident within fifty miles of Boston hereafter elected shall pay an Admission Fee of Ten dollars.

Every Fellow resident within fifty miles of Boston shall, and others may, pay such Annual Dues, not exceeding Fifteen dollars, as shall be voted by the Academy at each Annual Meeting, when they shall become due; but any Fellow shall be exempt from the annual payment if, at any time after his admission, he shall pay into the treasury Two hundred dollars in addition to his previous payments.

All Commutations of the Annual Dues shall be and remain permanently funded, the interest only to be used for current expenses.

Any Fellow not previously subject to Annual Dues who takes up his residence within fifty miles of Boston, shall pay to the Treasurer within three months thereafter Annual Dues for the current year, failing which his Fellowship shall cease; but the Council may suspend the provisions of this Article for a reasonable time.

Only Fellows who pay Annual Dues or have commuted them may hold office in the Academy or serve on the Standing Committees or vote at meetings.

Article 6. Fellows who pay or have commuted the Annual Dues and Foreign Honorary Members shall be entitled to receive gratis one copy of all Publications of the Academy issued after their election.

See Chap. x, art. 2.

Article 7. Diplomas signed by the President and the Vice-President of the Class to which the member belongs, and countersigned by the Secretaries, shall be given to Foreign Honorary Members and to Fellows on request.

Article 8. If, in the opinion of a majority of the entire Council, any Fellow or Foreign Honorary Member shall have rendered himself unworthy of a place in the Academy, the Council shall recommend to the Academy the termination of his membership; and if three fourths of the Fellows present, out of a total attendance of not less than fifty at a Stated Meeting, or at a Special Meeting called for the purpose, shall adopt this recommendation, his name shall be stricken from the Roll.

See Chap. iii.; chap. vi. art. 1; chap. ix, art. 1, 7; chap. x. art. 2.

CHAPTER III

Election of Fellows and Foreign Honorary Members

The procedure in the election of Fellows and Foreign Honorary Members shall be as follows:

Nominations to Fellowship or Foreign Honorary Membership in any Section must be signed by two Fellows of that Section or by three Fellows of any Sections, and sent to the Corresponding Secretary accompanied by a statement of the qualifications of the nominee and brief biographical data.

Notice shall be sent to every Fellow not later than the fifteenth of January in each year, reminding him that all nominations must be in the hands of the Corresponding Secretary before the fifteenth of February following.

A list of the nominees, giving a brief account of each, with the names of the nominators, shall be sent to every Fellow with a request that he return the list with such confidential comments and indications of preference as he may choose to make.

All the nominations, with any comments thereon and with expressions of preference on the part of the Fellows, shall be referred to the appropriate Class Committees, which shall canvass them, and report their recommendations in writing to the Council before the Stated Meeting of the Academy in April.

Elections of Fellows and Foreign Honorary Members shall be made by the Council before the Annual Meeting in May, and announced at that meeting.

Persons nominated in any year, but not elected, may be carried over to the list of nominees for the next year at the discretion of the Council, but shall not be further continued unless renominated.

See Chap. ii.; chap. vi. art. 1; chap. ix. art. 1.

CHAPTER IV

OFFICERS

ARTICLE 1. The Officers of the Academy shall be a President (who shall be Chairman of the Council), three Vice-Presidents (one from each Class), a Corresponding Secretary (who shall be Secretary of the Council), a Recording Secretary, a Treasurer, and a Librarian, all of whom shall be elected by ballot at the Annual Meeting, and shall hold their respective offices for one year, and until others are duly chosen and installed.

There shall be also twelve Councillors, one from each Section of each Class. At each Annual Meeting three Councillors, one from each Class, shall be elected by ballot to serve for the full term of four years and until others are duly chosen and installed. The same Fellow shall not be eligible for two successive terms.

The Councillors, with the other officers previously named, and the Chairman of the House Committee, *ex officio*, shall constitute the Council.

See Chap. x, art. 1.

ARTICLE 2. If any officer be unable, through death, absence, or disability, to fulfil the duties of his office, or if he shall resign, his place may be filled by the Council in its discretion for any part or the whole of the unexpired term.

ARTICLE 3. At the Stated Meeting in March, the President shall appoint a Nominating Committee of three Fellows having the right to vote, one from each Class. This Committee shall prepare a list of nominees for the several offices to be filled, and for the Standing Committees, and file it with the Recording Secretary not later than four weeks before the Annual Meeting.

See Chap. vi. art. 2.

ARTICLE 4. Independent nominations for any office, if signed by at least twenty Fellows having the right to vote, and received by the Recording Secretary not less than ten days before the Annual Meeting, shall be inserted in the call therefor, and shall be mailed to all the Fellows having the right to vote.

See Chap. vi. art. 2.

ARTICLE 5. The Recording Secretary shall prepare for use in voting at the Annual Meeting a ballot containing the names of all persons duly nominated for office.

CHAPTER V

THE PRESIDENT

ARTICLE 1. The President, or in his absence the senior Vice-President present (seniority to be determined by length of continuous fellowship in the Academy), shall preside at all meetings of the Academy. In the absence of all these officers, a Chairman of the meeting shall be chosen by ballot.

ARTICLE 2. Unless otherwise ordered, all Committees which are not elected by ballot shall be appointed by the presiding officer.

ARTICLE 3. Any deed or writing to which the Corporate Seal is to be affixed, except leases of real estate, shall be executed in the name of the Academy by the President or, in the event of his death, absence, or inability, by one of the Vice-Presidents, when thereto duly authorized.

See Chap. ii. art. 7; chap. iv. art. 1, 3; chap. vi. art. 2; chap. vii. art. 1; chap. ix. art. 6; chap. x. art. 1, 2; chap. xi. art. 1.

CHAPTER VI

THE SECRETARIES

ARTICLE 1. The Corresponding Secretary shall conduct the correspondence of the Academy and of the Council, recording or making an entry of all letters written in its name, and preserving for the files all official papers which may be received. At each meeting of the Council he shall present the communications addressed to the Academy which have been received since the previous meeting, and at the next meeting of the Academy he shall present such as the Council may determine.

He shall notify all persons who may be elected Fellows or Foreign Honorary Members, send to each a copy of the Statutes, and on their acceptance issue the proper Diploma. He shall also notify all meetings of the Council; and in case of the death, absence, or inability of the Recording Secretary he shall notify all meetings of the Academy.

Under the direction of the Council, he shall keep a List of the Fellows and Foreign Honorary Members, arranged in their several Classes and Sections. It shall be printed annually and issued as of the first day of July.

See Chap. ii. art. 7; chap. iii. art. 2, 3; chap. iv. art. 1; chap. ix. art. 6; chap. x. art. 1; chap. xi. art. 1.

ARTICLE 2. The Recording Secretary shall have the custody of the Charter, Corporate Seal, Archives, Statute-Book, Journals, and all literary papers belonging to the Academy.

Fellows borrowing such papers or documents shall receipt for them to their custodian.

The Recording Secretary shall attend the meetings of the Academy and keep a faithful record of the proceedings with the names of the Fellows present; and after each meeting is duly opened, he shall read the record of the preceding meeting.

He shall notify the meetings of the Academy to each Fellow and by mail at least seven days beforehand, and in his discretion may also cause the meetings to be advertised; he shall apprise Officers and Committees of their election or appointment, and inform the Treasurer of appropriations of money voted by the Academy.

After all elections, he shall insert in the Records the names of the Fellows by whom the successful nominees were proposed.

He shall send the Report of the Nominating Committee in print to every Fellow having the right to vote at least three weeks before the Annual Meeting.

See Chap. iv. art. 3.

In the absence of the President and of the Vice-Presidents he shall, if present, call the meeting to order, and preside until a Chairman is chosen.

See Chap. i.; chap. ii. art. 7; chap. iv. art. 3, 4, 5; chap. ix. art. 6; chap. x. art. 1, 2; chap. xi. art. 1, 3.

ARTICLE 3. The Secretaries, with the Chairman of the Committee of Publication, shall have authority to publish such of the records of the meetings of the Academy as may seem to them likely to promote its interests.

CHAPTER VII

THE TREASURER AND THE TREASURY

ARTICLE 1. The Treasurer shall collect all money due or payable to the Academy, and all gifts and bequests made to it. He shall pay all bills due by the Academy, when approved by the proper officers, except those of the Treasurer's office, which may be paid without such approval; in the name of the Academy he shall sign all leases of real estate; and, with the written consent of a member of the Committee on Finance, he shall make all transfers of stocks, bonds, and other securities belonging to the Academy, all of which shall be in his official custody.

He shall keep a faithful account of all receipts and expenditures, submit his accounts annually to the Auditing Committee, and render them at the expiration of his term of office, or whenever required to do so by the Academy or the Council.

He shall keep separate accounts of the income of the Rumford Fund, and of all other special Funds, and of the appropriation thereof, and render them annually.

His accounts shall always be open to the inspection of the Council.

ARTICLE 2. He shall report annually to the Council at its March meeting on the expected income of the various Funds and from all other sources during the ensuing financial year. He shall also report the names of all Fellows who may be then delinquent in the payment of their Annual Dues.

ARTICLE 3. He shall give such security for the trust reposed in him as the Academy may require.

ARTICLE 4. With the approval of a majority of the Committee on Finance, he may appoint an Assistant Treasurer to perform his duties, for whose acts, as such assistant, he shall be responsible; or, with like approval and responsibility, he may employ any Trust Company doing business in Boston as his agent for the same purpose, the compensation of such Assistant Treasurer or agent to be fixed by the Committee on Finance and paid from the funds of the Academy.

ARTICLE 5. At the Annual Meeting he shall report in print all his official doings for the preceding year, stating the amount and condition of all the property of the Academy entrusted to him, and the character of the investments.

ARTICLE 6. The Financial Year of the Academy shall begin with the first day of April.

ARTICLE 7. No person or committee shall incur any debt or liability in the name of the Academy, unless in accordance with a previous vote and appropriation therefor by the Academy or the Council, or sell or otherwise dispose of any property of the Academy, except cash or invested funds, without the previous consent and approval of the Council.

See Chap. ii. art. 4, 5; chap. vi. art. 2; chap. ix. art. 6; chap. x. art. 1, 2, 3; chap. xi. art. 1.

CHAPTER VIII

THE LIBRARIAN AND THE LIBRARY

ARTICLE 1. The Librarian shall have charge of the printed books, keep a correct catalogue thereof, and provide for their delivery from the Library.

At the Annual Meeting, as Chairman of the Committee on the Library, he shall make a Report on its condition.

ARTICLE 2. In conjunction with the Committee on the Library he shall have authority to expend such sums as may be appropriated by the Academy for the purchase of books, periodicals, etc., and for defraying other necessary expenses connected with the Library.

ARTICLE 3. All books procured from the income of the Rumford Fund or of other special Funds shall contain a book-plate expressing the fact.

ARTICLE 4. Books taken from the Library shall be receipted for to the Librarian or his assistant.

ARTICLE 5. Books shall be returned in good order, regard being had to necessary wear with good usage. If any book shall be lost or injured, the Fellow to whom it stands charged shall replace it by a new volume or by a new set, if it belongs to a set, or pay the current price thereof to the Librarian, whereupon the remainder of the set, if any, shall be delivered to the Fellow so paying, unless such remainder be valuable by reason of association.

ARTICLE 6. All books shall be returned to the Library for examination at least one week before the Annual Meeting.

ARTICLE 7. The Librarian shall have the custody of the Publications of the Academy. With the advice and consent of the President, he may effect exchanges with other associations.

See Chap. ii. art. 6; chap. x. art. 1, 2.

CHAPTER IX

THE COUNCIL

ARTICLE 1. The Council shall exercise a discreet supervision over all nominations and elections to membership, and in general supervise all the affairs of the Academy not explicitly reserved to the Academy as a whole or entrusted by it or by the Statutes to standing or special committees.

It shall consider all nominations duly sent to it by any Class Committee, and act upon them in accordance with the provisions of Chapter III.

With the consent of the Fellow interested, it shall have power to make transfers between the several Sections, reporting its action to the Academy.

See Chap. iii. art. 2, 3; chap. x. art. 1.

ARTICLE 2. Seven members shall constitute a quorum.

ARTICLE 3. It shall establish rules and regulations for the transaction of its business, and provide all printed and engraved blanks and books of record.

ARTICLE 4. It shall act upon all resignations of officers, and all resignations and forfeitures of Fellowship; and cause the Statutes to be faithfully executed.

It shall appoint all agents and subordinates not otherwise provided for by the Statutes, prescribe their duties, and fix their compensation. They shall hold their respective positions during the pleasure of the Council.

ARTICLE 5. It may appoint, for terms not exceeding one year, and prescribe the functions of, such committees of its number, or of the Fellows of the Academy, as it may deem expedient, to facilitate the administration of the affairs of the Academy or to promote its interests.

ARTICLE 6. At its March meeting it shall receive reports from the President, the Secretaries, the Treasurer, and the Standing Committees, on the appropriations severally needed for the ensuing financial year. At the same meeting the Treasurer shall report on the expected income of the various Funds and from all other sources during the same year.

A report from the Council shall be submitted to the Academy, for action, at the March meeting, recommending the appropriation which in the opinion of the Council should be made.

On the recommendation of the Council, special appropriations may be made at any Stated Meeting of the Academy, or at a Special Meeting called for the purpose.

See Chap. x. art. 3.

ARTICLE 7. After the death of a Fellow or Foreign Honorary Member, it shall appoint a member of the Academy to prepare a biographical notice for publication in the Proceedings.

ARTICLE 8. It shall report at every meeting of the Academy such business as it may deem advisable to present.

See Chap. ii. art. 4, 5, 8; chap. iv. art. 1, 2; chap. vi. art. 1; chap. vii. art. 1; chap. xi. art. 1, 4.

CHAPTER X

STANDING COMMITTEES

ARTICLE 1. The Class Committee of each Class shall consist of the Vice-President, who shall be chairman, and the four Councillors of the Class, together with such other officer or officers annually elected as may belong to the Class. It shall consider nominations to Fellowship in its own Class, and report in writing to the Council such as may receive at a Class Committee Meeting a majority of the votes cast, provided at least three shall have been in the affirmative.

See Chap. iii. art. 2.

ARTICLE 2. At the Annual Meeting the following Standing Committees shall be elected by ballot to serve for the ensuing year:

(i) *The Committee on Finance,* to consist of three Fellows, who, through the Treasurer, shall have full control and management of the funds and trusts of the Academy, with the power of investing the funds and of changing the investments thereof in their discretion.

See Chap. iv. art. 3; chap. vii. art. 1, 4; chap. ix. art. 6.

(ii) *The Rumford Committee,* to consist of seven Fellows, who shall report to the Academy on all applications and claims for the Rumford Premium. It alone shall authorize the purchase of books, publications and apparatus at the charge of the income from the Rumford Fund, and generally shall see to the proper execution of the trust.

See Chap. iv. art. 3; chap. ix. art. 6.

(iii) *The Cyrus Moors Warren Committee,* to consist of seven Fellows, who shall consider all applications for appropriations from the income of the Cyrus Moors Warren Fund, and generally shall see to the proper execution of the trust.

See Chap. iv. art. 3; chap. ix. art. 6.

(iv) *The Committee of Publication,* to consist of three Fellows, one from each Class, to whom all communications submitted to the

Academy for publication shall be referred, and to whom the printing of the Proceedings and the Memoirs shall be entrusted.

It shall fix the price at which the Publications shall be sold; but Fellows may be supplied at half price with volumes which may be needed to complete their sets, but which they are not entitled to receive gratis.

Two hundred extra copies of each paper accepted for publication in the Proceedings or the Memoirs shall be placed at the disposal of the author without charge.

See Chap. iv. art. 3; chap. vi. art. 1, 3; chap. ix. art. 6.

(v) *The Committee on the Library,* to consist of the Librarian, *ex officio,* as Chairman, and three other Fellows, one from each Class, who shall examine the Library and make an annual report on its condition and management.

See Chap. iv. art. 3; chap. viii. art. 1, 2; chap. ix. art. 6.

(vi) *The House Committee,* to consist of three Fellows, who shall have charge of all expenses connected with the House, including the general expenses of the Academy not specifically assigned to the care of other Committees or Officers.

See Chap. iv. art. 1, 3; chap. ix. art. 6.

(vii) *The Committee on Meetings,* to consist of the President, the Recording Secretary, and three other Fellows, who shall have charge of plans for meetings of the Academy.

See Chap. iv. art. 3; chap. ix. art. 6.

(viii) *The Auditing Committee,* to consist of two Fellows, who shall audit the accounts of the Treasurer, with power to employ an expert and to approve his bill.

See Chap. iv. art. 3; chap. vii. art. 1; chap. ix. art. 6.

ARTICLE 3. The Standing Committees shall report annually to the Council in March on the appropriations severally needed for the ensuing financial year; and all bills incurred on account of these Committees, within the limits of the several appropriations made by the Academy, shall be approved by their respective Chairmen.

In the absence of the Chairman of any Committee, bills may be approved by any member of the Committee whom he shall designate for the purpose.

See Chap. vii. art. 1, 7; chap. ix. art. 6.

CHAPTER XI

MEETINGS, COMMUNICATIONS, AND AMENDMENTS

ARTICLE 1. There shall be annually eight Stated Meetings of the Academy, namely, on the second Wednesday of October, November, December, January, February, March, April and May. Only at these meetings, or at adjournments thereof regularly notified, or at Special Meetings called for the purpose, shall appropriations of money be made or amendments of the Statutes or Standing Votes be effected.

The Stated Meeting in May shall be the Annual Meeting of the Corporation.

Special Meetings shall be called by either of the Secretaries at the request of the President, of a Vice-President, of the Council, or of ten Fellows having the right to vote; and notifications thereof shall state the purpose for which the meeting is called.

A meeting for receiving and discussing literary or scientific communications may be held on the fourth Wednesday of each month, excepting July, August, and September; but no business shall be transacted at said meetings.

ARTICLE 2. Twenty Fellows having the right to vote shall constitute a quorum for the transaction of business at Stated or Special Meetings. Fifteen Fellows shall be sufficient to constitute a meeting for literary or scientific communications and discussions.

ARTICLE 3. Upon the request of the presiding officer or the Recording Secretary, any motion or resolution offered at any meeting shall be submitted in writing.

ARTICLE 4. No report of any paper presented at a meeting of the Academy shall be published by any Fellow without the consent of the author; and no report shall in any case be published by any Fellow in a newspaper as an account of the proceedings of the Academy without the previous consent and approval of the Council. The Council, in its discretion, by a duly recorded vote, may delegate its authority in this regard to one or more of its members.

ARTICLE 5. No Fellow shall introduce a guest at any meeting of the Academy until after the business has been transacted, and especially until after the result of the balloting upon nominations has been declared.

Article 6. The Academy shall not express its judgment on literary or scientific memoirs or performances submitted to it, or included in its Publications.

Article 7. All proposed Amendments of the Statutes shall be referred to a committee, and on its report, at a subsequent Stated Meeting or at a Special Meeting called for the purpose, two thirds of the ballot cast, and not less than twenty, must be affirmative to effect enactment.

Article 8. Standing Votes may be passed, amended, or rescinded at a Stated Meeting, or at a Special Meeting called for the purpose, by a vote of two thirds of the members present. They may be suspended by a unanimous vote.

See Chap. ii. art. 5, 8; chap. iii. chap. iv. art. 3, 4, 5; chap. v. art. 1; chap. vi. art. 1, 2; chap. ix. art. 8.

STANDING VOTES

1. Communications of which notice has been given to either of the Secretaries shall take precedence of those not so notified.

2. Fellows may take from the Library six volumes at any one time, and may retain them for three months, and no longer. Upon special application, and for adequate reasons assigned, the Librarian may permit a larger number of volumes, not exceeding twelve, to be drawn from the Library for a limited period.

3. Works published in numbers, when unbound, shall not be taken from the Hall of the Academy without the leave of the Librarian.

4. The Council, under such rules respecting nominations as it may prescribe, may elect as Associates of the Academy a limited number of men of mark in affairs or of distinguished service in the community.

Associates shall be entitled to the same privileges as Fellows, but shall not have the right to vote.

The admission fee and annual dues of Associates shall be the same as those of Fellows residing within fifty miles of Boston.

5. Communications offered for publication in the Proceedings or Memoirs of the Academy shall not be accepted for publication before the author shall have informed the Committee on Meetings of his readiness, either himself or through some agent, to use such time as the Committee may assign him at such meeting as may be convenient both to him and to the Committee, for the purpose of presenting to the Academy a general statement of the nature and significance of the results contained in his communication.

RUMFORD PREMIUM

In conformity with the terms of the gift of Sir Benjamin Thompson, Count Rumford, of a certain Fund to the American Academy of Arts and Sciences, and with a decree of the Supreme Judicial Court of Massachusetts for carrying into effect the general charitable intent and purpose of Count Rumford, as expressed in his letter of gift, the Academy is empowered to make from the income of the Rumford Fund, as it now exists, at any Annual Meeting, an award of a gold and a silver medal, being together of the intrinsic value of three hundred dollars, as a Premium to the author of any important discovery or useful improvement in light or heat, which shall have been made and published by printing, or in any way made known to the public, in any part of the continent of America, or any of the American Islands; preference always being given to such discoveries as, in the opinion of the Academy, shall tend most to promote the good of mankind; and, if the Academy sees fit, to add to such medals, as a further Premium for such discovery and improvement, a sum of money not exceeding three hundred dollars.

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